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Y. Tachikawa

Liouville from 4D Gauge Theory, I

$\mathcal{N}=2$, $SU(2)$ 4 flavors

$$\tilde{\rho} = R_G \otimes R_A \otimes R_B \oplus R_G \otimes R_C \otimes R_D$$

$$SU(2)_G \times \underbrace{(SU(2)_A \times SU(2)_B \times SU(2)_C \times SU(2)_D)}_{SO(8)}$$

$\varepsilon_1, \varepsilon_2$

$$\begin{pmatrix} a & \\ & -a \end{pmatrix} \quad \begin{matrix} m_A & \\ & -m_A \end{matrix} \quad \begin{matrix} m_B & \\ & -m_B \end{matrix} \quad \cdots$$

adj $\pm 2a$

$\tilde{\rho}$ $\pm a \pm m_A \pm m_B$

$\pm a \pm m_C \pm m_D$

$$Z_{S_4} = \int_0^\infty da \, \frac{\prod_{\pm\pm} \Gamma_2(\pm m_A \pm m_B + a) \prod_{\pm\pm} \Gamma_2(\pm m_C \pm m_D + a)}{\Gamma_2(2a)\Gamma_2(-2a)\Gamma_2(2a)\Gamma_2(-2a)} \cdot \left| Z_{Nek}(\varepsilon_1, \varepsilon_2, a, m_{A,B,C,D}, q) \right|^2$$

$$\langle V_{m_A}(0) V_{m_B}(q) V_{m_C}(1) V_{m_D}(\infty) \rangle$$

$$= \int_0^\infty da\, C(m_A, m_B, a)\, C(a, m_C, m_D)\, |F|^2$$

$$C(a_1, a_2, a_3) \propto \frac{\Upsilon(2a_1)\Upsilon(2a_2)\Upsilon(2a_3)}{\Upsilon(a_1+a_2+a_3)\,\Upsilon(a_1+a_2-a_3)\,\Upsilon(a_1-a_2+a_3)\,\Upsilon(-a_1+a_2+a_3)}$$

$$\Upsilon(x) = \frac{1}{\Gamma_2(x)\Gamma_2(-x)}$$

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(1) geometric engineering

$\mathcal{N}=2$ 4d gauge theory $\longleftrightarrow$ type II string / non compact CY

~1997 $\quad F$ prepotential $\longleftrightarrow$ genus zero partition function

$\nwarrow Z$ $\quad$ when $\varepsilon_1+\varepsilon_2=0 \longleftrightarrow$ all genus partition function

$F = \varepsilon_1\varepsilon_2 \log Z$ $\quad \varepsilon_1 = -\varepsilon_2 = \hbar \longleftrightarrow$ top. string coupling

The noncompact CY is an $A_1^\#$ fibration on SW curve Σ
$\quad\uparrow$
$\mathbb{C}^2/\mathbb{Z}_2$

and the all-genus partition function

$\Leftrightarrow$ free fermion or chiral boson partition func. on Σ

Vafa, Dijkgraaf, ...

$\Sigma \to C$ ← Gaiotto curve

N 1

SW curve N fermions $\leftrightarrow$ N bosons.

Q. What happens when $\varepsilon_1 + \varepsilon_2 \neq 0$?

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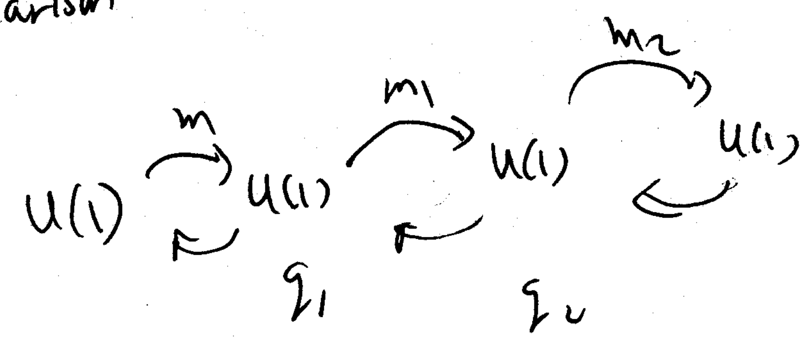
② Carlson-Okounkov 2008



$$Z_{Nek}(\varepsilon_1, \varepsilon_2) = \mathrm{tr}\, e^{m_1 \phi(z_1)} e^{m_2 \phi(z_2)} \cdots e^{m_n \phi(z_n)}$$

Q what happens when $U(N)$ instead of $U(1)$?

(3) Braverman 2003 (?)

