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Feynman graphs, Hall algebras and $\mathbb{F}_1$-linear categories

1. x $\mathbb{F}_1$
2. x Hall algebras
3. x Feynman graphs
4. x $Rep(Q, \mathbb{F}_1)$
5. x $Coh(X/\mathbb{F}_1)$

$\underline{\mathbb{F}_1}$ "$\lim_{q\to 1}$ linear alg$/\mathbb{F}_q$ = combinatorics of sets"

$$/\mathbb{F}_q \quad \# \underset{\substack{\shortparallel \\ GL(n,\mathbb{F}_q)/P}}{Gr(k,n)} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q!\,[k]_q!}$$

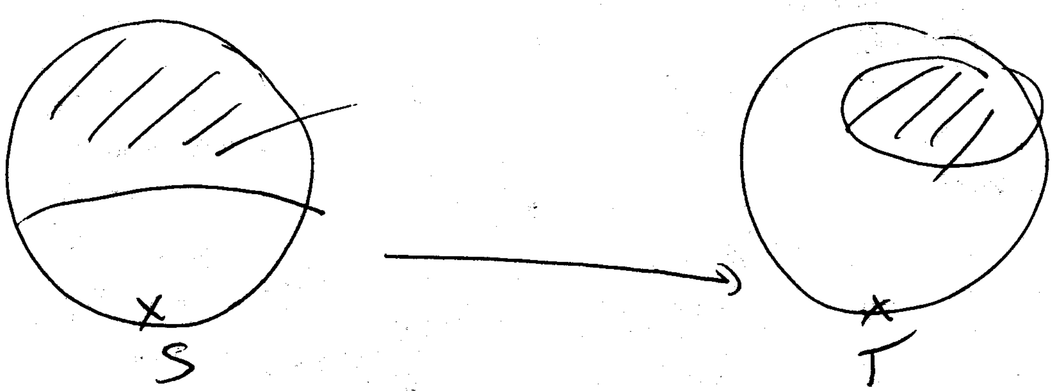
$$\lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}$$

Vect/$\mathbb{F}_1$

$$\mathrm{Ob}(\mathrm{Vect}/\mathbb{F}_1) = \{\text{pointed sets } (S, *)\}$$

$$\mathrm{Hom}((S,*),(T,*)) = \left\{ f \text{ basepoint preserving},\ f|_{S \setminus f^{-1}(*_T)} \text{ is injective} \right\}$$

$$T : V \to W \ /k$$
$$\cong$$
$$V/\ker T \hookrightarrow X$$



1) All maps have kernels + cokernels
2) $S \oplus T = S \vee T$
3) $S \otimes T = S \wedge T$
4) $f : S \to T \implies f^T : T \to S$

All of the categories in this talk are enriched over $\mathrm{Vect}/\mathbb{F}_1$

Hall Algebras $\quad \mathcal{C}$

$$\mathrm{Hecke}_{\mathcal{C}} = \left\{ (A \subset B) \;\; \begin{matrix} B \in \mathrm{Isom}(\mathcal{C}) \\ A \subset B \end{matrix} \right\}$$

$$\begin{array}{ccc} & \mathrm{Hecke}_{\mathcal{C}} & \\ \swarrow \pi_1 & \downarrow \pi_2 & \searrow \pi_3 \\ \mathrm{Isom}(\mathcal{C}) & \mathrm{Isom}(\mathcal{C}) & \mathrm{Isom}(\mathcal{C}) \\ \pi_1(A \subset B) = A & \pi_2(A \subset B) = B/A & \pi_3(A \subset B) = B \end{array}$$

$H_{\mathcal{C}} = \mathbb{C}[\mathrm{Isom}\,\mathcal{C}]$ w. finite support

$$f * g = \pi_{3*}\left(\pi_1^*(f)\, \pi^*(g)\right)$$

δ_A = delta fn supported at the isomorphism class of A.

