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Omega-backgrounds

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Nekrasov '02, '03
Nekrasov-Okounkov '03

$Z_{N=2}$

M = 4-manifold e.g. $M=\mathbb{R}^4$
G = compact Lie group
E = principal G-bundle over M
$\mathcal{A} = \{$space of connections on $E\}$
$G_\infty = \{$gauge transformations, $g(x)|_{x=\infty} = 1\}$

$$Z = \int_{\mathcal{A}/G_\infty}$$

$\mathcal{N}=2$ SYM with no matter fields

$$\mathcal{E} = \Omega^{2+} \otimes \mathrm{ad}(g) \qquad \begin{array}{c}\mathcal{E}\\ \downarrow \\ \mathcal{A}\end{array} \qquad \begin{array}{c}\mathcal{E}/G_\infty\\ \downarrow \\ \mathcal{A}/G_\infty\end{array} = \mathcal{V}$$

↑ self-dual 2-forms

Equivariant setup (for $M = \mathbb{R}^4$)

$$\underset{\substack{\parallel \\ SO(4)}}{\Omega} \times G$$

$T_{\Omega, G}$ = maximal torus

$= T^2 \times T^r$, $r = \mathrm{rank}(G)$

Let $\varepsilon_1, \varepsilon_2$ be coords on $\mathrm{Lie}(T^2)$

$a_1, \ldots, a_r$ be coords on $\mathrm{Lie}(T^r)$

$\mathcal{M}$ $\quad H^\bullet(\mathcal{M}) = \ker d / \mathrm{Im}\, d \qquad d^2 = 0, \; \Omega^\bullet(\mathcal{M}) \to \Omega^{\bullet+1}(\mathcal{M})$

$G \circlearrowright \mathcal{M}$ $\qquad \Omega^\bullet(\mathcal{M}) \otimes S^*(\mathfrak{g}^*)$

(Cartan model) $\qquad D_{eq} = d - \varphi^a i_{V_a}$

where $\{e_a\}$ = basis of $\mathfrak{g}$

V^a = vector fields

$\deg d = 1$

$\deg \varphi = 2$ $\qquad D^2 = -\varphi^a \mathcal{L}_{V^a}$

$\deg i_{V^a} = -1$

$$Z(\varepsilon_1,\varepsilon_2,a_1,\dots,a_r) = \sum_{k=c_2(E)=0}^{\infty} q^k \int_{\mathcal{A}/\mathcal{G}} e_{T(\Omega_{\mathcal{G}})}\left(\mathcal{E}/\mathcal{G}_\infty \to \mathcal{A}/\mathcal{G}_\infty\right)$$

equivariant Euler class

<u>Matter hypermultiplets</u>

R of G e.g.

$U(N_f)$ = flavor group

$$R = \bigoplus_{f=1}^{N_f} \mathbb{C}^r \qquad N_f = \#\text{ of flavors}$$

$$\begin{array}{ccc} S^- & & S^+ \\ \downarrow & & \downarrow \\ M & & M \end{array}$$

$D : \Gamma(S^-) \to \Gamma(S^+)$, Dirac operator

$\mathcal{N}=2$ matter

T_F = max. torus of flavor group

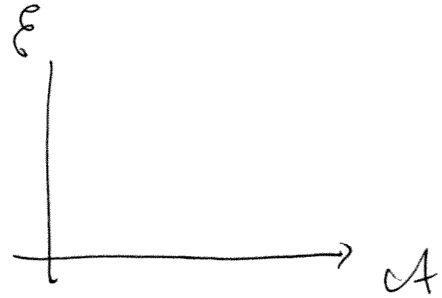
parameters $(m_1,\dots,m_f)$ (masses)

$$Z(\varepsilon_1,\varepsilon_2,a_1,\dots,a_r,m_1,\dots,m_f) = \sum_{k=0}^{\infty} q^k \int_{\mathcal{A}/\mathcal{G}_\infty} e_{T_{\Omega,G}}\left(\mathcal{E}/\mathcal{G}_\infty \to \mathcal{A}/\mathcal{G}_\infty\right) \cdot \frac{e_{T_{\Omega,G,F}}(S^-\otimes R \to \mathcal{A}/\mathcal{G}_\infty)}{e_{T_{\Omega,G,F}}(S^+\otimes R \to \mathcal{A}/\mathcal{G}_\infty)}$$