Z_{Nek} instanton partition functions

pure G ($\check{\rho}$ = empty)

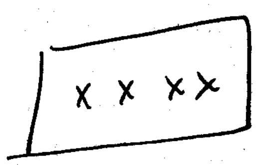
but with surface operator

$Z_{Nek} = \langle w | w \rangle$

$|w\rangle$ = Whittaker vector of affine G algebra.

Q. What happens without surface operator?

Conformal blocks



$L_n,\ n \in \mathbb{Z}$ $\quad [L_n, L_m] = (m-n) L_{m+n} + \frac{c}{12}(m^3 - m)\, \delta_{m,-n}$

$V_\Delta \ni |\Delta\rangle$ $\quad L_n |\Delta\rangle = 0 \quad n > 0$

$L_0 |\Delta\rangle = \Delta |\Delta\rangle$

$$L_0 (\cdots L_{-2}^{n_2} L_{-1}^{n_1} |\Delta\rangle) = (\Delta + \underbrace{\sum_i i n_i}_{k}) (\cdots L_{-2}^{n_2} L_{-1}^{n_1} |\Delta\rangle)$$

at "degree" k: # states = # partitions

$$V_\Delta^* \ni \langle \Delta | \qquad \langle \Delta | L_n = 0 \quad n<0$$

$$\langle \Delta | L_0 = \Delta \langle \Delta |$$

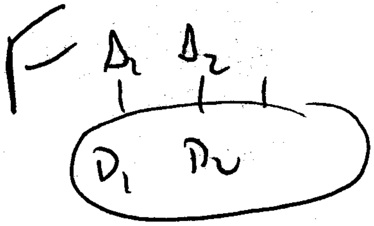
$$\langle \Delta | \Delta \rangle = 1.$$

$$V_\Delta = V_{\Delta_1, \Delta, \Delta_2} : V_{\Delta_2} \to V_{\Delta_1}$$

$$V_\Delta(z) = z^{L_0} V_\Delta z^{-L_0}$$

$$[L_m, V_\Delta(z)] = (1+m) z^m \Delta \cdot V_\Delta(z) + z^{m+1} \partial V_\Delta(z)$$

$$\langle \Delta_1 | V_\Delta(z) | \Delta_2 \rangle = z^{\Delta_1 - \Delta - \Delta_2}$$

F 

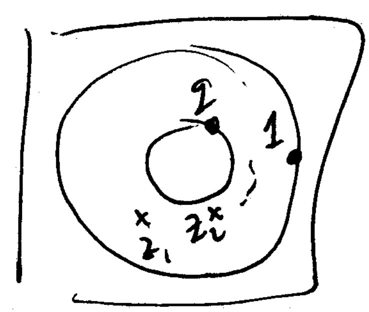
$$(q_1, q_2, \dots, q_n) = \mathrm{tr}\, q^{L_0} V_{D_1 \Delta_1 D_2}(z_1) V_{D_2 \Delta_2 D_3}(z_2) \cdots V_{D_n \Delta_n D_1}(z_n)$$

where $z_1 = q_1$

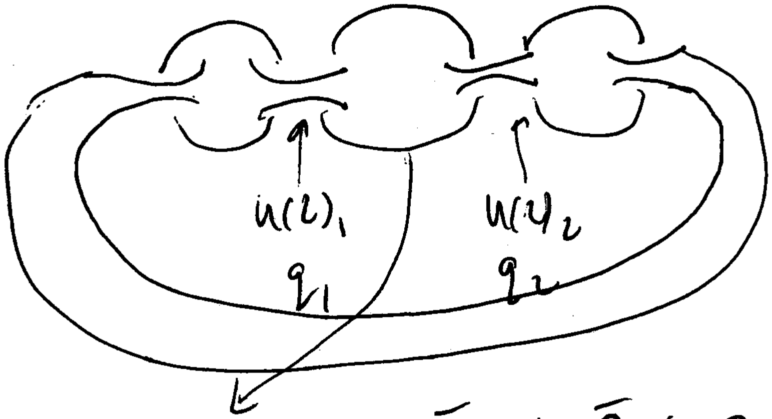
$z_2 = q_1 q_2$

$\vdots$

$q = z_n = q_1 q_2 \cdots q_n$



$z \sim z q.$



$$\tilde{\rho} = \mathcal{D}_1 \otimes \bar{\mathcal{D}}_2 \otimes \bar{\mathcal{D}}_1 \otimes \mathcal{D}_2$$

