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# Introduction to Liouville Theory, II

**Summary of 1st part:** Conformal invariance

$\Rightarrow$ can construct Liouv. th. correlation function in the form

$$\Big\langle \prod_{r=1}^{n} V_{\alpha_r}(z_r,\bar{z}_r) \Big\rangle_{C} = \int_{\mathbb{R}_+^{3g-3+n}} d\mu(P)\, |\mathcal{F}(P;b,\alpha;q)|^2$$

where $\mathcal{F}$ = power series of the form

$$\mathcal{F}(\ldots) = \prod_{t} N(\alpha_3^t,\alpha_2^t,\alpha_1^t) \sum_{\nu\in\mathbb{Z}_+^{3g-3+n}} g_\nu(P,b,\alpha)\,\cdot$$

$$\cdot\, q_1^{\Delta_{P_1}+n_1} \cdots q_h^{\Delta_{P_h}+n_h} \qquad (h = 3g+3-n)$$

Completely fixed by (CWI)

$$d\mu(P) = \prod_{r=1}^{h} dP_r\, P_r^2$$

$$\langle \Pi \cdots \rangle_{\mathcal{C}} = \int d\mu(P) \prod_t \underbrace{|N(\alpha_3^t, \alpha_2^t, \alpha_1^t)|^2}_{C(\alpha_3, \alpha_2, \alpha_1)} |\mathcal{G}(P, b, \alpha, q)|^2$$

$$\mathcal{G}_{\mathcal{C}_{0,3}}(v_{\alpha_3} \otimes v_{\alpha_2} \otimes v_1) \equiv 1$$

- Heavily dependent on a pants decomp.
- Only valid when $q$'s sufficiently small

Physical requirement:

Corr. fcts. real analytic fcts. on $\mathcal{M}_{g,n}$

Independent of pants decomposition (crossing symmetry / modular invariance)

Modular invariance possible thanks to existence of rels

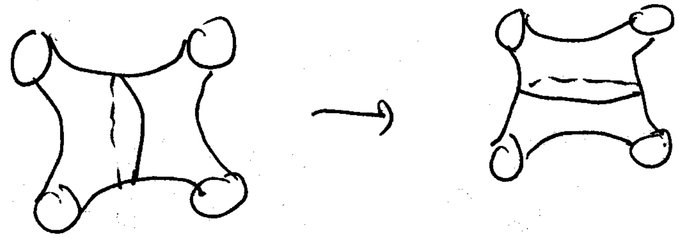
$$\mathcal{F}_2(P_2, b; \alpha, q_1) = \int_{\mathbb{R}_+^h} d\mu(P_1)\, f(P_2, P_1, b, \alpha)\, \mathcal{F}_1(P_1, b; \alpha, q_2) \qquad (F)$$

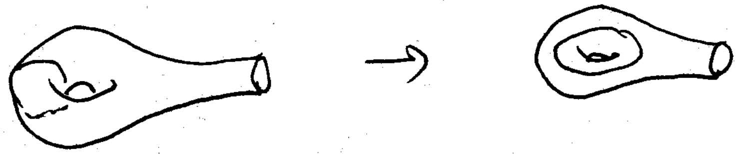
conf. blocks defined from 2nd pants decomp

gluing parameters from first pants decomp

Rem

(1) Rels. (F) follow from cases $\mathcal{C} = \mathcal{C}_{0,4}$ and $\mathcal{C} = \mathcal{C}_{1,1}$.

a)  (J. T. '01, '03)

b)  (can be reduced to case (a)) (Hadasz et al. '10)

Relations : root for a harmonic analysis of $\mathrm{Diff}(S_1)$ / Vir.

Indeed, there exist Sub$^{(*)}$-Hilbert space of space of conformal blocks spanned by $F_{\mathcal{C},P}$ for any decomp. $\lhd$

Thm (U) There is a canonical choice of $C(\alpha_3, \alpha_2, \alpha_1) = |N(\alpha_3, \alpha_2, \alpha_1)|^2$

s.t. (F) ⇝ unitary operator on $L^2(\mathbb{R}_+^{3g-3+n}, d\mu)$

Thm U ⇒ Modular invariance

$$\int d\mu(P_1)\, |\mathcal{F}_2(P_2; q_2)|^2 = \int d\mu(P_1)\, d\mu(P_1') \int d\mu(P_2)$$

