23 August 2010

I - Mirković

Travkin's toy model for quantum geometric Langlands

$$\mathrm{Coh}(\mathcal{L}S_G) \simeq \mathrm{mod}(\mathcal{D}_{Bun_{G^\vee}})$$

$$\begin{tikzcd}
\mathcal{L}S_G \arrow[d] & T^*Bun_G \arrow[l, "\text{comm}"', "\text{def.}"] \arrow[d] \arrow[r, phantom, "\simeq"] & T^*Bun_{G^\vee} \arrow[r, "nc"] \arrow[d] & \mathcal{D}_{Bun_{G^\vee}} \arrow[d] \\
\mathbb{B}_G^{(1)} & \mathbb{B}_G \arrow[l, two heads] \arrow[r, phantom, "\simeq"] & \mathbb{B}_{G^\vee} \arrow[r] & \mathbb{B}_{G^\vee}^{(1)}
\end{tikzcd}$$

$$\mathbb{B}_G \supseteq \mathbb{B}_{ar} \subseteq \quad \text{Arinkin}$$

Kapustin-Witten (branes)

Bezrukavnikov-Braverman-Travkin $(p>0)$

Outline

I $\quad p>0$ as criticality $\quad$ Bezr-Mirk-R, Bezr-Braverman, Travkin

II $\quad$ GL for $p>0$

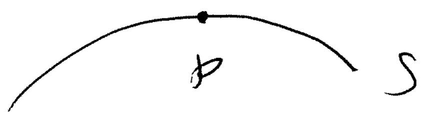
III $\quad$ qGL for $p>0$

$\boxed{p>0}$

char(k) = p. (char($\mathbb{C}$) = 0)

$k \supseteq \{0,1,\dots,p-1\}$, $p>0$

I.



generic point.

special point by prime.

Collapse: char(k) = p > 0

Calculus: $df=0 \iff f\in \mathcal{O}_X^p \not\Rightarrow f$ const.

Rep theory: irred. reps are smaller.

$G = SL_2$ by $\begin{pmatrix}1&\\&-1\end{pmatrix} = \begin{pmatrix}1&\\&1\end{pmatrix}\in Z(G)$
$p=2$.

<u>accidents</u>

Simpler: new spaces $\mathcal{O}_X^p \subseteq \mathcal{O}_X$ $(a+b)^p = a^p+b^p$.

$X \twoheadrightarrow X^{(1)}$

quantization collapses in new direction:

$$Z(\mathcal{D}_X) = \begin{cases} \text{Const.} & p=0 \\ \mathcal{O}_{T^*X^{(1)}} \simeq (\mathcal{O}_{T^*X})^p & p>0\end{cases}$$

③

$$[\partial, f^p] = \partial(f^p) = \underset{\overset{\shortparallel}{0}}{p} f^{p-1} \cdot f' = 0$$

~~$\partial_i^p \mapsto \partial$~~

$$\partial_i^{(1)} \mapsto \partial_i^p.$$

$$T_{X^{(1)}} \ni \partial^{(1)} \mapsto \partial^p - \partial^{[p]} \in Z(\mathcal{D}_X).$$

$$Z[U_k(\mathfrak{g})] = \begin{cases} \text{const. } k \neq \mathbb{F}_p \\ \text{interesting} \end{cases}$$

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a) $\mathcal{D}_X$ is an Azumaya algebra over $T^*X^{(1)}$

b) Azumaya alg. $\mathcal{D}_X$ splits on some Lagrangian $T^*_Y(X)^{(1)} = L$.

c) not split

Azumaya algebra A on Y:

Vector bundle of algebras

$y \in Y$, $A_y \simeq M_{n \times n}$ $\qquad V' = \mathcal{L} \otimes V$

$A \simeq \mathrm{End}(V)$ vect. b

<u>V splitting</u> $\quad \mathrm{Coh}(Y) \xrightarrow{\sim} \mathrm{mod}(A)$

$$\mathcal{F} \mapsto \mathcal{F} \otimes V$$

$X = \mathbb{A}^1$ $\quad D_X = \bigoplus k\, x^i \partial^j$

$\cancel{D_X}$

$$D_X \Big|_{\substack{x^p=0 \\ \partial^p=0}} = \bigoplus_{i,j<p} k\, x^i \partial^j$$

dim is p^2.

acts on $k[x]/x^p$, dim $= p$.

$$\Rightarrow D_X \Big|_{x^p = \partial^p = 0} \simeq M_{p \times p}.$$

$Y = X$, $Z = X^{(1)}$

$\mathcal{O}_X$ rank p vect. bundle on $X^{(1)}$

$\mathcal{O}_X = k[x]$ $\quad \mathcal{O}_{X^{(1)}} = k[x^p]$.

