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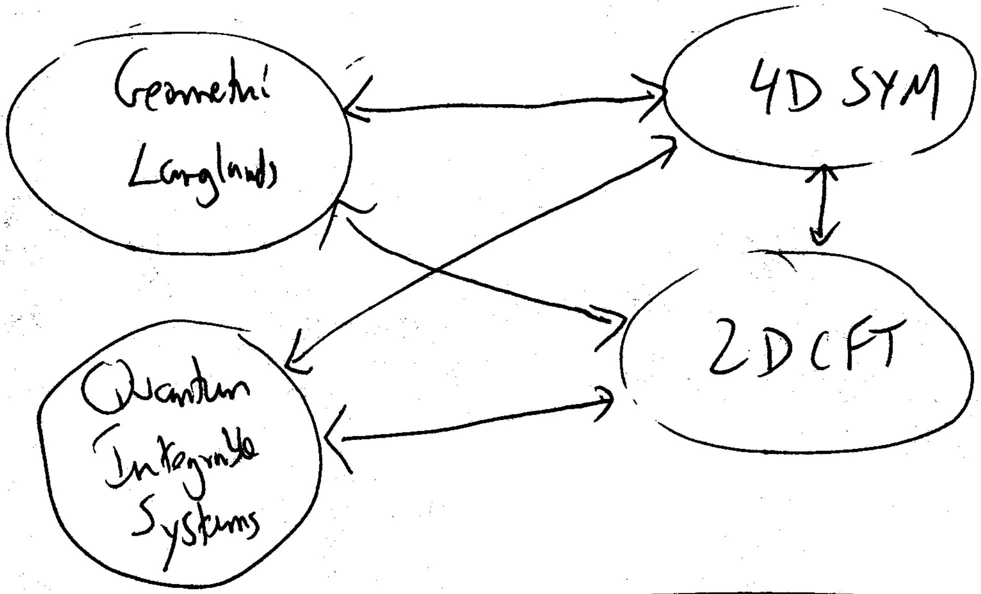
Geometric Langlands Program, 2D conformal field theory, and integrable systems



Nekrasov-Shatashvili

Nekrasov-Witten

2D CFT: device for producing $\mathcal{D}$-modules on various moduli spaces

Categorical Langlands correspondence

C = compact Riemann surface

G = reductive Lie group $/\mathbb{C}$.

${}^L G$ = Langlands dual group.

Bun_G = moduli stack of G-bundles on C

$\mathrm{Loc}_{{}^L G}$ = - - - - flat ${}^L G$-bundles on C $\quad \mathcal{E} = (E, \nabla)$

$$
\begin{tikzcd}
\boxed{\begin{array}{c}\mathcal{O}\text{-modules}\\ (\text{coherent sheaves})\\ \text{on } \mathrm{Loc}_{{}^LG}\end{array}} \arrow[r, leftrightarrow] \arrow[d, phantom, "\wr"] \arrow[dr, leftrightarrow] & \boxed{\begin{array}{c}\mathcal{D}\text{-modules}\\ \text{on } \mathrm{Bun}_G\end{array}} \arrow[d, leftrightarrow, "\mathrm{KW}"] \\
\boxed{\text{B-branes}} & \boxed{\text{A-branes}}
\end{tikzcd}
$$

Kapustin–Witten:

S-duality in $\mathcal{N}=4$ SYM.

$\rightsquigarrow$

mirror symmetry

$$\mathcal{E} \in \mathrm{Loc}_{{}^LG} \qquad \mathcal{O}_{\mathcal{E}} \longleftrightarrow \mathcal{F}_{\mathcal{E}} \in \mathcal{D}\text{-module on } \mathrm{Bun}_G$$

$\mathcal{E} = (E, \nabla)$

E: ${}^LG$-bundle; ∇: flat connection

$\mathcal{O}_{\mathcal{E}} \longmapsto \mathcal{F}_{\mathcal{E}}$ has special property: Hecke eigensheaf on Bun_G with respect to $\mathcal{E}$

$$H_{\rho,x}(\mathcal{F}_{\mathcal{E}}) = \rho \otimes \mathcal{F}_{\mathcal{E}}, \quad \rho \in \mathrm{Rep}({}^LG), \quad x \in C.$$

$H_{\rho,x}$: Hecke functors.

