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Twisted T-duality and quantization

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CDR Vertex Algebras

$M =$ manifold
$B \in \Omega^2(M, S^1)$ $\rightsquigarrow$ (Courant)-Dorfman bracket on $C^\infty(TM \oplus T^\vee M)$

$$[X+\xi, Y+\eta]_{CD} := [X,Y] + \mathcal{L}_X \eta - \mathcal{L}_Y d\xi + \iota_X \iota_Y dB.$$

$\rightsquigarrow$ extend to $\underset{f}{C^\infty(M)} \oplus \underset{X}{C^\infty(TM \oplus T^\vee M)}$

$$[X, f]_{CD} := X(f)$$

<u>Def</u> $CD(M,B) := \left(C^\infty(M) \oplus C^\infty(TM \oplus T^\vee M), [\ ]_{CD} \right)$

$\rightsquigarrow$ quantization of $C^\infty(T_B^\vee LM)$

$$LM = \mathrm{Maps}(S^1, M)$$

$$[A_1(z_1), A_2(z_2)]_{CDR} := \frac{[A_1, A_2]_{CD}(z_2)}{z_1 - z_2} + \frac{\langle A_1, A_2\rangle(z_2)}{(z_1-z_2)^2}$$

$\langle\,,\,\rangle$ = tautological pairing

$\rightsquigarrow$ vertex algebra

$CDR(M, B)$

Reminder A vertex algebra is a vectorspace V and a map

$$V \to \mathrm{End}(V((z)))^+$$
$$a \mapsto a(z)$$

satisfying axioms:

$a(z)b \in V[[z]][z^{-1}]$.

$$0 = [a_1(z_1), a_2(z_2)](z_1 - z_2)^N \quad , N \gg 0$$

⋮

(more axioms)

Remark Can add a Hamiltonian to the picture $\rightsquigarrow$ Dynamics.

~~Also~~ Can add susy $\rightsquigarrow$ GC, GK, GCY

} Heluani, Zabzine

add winding

$CDR(M,B)$ is a good quantization of $T_B^\vee LM$
only if $\pi_1(M)=1 \longleftrightarrow \pi_0(LM)=\{1\}$

$$T^\vee LM = \coprod_{w\in(\pi_1(M))^2} T^\vee L_w M$$ winding sectors

example: $M = S^1 = \mathbb{Z}\backslash\mathbb{R}$ $(B=0)$

$$T^\vee L S^1 = \coprod_{n\in\mathbb{Z}} T^\vee L_n S^1$$

$$x(z) = W\log(z) + \sum_{n\in\mathbb{Z}} x_n z^n.$$

$$\dot{x}(z) = \sum_{n\in\mathbb{Z}} \dot{x}_n z^n \qquad [\dot{x}(z_1), x(z_2)] = \frac{1}{z_1 - z_2}$$

$$x, \dot{x} \circlearrowright C^\infty(L_{ag} \subset T^*LS^1) = \bigoplus_{W\in\mathbb{Z}} C^\infty_N(S^1)\otimes \mathbb{C}[x_n]$$

$$= \bigoplus_W \widehat{\mathbb{C}_n(\mathbb{Z})} \otimes \text{Fock}$$

$$\widetilde{CDR}(S^1) = \bigoplus_w \widehat{\mathbb{C}[\mathbb{Z}]}_w \otimes \text{Fock} \supset \widehat{\mathbb{C}_0(\mathbb{Z})} \otimes \text{Fock} = CDR(S^1)$$

<u>Remark</u> $\widetilde{CDR}(S^1) = \widehat{\mathbb{C}[\mathbb{Z} \oplus \mathbb{Z}^\vee]} \otimes \text{Fock}$

= lattice vertex algebra associated to $\mathbb{Z} \oplus \mathbb{Z}^\vee, \langle\, , \rangle$

$SL_2(\mathbb{Z}) \circlearrowright \widetilde{CDR}(S^1)$

$\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \in SL_2(\mathbb{Z})$ acts by T-duality

$$\widetilde{CDR}(S^1) \cong \widetilde{CDR}(S^{1\vee})$$

$$(w, \dot{x}) \longleftrightarrow (-\dot{x}, w)$$

<u>Kapustin-Orlov:</u>

$S^1 \rightsquigarrow M = \Gamma \otimes \mathbb{R} / \Gamma \qquad \Gamma \cong \mathbb{Z}^l$

add $dB = 0$, $G\mathbb{C}$, ..., MS

<u>(Topological) T-duality</u> (Bouwknegt, Hannabuss, Mathai, ...)