$\Rightarrow$ quantization collapses + remaining nc is thin.

Applications: $\mathrm{Rep}(\mathfrak{g})$, $p > 0$

- "Critical RTs" quantum groups at $\sqrt{1}$
- application to $p = 0$ by reduction to $p > 0$

e.g. Bezrukavnikov-Okounkov relating RT to quantum cohomology of symplectic resolutions.

"Azumaya algebra A is a cohomology class on Y".

Geometric realization of coh. classes:

$$\alpha \in H^P(Y, ?)$$

sheaf on Y (of categories) = local trivialization of α.

$H^1(Y, G)$, G-torsors
$$\begin{array}{c} P \\ \downarrow \\ Y \end{array}$$

$A \rightsquigarrow \mathcal{G}(A)$ = sheaf of splittings of A

"spaces", BG_m-torsors, "gerbes"

torsor for Pic_Y

$$\in H^1(Y, \mathrm{Pic}_Y) = H^1(Y, B(G_m)) = H^2(Y, G_m)$$

$$B(G_m) = G_m[1]$$

$$Y \to B(G_m) \text{ group.}$$

$$B(G_m) = \bullet / G_m$$

$$\mathrm{mod}(A) = \mathrm{Coh}[\mathcal{G}(A)]$$

nc

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disorder operators: 't Hooft

$$\begin{array}{ccccc} & & \tilde{M} & \text{gerbe} & \\ & & \downarrow & & \\ D & \hookrightarrow & M & \hookleftarrow & M-D \end{array}$$

K-theory twisted

Our quantization is thin, and on a twisted space.

$\mathrm{Rep}(\mathfrak{g})$.

acts:

$G, \mathfrak{g} \to \mathrm{Vect}$

$\mathcal{B} = G/B$

$U\mathfrak{g} \to \mathcal{D}_{\mathcal{B}}$.

$\mathrm{mod}(\mathcal{D}_{\mathcal{B}}) \xrightarrow[\approx]{\Gamma} \mathrm{mod}(\mathfrak{g})$

$\mathcal{D}^{\lambda}_{\mathcal{B}}$ $\quad\quad$ $U^{\lambda}\mathfrak{g}$

D^b

$\rho(\lambda)$

$$\begin{array}{l} \mathrm{Rep}(\mathfrak{g}) \\ \updownarrow \\ \mathrm{mod}(\mathcal{D}_{\mathcal{B}}) \\ \updownarrow \, p>0 \\ \mathrm{Coh}(T^*\mathcal{B}^{(1)}) \end{array}$$

Beilinson–Bernstein localization

Springer fibers

$$\begin{array}{ccc} T^*\mathcal{B} & \supset & \mathcal{B}_x \\ \downarrow & & \downarrow \\ \mathcal{N} & \ni & x \end{array}$$

$\mathcal{D}_{\mathcal{B}}$ splits on $\widehat{\mathcal{B}}_x$

$Z[U(\mathfrak{g})] \supseteq \mathcal{O}(\mathfrak{g}^{*(1)})$ $\qquad T^*G$

$x \in \mathfrak{g}^*, \quad U_x\mathfrak{g}$

$$\begin{array}{l} \mathrm{Rep}(\mathfrak{g}) \\ \updownarrow \\ \mathrm{mod}(\mathcal{D}_{\mathcal{B}}) \\ \updownarrow \; p > 0 \\ \mathrm{Coh}(T^*\mathcal{B}^{(1)}) \\ \updownarrow \; p > 0 \rightsquigarrow p = 0 \\ p = 0 \;\; \mathrm{Coh}(T^*\mathcal{B}) \\ \updownarrow \; \text{Langlands (Bezrukavnikov)} \\ \mathrm{Perv}(\mathcal{B}^\vee_{G((z))}) \end{array}$$

