Computation of the mean reversal rate of geodynamo models

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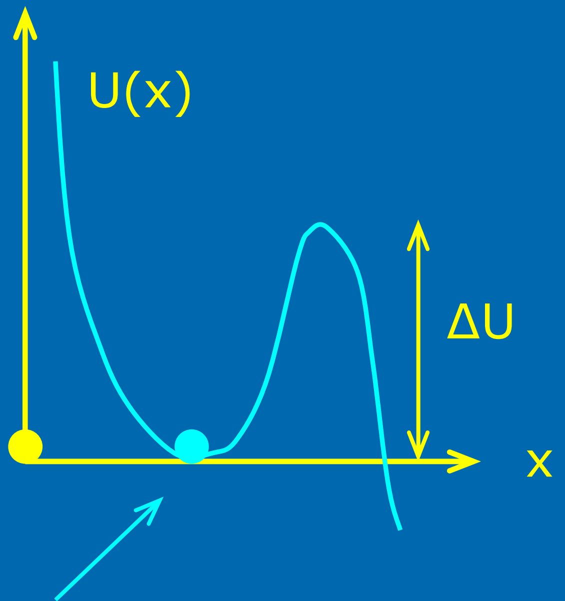
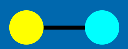


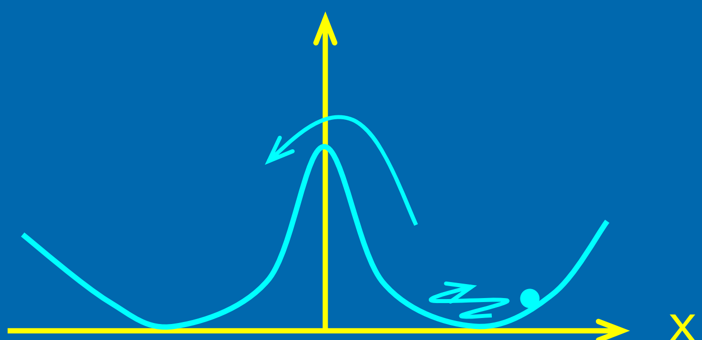
F = - ∂U/∂x + Brownian force

History

Kramers' escape problem (1940)

Dissociation of molecule in two parts 

$T_{esc} \sim T_{osc} \exp(\Delta U/kT)$



Mean reversal rate problem:
bistable, and no thermal kicks

[Hoyng et al. (GAFD 94 2001
(reversals of mean field)

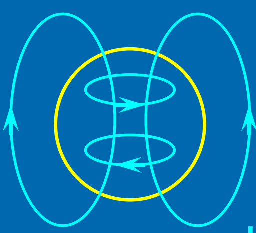
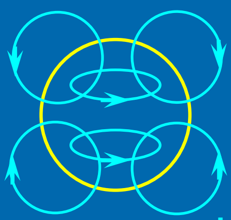
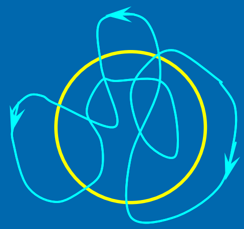
Hoyng & Duistermaat (HD model)
(EPL 68 2004; dipole + 2 overtones)]

Mean reversal rate: setup

$\partial B/\partial t = \nabla \times [U \times B - \eta \nabla \times B]$

$U = v + u(t)$ v = mean flow, $u(t)$ = convection (given)

$B = \Sigma_k a^k(t)\, b^k(r)$ B is decomposed in set of base functions b^k

$B = a^0$  $+ a^1$  $+ \cdot\cdot\cdot\; a^i$ 

b^0 b^1 b^i

dipole overtones

$$\partial_t a^i = [R^{ik} + V^{ik}(t)]\, a^k$$

$$R^{ik} = \int_V \hat{j}^{\,i} \cdot (v - \eta\nabla) \times b^k \; d^3r$$

$$V^{ik}(t) = -\int_V u(t) \cdot \hat{j}^{\,i} \times b^k \; d^3r$$

- choice of b^k and adjoints $\hat{b}^k$:
  eigenfunctions of R: $R^{ik} = \lambda^k \delta^{ik}$
- physics in terms of interaction of global modes
- reversal = sign flip of $a^0(t)$