$G = U(1)^r$ $\quad D\alpha = 0$

$$\int_M \alpha = \int_F \frac{\iota_F \alpha}{e(N_F)}$$

F = set of fixed pts



$$\int_{\mathcal{E}\times\mathcal{A}} \varphi(E)$$

$QA_\mu = \psi_\mu \qquad QH^+_{\mu\nu} = [\varphi, \chi^+_{\mu\nu}]$

$Q\psi_\mu = -D\varphi \qquad Q\chi^+_{\mu\nu} = H^+_{\mu\nu}$

$Q\varphi = 0$

$Q\bar{\varphi} = \lambda \qquad Q^2 = [\varphi, \cdot\,]$

$Q\lambda = [\varphi, \bar{\varphi}]$

$$\int DA\, D\varphi\, D\bar{\varphi}\, D\psi\, D\chi\, DH\, D\lambda\, e^{Q\left((\psi_\mu, \overline{D\varphi}) + (\chi^+_{\mu\nu}, H^+_{\mu\nu} + iF^+_{\mu\nu}) + (\lambda, [\varphi, \bar{\varphi}])\right)}$$

$$+ \tau \int F \wedge F$$

$(F^+)^2$

$A = 0, \varphi = 0, H^+_{\mu\nu} = 0$

$$\int_{\mathcal{A}/\mathcal{G}_\infty} e(\Omega^{2+}) = \frac{e(\Omega^{2+})}{e(\Omega^1)}\, e(\Omega^0)$$

$$C = \left(\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^+} \Omega^{2+}\right)$$

$$ch_{\varepsilon_i, a_i} = \text{ind}(C) = \sum \frac{ch(\Omega^0 - \Omega^1 + \Omega^{2+}|_{x=0})}{\det(1 - g(\varepsilon_1, \varepsilon_2))|_{\mathbb{R}^4}}$$

Atiyah-Singer

$$\mathbb{R}^4 = \underset{\varepsilon_1}{\underset{\circlearrowright}{\mathbb{R}^2}} \oplus \underset{\varepsilon_2}{\underset{\circlearrowright}{\mathbb{R}^2}}$$

Ω^0 wts 0

Ω^1 wts $\pm\varepsilon_1, \pm\varepsilon_2$

Ω^{2+} wts $+(\varepsilon_1+\varepsilon_2), -(\varepsilon_1+\varepsilon_2), 0$

$$ch_{\varepsilon_i, a_i} = \frac{1-\left(e^{i\varepsilon_1}+e^{-i\varepsilon_1}+e^{i\varepsilon_2}+e^{-i\varepsilon_2}\right) + \left(1+e^{i(\varepsilon_1+\varepsilon_2)}+e^{-i(\varepsilon_1+\varepsilon_2)}\right)}{(1-e^{i\varepsilon_1})(1-e^{-i\varepsilon_1})(1-e^{i\varepsilon_2})(1-e^{-i\varepsilon_2})}$$

$$= \frac{1}{(1-e^{-i\varepsilon_1})(1-e^{-i\varepsilon_2})} + \frac{1}{(1-e^{i\varepsilon_1})(1-e^{i\varepsilon_2})}$$

$$= \left(\sum_{\substack{n_1\geq 0\\ n_2 \geq 0}} e^{-n_1 i\varepsilon_1 - n_2 i\varepsilon_2} + \sum e^{-(n_1+1)i\varepsilon_1-(n_2+1)i\varepsilon_2}\right)$$

multiply by $\left(\sum_\alpha e^{\alpha(a)}\right)$

$$\frac{e(\Omega^{2T})e(\Omega^0)}{e(\Omega^1)} = \prod_{\alpha} \prod_{n_1,n_2 \geq 0} (\alpha(a) - n_1\varepsilon_1 - n_2\varepsilon_2)(\alpha(a) - n_1\varepsilon_1 - n_2\varepsilon_2 - \varepsilon_1 - \varepsilon_2)$$

$$= Z_{pert}(\varepsilon_1, \varepsilon_2\, a_1, \ldots, a_r)$$

$$\prod_{\substack{n_1 \geq 0 \\ \vdots \\ n_N \geq 0}} (z - n_1\varepsilon_1 \cdots - n_N\varepsilon_N) =$$ ~~generalized~~ (inverse of) multiple gamma function

(Barnes, 1901)

$$= \Gamma^{-1}_{\varepsilon_1, \ldots, \varepsilon_N}(z)$$

So $Z_{pert}(\varepsilon_1\, \varepsilon_2\, a_1 \ldots a_r)$ = product of double gamma function.

$$\left.\begin{matrix} a \to \infty \\ \varepsilon_1 \varepsilon_2 \to 0 \end{matrix}\right| \quad \exp((\alpha, a)^2 \log(a\,\alpha))$$

$$\int_M \alpha = \sum_{\text{fixed}} \frac{e(\ )(\ )}{(\ )} = Z_{pert} \cdot Z_{inst}(q)$$