$$\delta_A * \delta_C = \sum_{\substack{A \subseteq B, \\ B/A = C}} \delta_B$$

$$C_{AC}^{B} = \#\{0 \to A \to B \to C \to 0\}$$

$$= \frac{\sum C_{AC}^{B}\, \delta_B}{|\mathrm{Aut}\,A|\,|\mathrm{Aut}\,B|}$$

If C has $\oplus$, $\Delta(f)(A,B) = f(A \oplus B)$

In our cases $\Rightarrow$ will get a co-commutative Hopf algebra

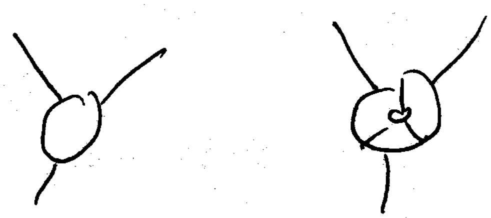
$$= U(n_{\mathcal{C}})$$

Hall Lie algebra of $\mathcal{C}$

③ <u>Feynman graphs.</u>

φ^3 (in 6-dim)

$\mathcal{C}_{\varphi^3}$ $\quad Ob(\mathcal{C}_{\varphi^3}) = \{$ trivalent graphs with at least one loop $\}$



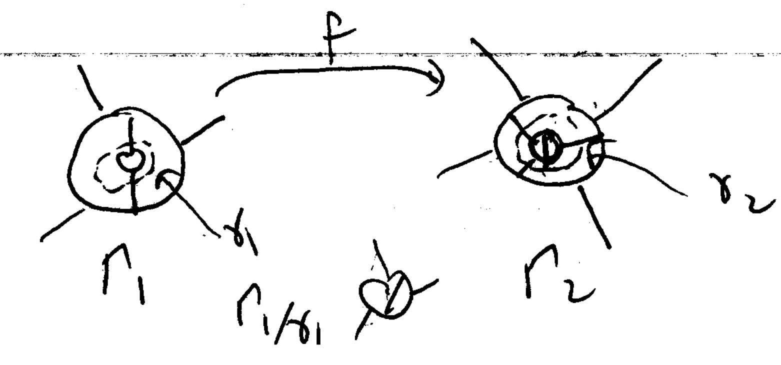
$$\mathrm{Hom}(\Gamma_1, \Gamma_2) = \{(\gamma_1, \gamma_2, f) \mid \gamma_1 \subseteq \Gamma_1,\ \gamma_2 \subseteq \Gamma_2,$$

each component of γ_i has 2 or 3 external legs

$$f : \Gamma_1/\gamma_1 \xrightarrow{\sim} \gamma_2 \}$$

Example

$\mathcal{E}_f$:



Properties of $C\varphi^3$

(1) All maps have kernels + cokernels

(2) $\emptyset$

(3) $\oplus = \amalg$

$\mathrm{Hom}(\Gamma_1, \Gamma_2) \in \mathrm{Ob}(\mathrm{Vect}/\mathbb{F}_1)$

$\exists$ forgetful ~~free~~ functor

$\mathfrak{F} : C\varphi^3 \to \mathrm{Vect}/\mathbb{F}_1$

$\mathfrak{F}(\Gamma) = \{ \text{subgraph of } \Gamma, \emptyset \}$

(Joint with Kobi Kremnizer)

We can define the Hall algebra of $C\varphi^3$, called $H_{C\varphi^3} = U(n_{C\varphi^3})$

$\Gamma_1 \triangleright \Gamma_2 = \sum$ all insertions of $\Gamma_1 \to \Gamma_2$

$[\Gamma_1, \Gamma_2] = \Gamma_1 \triangleright \Gamma_2 - \Gamma_2 \triangleright \Gamma_1$

$U(n_{C\varphi^3})$ = dual to Connes-Kreimer Hopf algebra

Q = quiver.