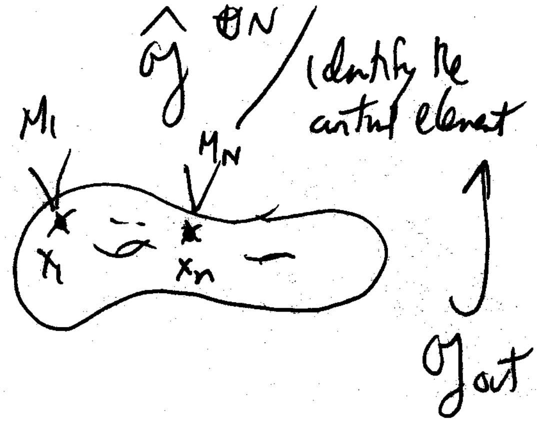
$\mathcal{D}$-sheaf of differential operators on Bun_G

2D CFT with $\hat{\mathfrak{g}}$-symmetry (affine Kač-Moody algebra) $\rightsquigarrow$ $\mathcal{D}$-modules on Bun_G

$M_1, \dots, M_N$: $\hat{\mathfrak{g}}$-modules on level k

$\varphi : M_1 \otimes \dots \otimes M_N \to \mathbb{C}$ is a conformal block if

$\varphi(g \cdot v) = 0$,



$\mathcal{C}(C, (x_i), (M_i))$ = vector space of conformal blocks

$H(C, (x_i), (M_i)) = M_1 \otimes \dots \otimes M_N / \mathfrak{g}_{out}$ — dual space, space of coinvariants

Dependence on $P \in \mathrm{Bun}_G$.

$\mathfrak{g}_{out}(P) = \Gamma(C \setminus \{x_1, \dots, x_n\}, \mathfrak{g}_P)$

$\cap$

$\hat{\mathfrak{g}}^{\oplus N}$ acts on $M_1 \otimes \dots \otimes M_N$.

$$H(C, (x_i), (M_i), P) = M_1 \otimes \dots \otimes M_N / \mathfrak{g}_{out}(P)$$

vector space depends on P $\rightsquigarrow$ combine into a quasi-coherent sheaf on Bun_G.

This sheaf carries a (projectively) flat connection

$\Downarrow$

Get a $\mathcal{D}$-module on Bun_G.

Want to get $\mathcal{F}_E$ - Hecke eigensheaves on Bun_G.

Take $k = -h^\vee$ ($h^\vee$ = dual Coxeter number)

$J^a(z)$ = currents of $\hat{\mathfrak{g}}$, $\{J^a\}$ - basis of $\mathfrak{g}$

$$J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1}$$

$$J^a \otimes t^n \in \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$$

$$S(z) = \frac{1}{2} \sum_a : J^a(z) J_a(z) : = \sum_{n\in\mathbb{Z}} S_n z^{-n-2}$$

$$[S_n, J^a_m] = -(k+h^\vee) m J^a_{n+m}$$

$$-t^{n+1}\frac{\partial}{\partial t} \cdot t^m = -m t^{n+m}$$

$$\updownarrow$$

$$L_n$$

if $k \neq -h^\vee$, $L_n := \frac{S_n}{k+h^\vee}$ $\rightsquigarrow$ action of Virasoro algebra

For $k = -h^\vee$, $[S_n, \hat{\mathfrak{g}}] = 0$. ← central elements

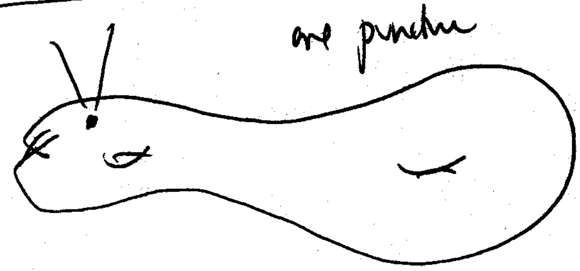
$S(z) J^a(w) \sim$ regular

← central elements in the chiral algebra

<u>Thm</u> (Feigin–Frenkel, '91)

1) Center (of the chiral algebra) is trivial for $k \neq -h^\vee$
2) For $k = -h^\vee$, center is freely generated by

$$S(z) = S_1(z), S_2(z), \ldots, S_\ell(z) \qquad (\ell = \mathrm{rank}(\mathfrak{g}))$$