$T = \mathbb{R}^l / \mathbb{Z}^l, \quad \mathfrak{t} = Lie(T)$

$\beta \in \Lambda^2 \mathfrak{t} \rightsquigarrow T_\beta$ n.c. torus $\quad q^{x_i} q^{x_j} = q^{\beta_{ij}} q^{x_j} q^{x_i}$

$\phi \in \Lambda^3 \mathfrak{t} \rightsquigarrow T_\phi$ non-associative torus

$$(g^{X_i} g^{X_j}) g^{X_k} = g^{\phi_{ijk}} g^{X_i} (g^{X_j} g^{X_k})$$

(E, H) - twisted bundle

$$\begin{array}{c} T \hookrightarrow E \\ \downarrow \\ M \end{array}$$

$H \in H^3(E, \mathbb{Z})$

$H = \sum_{i=0}^{3} H_i$

$H_i \in \Omega^i(M, \Lambda^{3-i} \mathfrak{t})$

T-duality

1. $H_1 = H_0 = 0$

$(E, H) \longleftrightarrow (E^\vee, H^\vee)$

$c_1(E^\vee) = H_2$

$H^\vee = H_3 + c_1(E)$

2. $H_0 = 0, H_1 \neq 0$

$(E, H) \longleftrightarrow$

$$\begin{array}{c} T^\vee_{H_1} \hookrightarrow E^\vee \\ \downarrow \\ M \end{array}$$

NC torus bundle

③ $H_0 \neq 0$

$$(E,H) \longleftrightarrow (T^\vee)_{H_0} \hookrightarrow \begin{array}{c} E \\ \downarrow \\ M \end{array}$$

n.a. torus bundle

<u>Main Example</u>

$$G_1(\mathbb{R}) = \mathbb{R}^3 \supset \mathbb{Z}^3 = G_1(\mathbb{Z}) \rightsquigarrow E_1 = G_1(\mathbb{Z}) \backslash G_1(\mathbb{R}) = \underset{x_1}{S^1} \times \underset{x_2}{S^1} \times \underset{x_3}{S^1}$$

$$B_1 = x_3\, dx_1\, dx_2$$

$$dB_1 = H.$$

$$G_2(\mathbb{R}) = \mathrm{Heis}(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix} \,\middle|\, a,b,c \in \mathbb{R} \right\}$$

$$\supseteq \mathrm{Heis}(\mathbb{Z}) = G_2(\mathbb{Z})$$

$$\rightsquigarrow E_2 = G_2(\mathbb{Z}) \backslash G_2(\mathbb{R})$$

Heisenberg nilmanifold

$$\begin{array}{ccc} x_3\; S^1 & \to & E_2 \\ & & \downarrow \end{array}$$

$x_1 \; x_2$

$$\begin{array}{c} \to E_2 \\ \downarrow \\ S^1 \end{array}$$

$$\begin{tikzcd}
(E_2,0) \arrow[r, leftrightarrow, "H=H_2"] & (E_2,H) \arrow[r, leftrightarrow, "H=H_0"] \arrow[d, leftrightarrow, "H=H_1"] & (S^1\times S^1\times S^1)_{H_0} \\
& (S^1\times S^1)_{H_1} \hookrightarrow E_3 \arrow[d] & \\
& S^1 &
\end{tikzcd}$$

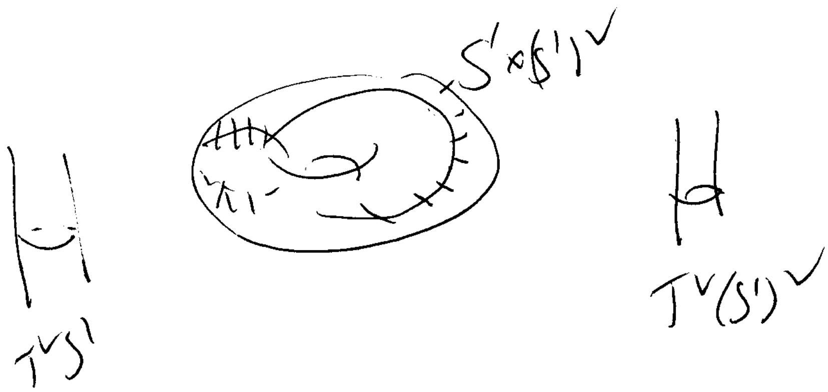
n.a. torus

n.c. torus bundle

Doubled formalism (C. Hull et al.)