<u>Recollections on $\mathrm{Rep}(Q, \mathbb{F}_q)$</u>

Can define $H_{\mathrm{Rep}(Q,\mathbb{F}_q)}$

<u>Thm</u> (Ringel, Green)

$\exists$ an injective Hopf algebra homomorphism $U_q(n) \hookrightarrow H_{\mathrm{Rep}(Q,\mathbb{F}_q)}$

$\left(\mathfrak{g}_Q = n_- \oplus h \oplus n_+ \right)$

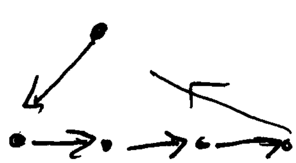
<u>$\mathrm{Rep}(Q, \mathbb{F}_1)$</u> $\overset{\text{def}}{=} \mathrm{Rep}(Q, \mathrm{Vect}/\mathbb{F}_1)$

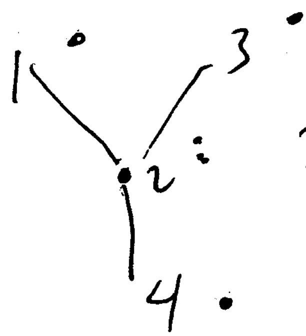
Jordan-Hölder; Krull-Schmidt: every rep of $Q/\mathbb{F}_1$ is a direct sum of indecomposables,

<u>Thm</u> (☺)

$\exists$ a Hopf alg. hom. $\psi : U(n) \to H_{\mathrm{Rep}(Q,\mathbb{F}_1)}$.

1) type A, ψ is an isomorphism.

2)  $U(L\mathfrak{sl}_n^+) \hookrightarrow H_{\mathrm{Rep}(Q,\mathbb{F}_1)} \cong U(L\mathfrak{gl}_n^+)$

3) 

: the maximal root of D_4 has multiplicity $\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$

Recollection on $Coh(X/\mathbb{F}_q)$

X = smooth projective curve$/\mathbb{F}_q$.

$Coh(X)$ – nice hereditary category.

Thm (Kapranov)

$$U_v(L\mathfrak{b}^+) \to H_{Cat(\mathbb{P}^1/\mathbb{F}_q)}$$

$$v = q^{1/2}$$

$$L\mathfrak{b}^+ \subseteq L\mathfrak{sl}_2.$$

$$\{e \otimes t^n, h \otimes t^l\}_{n \in \mathbb{Z},\ l > 0}.$$

Schemes/$\mathbb{F}_1$ (Dietmar; Toën-Vaquié; Soulé; Berger)

A = unital commutative monoid

$\mathfrak{a} \subseteq A$ is an ideal if $\mathfrak{a} A \subseteq \mathfrak{a}$.

prime ideal defined as usual.

$\mathrm{Spec}\, A = \{ \mathfrak{p} \mid \text{prime ideals in } A \}$.

$(\mathrm{Spec}\, A, \mathcal{O}_A) \leftarrow$ affine scheme/$\mathbb{F}_1$.

A general scheme/$\mathbb{F}_1$ is obtained by gluing affines.

S is a set $\to \mathbb{Z}[S]$.

$$\langle 1, t^{-1}, t^{-2}, \dots \rangle \hookrightarrow \langle t, t^{-1} \rangle \hookleftarrow \langle 1, t, t^2, \dots \rangle$$

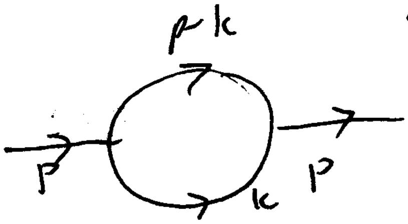
$$V = \mathbb{A}^1/\mathbb{F}_1 \leftarrow U \cap V \rightarrow U = \mathbb{A}^1/\mathbb{F}_1$$

$$\mathbb{P}^1 = \{0, \infty, \eta\}$$

Thm (☺)

$$U(L\mathfrak{f}_t) \hookrightarrow H_{\mathrm{Coh}(\mathbb{P}^1/\mathbb{F}_1)}$$

$\dim(X) > 1.$



$$I_{SU}(C) = A^N \int \frac{1}{(k^2+m^2)((p-k)^2+m^2)}$$