$$\cdot\, f(P_2, P_1, b, \alpha)^* f(P_2, P_1', b, \alpha)$$

$$\cdot\, (\mathcal{F}(P_1, q_1))^* \mathcal{F}(P_1', q_1)$$

$$= \int d\mu(P_1)\, |\mathcal{F}_1(P_1, q_1)|^2$$

Remarks

(i) $|N(\alpha_3,\alpha_2,\alpha_1)|^2 = C(\alpha_3,\alpha_2,\alpha_1)$

Norm of $F_{C_{0,3},(\alpha_3,\alpha_2,\alpha_1)}$

DOZZ formula

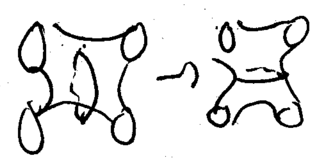
(ii) $$C(\alpha_3,\alpha_2,\alpha_1) \propto \frac{\Upsilon_0\,\Upsilon(2\alpha_3)\,\Upsilon(2\alpha_2)\,\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1+\alpha_2+\alpha_3-Q)\,\Upsilon(\alpha_1+\alpha_2-\alpha_3)\cdot\Upsilon(\alpha_1+\alpha_3-\alpha_2)\,\Upsilon(\alpha_2+\alpha_3-\alpha_1)}$$

where $\log\Upsilon(x) =$

$$\int \frac{dt}{t}\left(\frac{Q}{2}-x\right)^2 e^{-t} - \frac{\left(\sinh\left(\frac{Q}{2}-x\right)t\right)^2}{\sinh\frac{bt}{2}\,\sinh\frac{t}{2b}}$$

proportionality constant: $\left(\pi\mu\,\gamma(b^2)\,b^{2-2b^2}\right)^{\frac{Q-\alpha_1-\alpha_2-\alpha_3}{b}}$

$(Q = b+b^{-1})$



$-b^2 \to \frac{1}{k+2}$

Remark $C(\alpha_3,\alpha_2,\alpha_1) = f\left(\frac{iQ}{2}, i\left(\frac{Q}{2}-\alpha_3\right), b, \alpha_2,\alpha_2,\alpha_1,\alpha_1\right)$

Relation to gauge theory $\boxed{SU(2)}$

Gaiotto $\rightarrow$  $_e$ $\longmapsto$ $\mathcal{G}_e$ (Gaiotto)

Pestun, AGT $\Big\} \rightarrow$ $Z_{\mathcal{G}_e}(S_4) = \int d\nu(a) \, |Z(a, \epsilon_1, \epsilon_2, m, \tau)|^2$

$\epsilon_1 = \epsilon_2$

AGT: $Z = \mathcal{F}$ $\quad a \to P, \quad \epsilon_1,\epsilon_2,m \to \alpha, \quad \tau \to q$

$$Z_{\mathcal{G}_e}(S_4) \propto \langle \prod_i V_{\alpha_1} \cdots V_{\alpha_n} \rangle_{Liou}$$

Recall: Choice of pants decomposition $\rightsquigarrow$ Lagrangian for $\mathcal{G}_e$

& S-duality $\Rightarrow$ Invariance of $Z_{\mathcal{G}_e}(S_4)$ on pants decomp.

$\overset{(AGT)}{\Longleftrightarrow}$ Modular invariance of Liouville corr. fcts

## (C) Relation to quantum Teichmuller theory

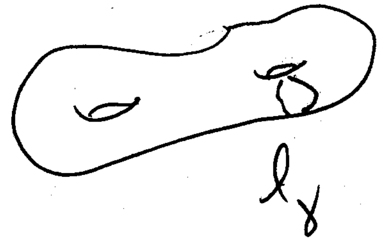
Liouville action $S_L[\phi_\alpha]$ at solution of EOM

Function on Teichmuller space $T_{g,n}$

Define natural sympl. form

$\omega_L = 2i \, \partial\bar{\partial} S_L[\phi_\alpha]$, $\quad d = \partial + \bar{\partial}$, de Rham differential on $T_{g,n}$

Natural observables: length function (wrt $e^{2\varphi} dz d\bar{z}$)




$$L_\gamma = 2\cosh\frac{\ell_\gamma}{2}$$

$\omega_L$ induces Poisson brackets for $L_\gamma$'s

$$\{L_\gamma L_{\gamma'}\} = \text{something known.}$$

Quantization:

Algebra $\mathcal{A}_b$ generated by quantum length ops. $\hat{L}_\gamma$

$$[\hat{L}_\gamma, \hat{L}_{\gamma'}] = \underbrace{b^2}_{\hbar}\{L_\gamma, L_{\gamma'}\} + O(b^4)$$

$\mathcal{A}_b$ realized on $L^2(\mathbb{R}^{3g-3+n})$

Main result:

$$\mathcal{F}(P, b, \alpha, q) = \Psi_q(\gamma) \equiv \langle \gamma | q \rangle$$

wave functions in quantum Teichmuller theory.

$$\cancel{\alpha = \frac{Q}{2} + i\frac{P}{4\pi b}} \qquad P = \frac{\ell}{4\pi b}$$

where

- $\langle l|$ ← Eigenstate of $\hat{L}_\gamma$ for all $\gamma$'s defining a pants decomp.
- $|q\rangle$ ← Eigenstate of $\hat{q}_r$, operator obtained by quantizing $q_r$

Relation to gauge theory (AGT)

$Z_{\mathcal{G}_c}\Big($  $L_\gamma\Big) = \langle q|\hat{L}_\gamma|q\rangle$

Wilson loop.

$\langle$  $|L_\gamma|$  $\rangle$