Hodge theory

Geometric Langlands

sp. curve

$$\begin{array}{ccccc} T^*\mathrm{Bun}_G & \ni (E,\varphi) & \longleftrightarrow & (C \leftarrow \mathcal{L}) \\ \downarrow & \downarrow & & \\ \mathbb{B}_G & \ni \mathrm{spec}(\varphi) \subseteq T^*C & & \end{array}$$

$$\begin{array}{ccc} & & \mathcal{D}_{\mathrm{Bun}_G} = \mathcal{G}(\mathcal{O}_{\mathrm{Bun}_G}) \\ & & \downarrow \\ \mathbb{B}_{G^\vee}^{(1)} & \longleftarrow & T^*\mathrm{Bun}_G^{(1)} \end{array}$$

$\mathcal{LS} \ni (E, \nabla)$

$p\text{-curv}(E,\nabla) \in \mathrm{End}(E) \otimes \Omega^1_{C^{(1)}}$

action $\mathcal{J}_{C^{(1)}} \overset{\iota}{\hookrightarrow} Z(\mathcal{D}_C)$ on E.

$\dashrightarrow \mathcal{J}_{C^{(1)}} \to \mathrm{End}_{\mathcal{D}_C}(E)$

$$\begin{array}{c} \mathcal{LS}_C \\ \mathrm{Spec}(p\text{-curv}) \downarrow \\ \mathbb{B}_G^{(1)} \end{array}$$

Duality :

$$\hat{A} = \mathrm{Hom}(A, BG_m)$$

abelian group stack

$$\hat{\hat{A}} \xleftarrow{\cong} A$$

$$\mathcal{P} \to A \times \hat{A} \to BG_m$$

$$\mathrm{Coh}(A) \qquad \mathrm{Coh}(\hat{A})$$

$$T^*\mathrm{Bun}_G^{(1)} = \mathrm{Pic}(\mathcal{C})^{(1)}$$ ← universal spectral curve

$$\mathcal{G}(\mathcal{D}_{\mathrm{Bun}_G}) = \text{gerbe on } T^*\mathrm{Bun}_G^{(1)}$$

$$0 \to BG_m \to \mathcal{G}(\mathcal{D}_{\mathrm{Bun}_G}) \to \mathrm{Bun}_G \to 0$$

group

$$\mathcal{L}\mathcal{G}_G = \text{torsor for } T^*\mathrm{Bun}_G^{(1)} = \mathrm{Pic}(\mathcal{C})^{(1)}$$

$$\mathcal{L}\mathcal{G} \ni (E, \nabla) \to C' \qquad \text{rank } p \text{ vect. bundle}$$

torser for $T^*\mathrm{Bun}_{G^\vee}^{(1)}$ $\mathcal{L}\mathfrak{g}_{G^\vee}$

gerbe ext. of $T^*\mathrm{Bun}_{G}^{(1)}$ $\mathcal{G}(\mathrm{Diff}_{\mathrm{Bun}_{G^\vee}})$

$G - GL$

$\mathrm{irred}(\mathcal{D}_{\mathrm{Bun}_G}^{\det^c}) \simeq \mathrm{irred}(\mathcal{D}_{\mathrm{Bun}_{G^\vee}}^{\det^{1/c}})$

Feigin - Stoyanovsky

Kapustin - Witten. $c=0, \infty$

$\check{c}=0, \infty$.

$$\begin{array}{ccc} A & & \\ \downarrow & \text{abelian variety} & \\ X & & \end{array} \qquad \begin{array}{cc} \mathcal{G} & \text{gerbe} \\ \downarrow & \\ P & \text{torser} \end{array}$$

$$\xrightarrow{\text{dualize}} \quad \begin{array}{c} \hat{\mathcal{G}} \\ \downarrow \\ \hat{P} \end{array} \quad \text{over} \quad \begin{array}{c} \hat{A} \\ \downarrow \\ X \end{array}$$

$$\mathrm{Coh}(\mathcal{G}) \simeq \mathrm{Coh}(\hat{\mathcal{G}})$$

G, c $\quad$ $\check{G}, \check{c} = 1/c$

$$\mathrm{mod}\left(\mathcal{D}^{\det^c}_{\mathrm{Bun}_G}\right)$$

$$\parallel$$

$$\mathrm{Ug}(\mathcal{D}\cdots)$$

$$\downarrow$$

$$\lambda\text{-}\mathrm{Con}(\det)^{(1)}$$

$$\downarrow$$

$$\mathbb{B}_G^{(1)(1)} = \mathbb{B}_G^{(2)}$$

$$\lambda = c^p - c$$

dual picture,
same data