$V_{-h^\vee}(\mathfrak{g})$ = vacuum module of $\hat{\mathfrak{g}}$

$\ni v$.

$$J^a_n v = 0 \quad \forall n \geq 0.$$

$$V_\chi = V_{-h^\vee}(\mathfrak{g}) \Big/ \Big(S_i(z) = \chi_i(z) \Big), \qquad \chi_i(z) = \sum \underbrace{\underset{\mathbb{C}}{\chi_{i,m}}}_{\in \mathbb{C}[[z]]} z^{-m-\deg S_i}$$

$\chi = (\chi_i(z))$

$$S_{i,m} = \chi_{i,m}.$$

(3) Center $\mathfrak{z}(\mathfrak{g}) \simeq \mathrm{Fun}(\mathrm{Op}_{{}^LG}(D_x))$

= classical limit of W-algebra associated to ${}^L\mathfrak{g}$ " $W_\infty({}^L\mathfrak{g})$

$W_b(\mathfrak{g})$

b - complex parameter.

$W_b(\mathfrak{sl}_2)$ - Virasoro algebra $c = 1 + 6(b+\frac{1}{b})^2$

$$\begin{array}{ccc} \text{center of chiral alg. of } \hat{\mathfrak{g}}_{-h^\vee} \ni \mathfrak{z}(\mathfrak{g}) & \simeq & W_\infty({}^L\mathfrak{g}) \\ \uparrow {\scriptstyle b\to\infty} & & \uparrow {\scriptstyle \frac{1}{b}\to\infty} \\ W_b(\mathfrak{g}) & \simeq & W_{1/b}({}^L\mathfrak{g}) \end{array}$$

Lectures by Frenkel : hep-th/0512...

$b = \frac{\varepsilon_1}{\varepsilon_2}$, $b \to 0$ is $\varepsilon_1 \to 0$

$\rightsquigarrow$ $\mathcal{D}$-module on Bun_G (twisted by $K^{1/2}$)

(generally get twisted $\mathcal{D}$-modules)

$\chi \in Op_{^LG}(D) \xrightarrow{\text{Beilinson-Drinfeld}}$ $\neq 0$ iff χ can be extended to a ^LG oper on C

$\mathcal{F}_{\mathcal{E}}$

Any ^LG oper $\rightsquigarrow$ flat ^LG-bundle.

$$\chi \longmapsto \mathcal{E}$$

Quantization of the Hitchin system

Global differential operators acting on $K^{1/2}$ on Bun_G

$$D_{1/2} \simeq Fun(Op_{^LG}(C)) \simeq \mathbb{C}[D_1, \dots, D_M]$$

$M = \dim Bun_G$

$$\uparrow$$

$$\mathfrak{z}(\hat{\mathfrak{g}}) \simeq Fun(Op_{^LG}(D_x))$$

$\chi \in Op_{^LG}(C) \rightsquigarrow \mathcal{D}_{1/2} / \mathcal{D}_{1/2} \cdot I_\chi = \mathcal{F}_{\mathcal{E}}$

sheaf of diff. ops. on $K^{1/2}$

I_χ = maximal ideal in $D_{1/2}$ corresponding to χ.

System of PDE:

$\chi \in Op_{{}^L G}(C)$

$D_i(\Psi) = \lambda_i \Psi$

λ_i = value of D_i at χ viewed as a function on $Op_{{}^L G}(C)$.

$\mathrm{Hom}(\mathcal{F}_\mathcal{E}, S)$ = space of solutions in S.

particular class of $\frac{1}{2}$-forms

Choice of S $\leftarrow$ choice of a brane in $T^* \mathrm{Bun}_G$

Space of solutions: $\mathrm{Hom}(B_{c.c.}, B)$

canonical coisotropic brane.

~~N~~ekrasov-Witten

Specialized to the case $C = \mathbb{P}^1$.

<u>Gaudin model</u>

$\mathrm{Hom}(\mathcal{D}_{K^{1/2}}, S) \to \mathrm{Hom}_A(B_{c.c.}, B) = \mathrm{Hom}(\mathcal{O}_{Op_{{}^L G}}(C), B')$

$\parallel$

$\mathrm{Hom}(\mathcal{O}(Op_{{}^L G}(C)), \mathcal{A})$

A-brane

B-brane

$\widehat{\mathcal{M}}_H({}^L G)$