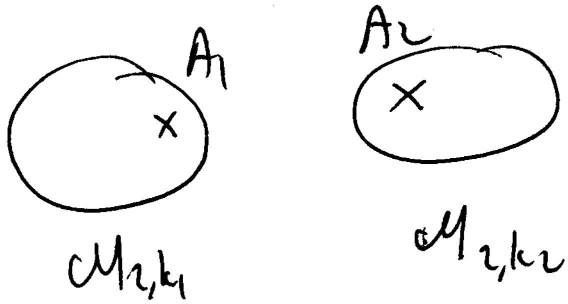
$\mathcal{M}_{N,k}$ = the moduli space of SU(N) instantons w/ inst #k

$$Z_{Nek}(\varepsilon_1, \varepsilon_2, a_1 a_2 \cdots a_n, m_1 m_2 \cdots m_2, q_1 \cdots q_n)$$

U(2) $(a_1\ a_1)$. punctures m_1, m_2, m_3.

$$Z_{Nek} = \sum_{k_1, k_2, \dots, k_n} q_1^{k_1} q_2^{k_2} \cdots q_n^{k_n} \int_{\mathcal{M}_{2,k_1}} \int_{\mathcal{M}_{2,k_2}} \cdots \int_{\mathcal{M}_{3,k_3}} \cdots$$

$$\cdot\, e(\mathrm{Ind}\, \not{D}_{\rho_{12}})\, e(\mathrm{Ind}\, \not{D}_{\rho_{23}})$$



$$\not{D} = \gamma^\mu(\partial_\mu + A_{1,\mu} - A_{2,\mu})$$

Ind $\not{D}_{\rho_{12}}$ is a vector bundle (at smooth points)

$$= \sum_{k_1 \cdots k_N} q_1^{k_1} q_2^{k_2} \cdots q_N^{k_N} \sum_{\substack{(Y_1^{(1)}, Y_1^{(2)}) \\ |Y_1^{(1)}|+|Y_1^{(2)}| = k_1}} \sum_{(Y_2^{(1)} Y_2^{(2)})} \cdots \sum_{(Y_n^{(1)} Y_n^{(2)})}$$

$$\frac{Z_H(Y_1^{(1)}, Y_1^{(2)}, Y_2^{(1)}, Y_2^{(2)}, a_1, a_2, m_1)\, Z_H(Y_2^{(1)} Y_2^{(2)} Y_3^{(1)} Y_3^{(2)} a_2 a_3 m_2) \cdots}{Z_V(Y_1^{(1)}, Y_1^{(2)}, a_1)\, Z_V(Y_2^{(1)}, Y_2^{(2)} a_2) \cdots}$$

$\mathcal{H}_a$: spanned by $|Y^{(1)} Y^{(2)}\rangle$ $\quad \oplus \int_{M_{2,k}} \nu(\varepsilon) \, \mathcal{H}(\varepsilon)$

$$\langle Y^{(1)} Y^{(2)} | Y'^{(1)} Y'^{(2)} \rangle = \delta_{Y^{(1)} Y'^{(1)}} \delta_{Y^{(2)} Y'^{(2)}} \frac{1}{Z_V(Y^{(1)}, Y^{(2)}, a)}$$

"N" = operator counting boxes

$$N |Y^{(1)} Y^{(2)}\rangle = (|Y^{(1)}| + |Y^{(2)}|) |YY\rangle$$

$$\Phi_{a_1, m, a_2} : \mathcal{H}_{a_2} \to \mathcal{H}_{a_1}$$

$$\Phi_{a_1, m, a_2} |Y_2^{(1)} Y_2^{(2)}\rangle_{a_2} = \sum_{Y_1^{(1)} Y_1^{(2)}} \frac{Z_H(Y_1^{(1)} Y_1^{(2)} Y_2^{(1)} Y_2^{(2)}, a_1, a_2, m)}{Z_V(Y_2^{(1)} Y_2^{(2)}\, a_2)} |Y_1^{(1)} Y_1^{(2)}\rangle$$

$$Z_{Nek} = \mathrm{tr}\, q^N \Phi_{a_1, m_1 a_2}(z_1) \Phi_{a_2 m_2 a_3}(z_2) \cdots \Phi_{a_n m_n a_1}(z_n)$$

$$\Phi_m(z) = z^N \Phi_m z^{-N}$$

$\mathcal{H}_a$: parameterized by TWO Young diagrams.

$$\mathcal{H}_a \simeq \text{Fock} \otimes V_{\Delta(a)}$$

$$\Phi_{a_1, m, a_2} \simeq e^{m\phi} \otimes V_{\Delta(a_1), \Delta(m), \Delta(a_2)}$$

$$\Delta(a) = \frac{1}{4}\frac{(\varepsilon_1+\varepsilon_2)^2}{\varepsilon_1\varepsilon_2} - \frac{a^2}{\varepsilon_1\varepsilon_2}$$

$$c = 1 + 6\frac{(\varepsilon_1+\varepsilon_2)^2}{\varepsilon_1\varepsilon_2}$$

$$\Phi \simeq e^{m\phi} \times V$$