HD model

nonlinearity by hand

$V^{ik}(t)$ uncorrelated & equal r.m.s. magnitude

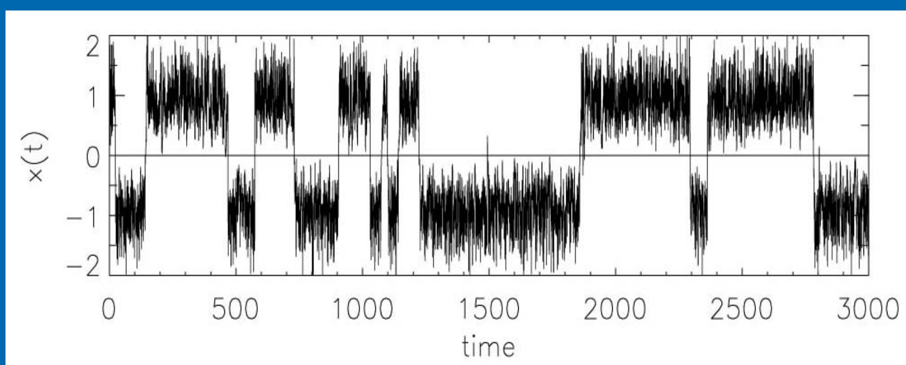
1 complex overtone

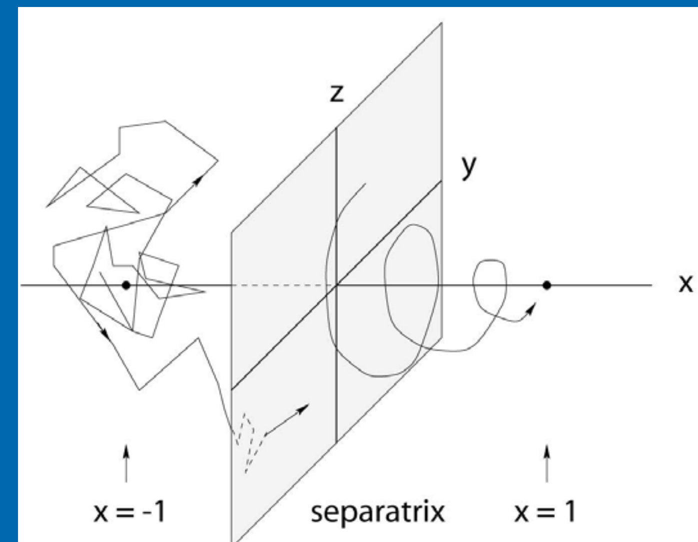
[must be removed for geodynamo model]

$$\partial_t x = (1-x^2)x + V^{00}x + V^{01}y + V^{02}z$$

$$\partial_t y = -ay - cz + V^{10}x + V^{11}y + V^{12}z$$

$$\partial_t z = cy - az + V^{20}x + V^{21}y + V^{22}z$$





3D escape problem

$\dot{x} = [\cdots]\, x \;\rightarrow\; \partial_t \log x = \cdots \;\rightarrow$ no reversals

$x \downarrow$ slowly $\;\rightarrow\; y, z$ (:) $x \;\rightarrow\; \dot{x}$ (:) $x \;\rightarrow$ no reversals

$x \downarrow$ fast $\;\rightarrow\; \dot{x} \approx V^{01}y + V^{02}z \;\rightarrow$ reversal possible

duration reversal ~ decay time first overtone

holds for any dynamo model

Computation mean reversal time in HD model

P(x,y,z,t) = probability distribution

T(x,y,z) = mean time to reach separatrix starting in x,y,z

$\dot{x} = \cdot\cdot\ ;\ \dot{y} = \cdot\cdot\ ;\ \dot{z} = \cdot\cdot$ $\longrightarrow$ Fokker-Planck Eq. for P(x,r,t) :

$$\frac{\partial P}{\partial t} = \left[-\frac{\partial}{\partial x}(1-x^2)x + \frac{a}{r}\frac{\partial}{\partial r}r^2 + \frac{1}{2}D(x^2+r^2)\left(\frac{\partial^2}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\right)\right]P$$

$r = (y^2+z^2)^{1/2}$; $D = \langle (V^{ij})^2 \rangle \tau_c$

$\partial_t P = MP$ $\rightarrow$ $M^\dagger T = -1$

Mean time between reversals $T_{rev} = 2T(1,0)$

Symmetries in the problem allow finding approximate value T_{rev}

Geodynamo: many overtones, nonzero cross correlations between V^{ij} (symmetries are gone). Need alternative approach: reduce to 1D problem by projecting out the overtones

Generalisation to many overtones + correlations

$$\dot{a} = (1-a^2)a + V^{00}a + \Sigma_{k\geq 1} V^{0k} a^k \qquad (a = a^0)$$

$$\dot{a}^i = \lambda^i a^i + V^{i0} a + \Sigma_{k\geq 1} V^{ik} a^k$$

variability dipole between reversals

reversals

systematic evolution

overtone excitation

overtone mixing

- Take $\langle V^{in}(t) V^{jm}(t-s)\rangle = D\,\delta^{ij}\delta^{nm}\,\delta(s)$ [correlations: $D \rightarrow D^{ijnm}$]