Recall: $\widetilde{CDR}(S^1) \cong \mathbb{C}[\widehat{\partial \oplus \partial^\vee}] \otimes \text{Fock}$

$\overset{\text{F.T.}}{\cong} C^\infty(\underbrace{S^1 \times (S^1)^\vee}_{\text{doubled torus}}) \otimes \text{Fock}$



$\dot{x}(z) = \partial x^*(z)$

$\rightsquigarrow x^*(z)$ coordinate

Main example

$$G(\mathbb{R}) = \mathbb{R}^3 \oplus (\mathbb{R}^3)^\vee$$

w/ mult. $(x,x^*)(y,y^*) = (x+y,\ x^*+y^*+H^+xy)$

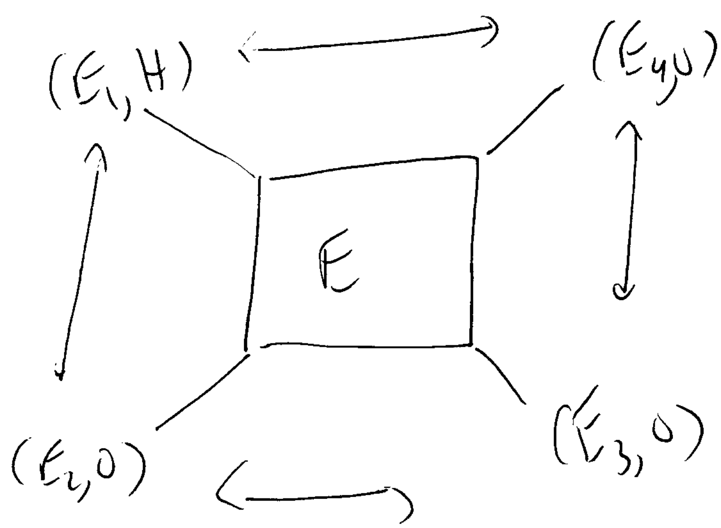
$$H^+ : \mathbb{R}^3 \otimes \mathbb{R}^3 \to (\mathbb{R}^3)^\vee$$

$$\mathfrak{g} = Lie(G(\mathbb{R}))$$

$\rightsquigarrow \quad E = G(\mathbb{Z}) \backslash G(\mathbb{R})$ 2-step nilmanifold

$$\begin{array}{c} T^3 \hookrightarrow E \\ \qquad\quad \downarrow \\ \qquad\quad T^3 \end{array}$$

"doubled twisted torus".



$\circlearrowleft SL_3(\mathbb{Z})$

Back to extended CDR

$$CD(E_1, B_1) \cong C^\infty(E_1) \oplus (C^\infty(E_1) \otimes \mathfrak{g})$$

$$CD(E_2, 0) \cong C^\infty(E_2) \oplus (C^\infty(E_2) \otimes \mathfrak{g})$$

$S^1 \hookrightarrow E_2$
&
$S^1 \times S^1$

$$C^\infty(E_1) \simeq \widehat{\mathbb{C}[\partial^3]}$$

$$C^\infty(E_2) \cong \bigoplus_{n \in \mathbb{Z}} C^\infty(\mathcal{L}^{\otimes n}) \qquad \sum f^{(nm)} e^{i m x_2}$$

<u>Remark</u> No natural isomorphism

$$CD(E_1, B_1) \xrightarrow{\simeq} CD(E_2, 0)$$

<u>Way out</u> add windings in "dualized direction"

$$(x_3, x_3^*)$$

$$\widetilde{CD}(E_2, 0) = \left(\bigoplus_{x_6 \in \mathbb{Z}} C^\infty(M)\right) \otimes \left(C^\infty(E) \otimes \mathfrak{g}\right)$$

$$\widetilde{CD}(E_1, B_1) = \left(\bigoplus_{n \in \mathbb{Z}} C^\infty_w(E_1, B_1)\right) \otimes (\mathfrak{g} \otimes C^\vee(\mathbb{Z}))$$

$C^\infty_w(E_1, B_1) = C^\infty(\mathcal{L}_w)$, $\mathcal{L}_w$ l.b. on E_1
$c_1(\mathcal{L}_w) = c_w H$.

Natural isomorphism

$$\widetilde{D}(E_1, B_1) \cong \widetilde{CD}(E_2, 0)$$

<u>Thm</u> (Aldi, Helvaci)

$$\widetilde{CDR}(E_1, B_1) \cong \widetilde{CDR}(E_2, 0).$$

<u>Remark</u>
Can further extend to add winding in two directions

$$\left(\cong \text{"} \widetilde{CDR}(E_{3,0}) \text{"} \right)$$