- $\partial_t P = MP$ $\rightarrow$ integrate over $a^1, a^2, ..$ $\rightarrow$ equation for $p(a)$ only:

$$\partial_t p = -\frac{\partial}{\partial a}(1-a^2)a\,p + \frac{1}{2} D \frac{\partial^2}{\partial a^2}(a^2 + \langle r^2\rangle)\,p \qquad r^2 = \Sigma_{k\geq 1}(a^k)^2$$

- $\partial_t p = \mathrm{M}\,p$ $\rightarrow$ $\mathrm{M}^{\dagger} T = -1$ is ODE for T:

$$(1-a^2)a\frac{dT}{da} + \frac{1}{2} D (a^2 + \langle r^2\rangle)\frac{d^2T}{da^2} + 1 = 0$$

find $\langle r^2\rangle$ near $a=0$ by ignoring mixing term

Numerical results

9 overtones

no correlations between $V^{ij}(t)$

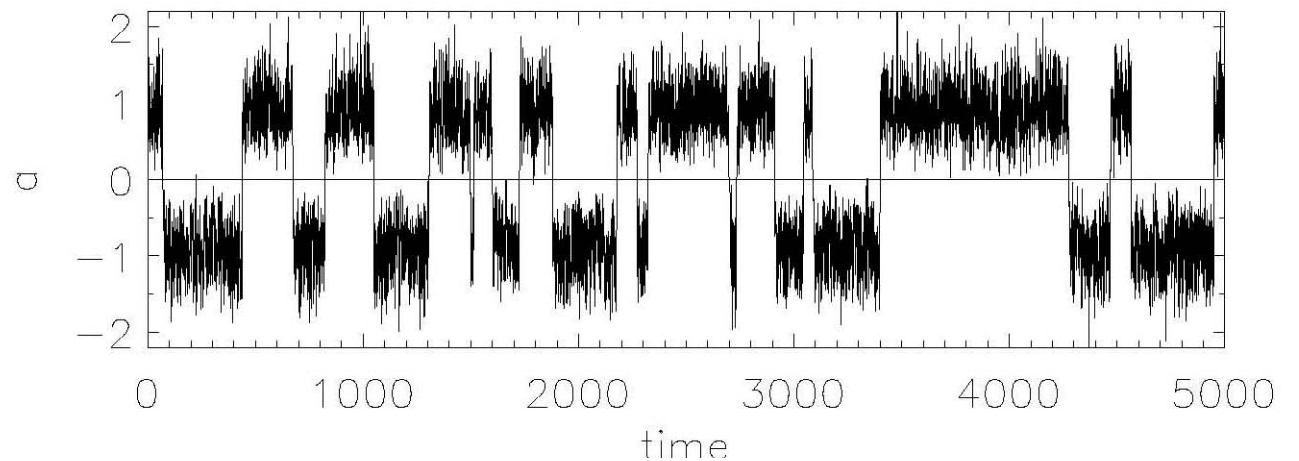
T_{rev} = 146 (theor),

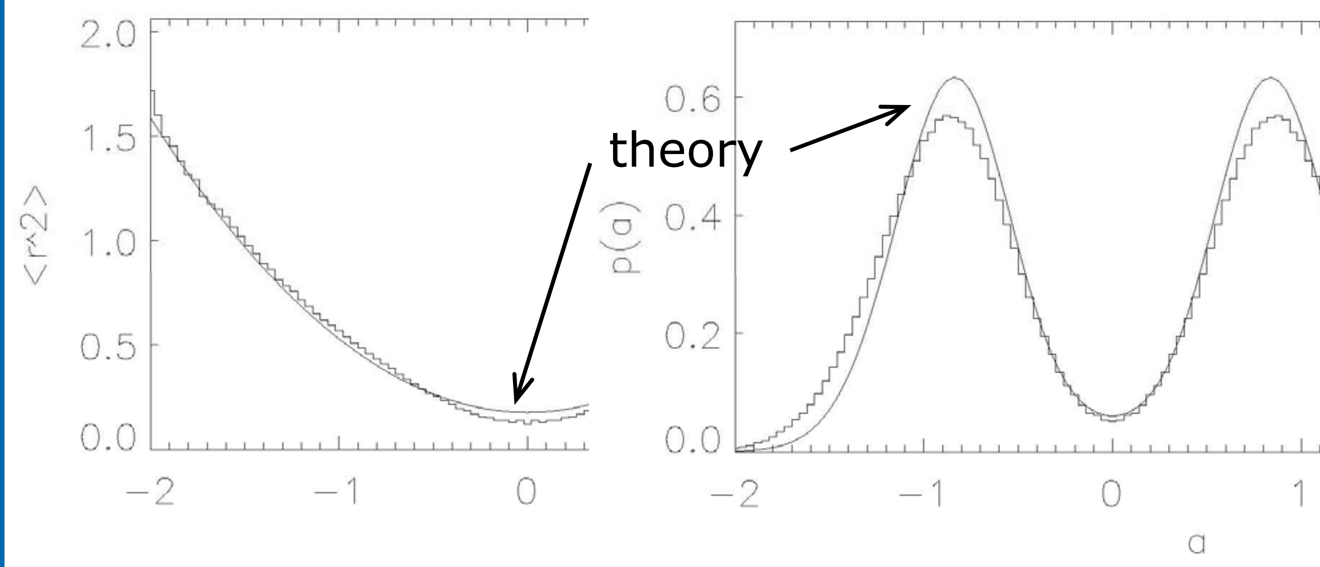
meas: 194 ± 27

(usually agreement to within 10-40%)

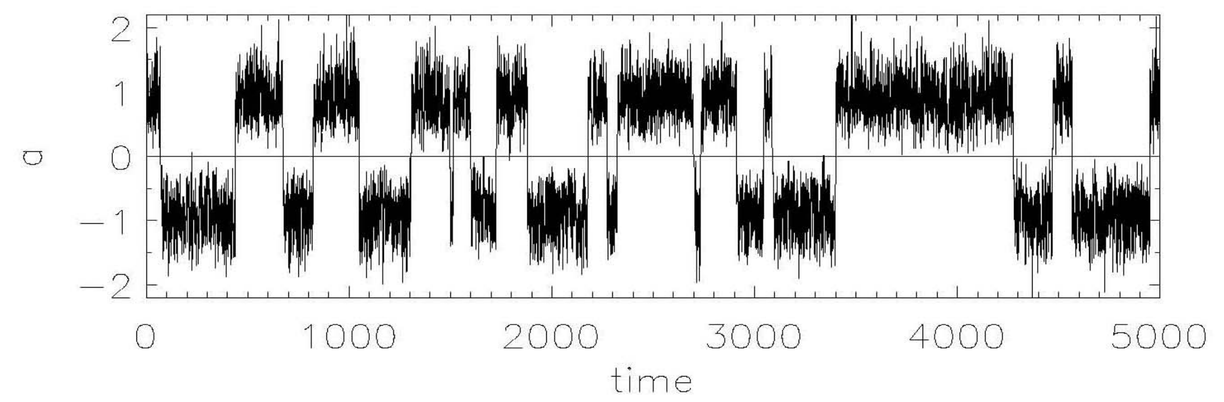
$T_{rev} \sim \exp(K/D)$

K function of D & $\{\lambda^k\}$





What causes a reversal?



$\dot{a} = (1-a^2)a + V^{00} a + \Sigma_{k\geq 1} V^{0k} a^k$

$\dot{a}^i = \lambda^i a^i + V^{i0} a$

($a = a^0$; mixing term ignored)

Three-step process:

1. jump in one or more $V^{i0}(t)$ makes corresponding a^i temporarily large;
2. jump in $V^{00}(t)$ makes the dipole amplitude a(t) collapse;
3. $V^{0k}(t)a^k$ pushes a(t) to the other basin of attraction.

Step 1 and 2 need to be cotemporal within overtone decay time → explains why reversals are rare, despite large fluctuations in a(t)

In a numerical model the jumps may be connected to properties of u(t) via

$$V^{ik}(t) = -\int_V u(t) \cdot \hat{j}^{\,i} \times b^k \; d^3r$$

Implementation

Collaboration with Dieter Schmitt, Martin Schrinner,
Johannes Wicht MPS / Lindau

Numerical model

Velocity averages $U = v + u(t)$

Find eigenmodes $b^n(r)$, adjoints $\hat{b}^n(r)$ & eigenvalues λ^n of R

Compute $V^{ik}(t) = -\int_V u(t) \cdot \hat{j}^{\,i} \times b^k \, d^3r$

Compute $\int_0^\infty \langle V^{ij}(t) V^{kl}(t-\tau) \rangle \, d\tau$

Compute D and $\langle r^2 \rangle$ $\rightarrow$ p(a) and T_{rev}

Discussion

Numerical tests of theory extended HD model encouraging

But: quenching of fundamental mode and diffusion coefficients