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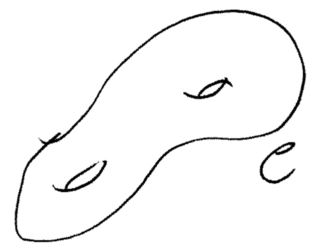
Remarks on gauge theory, quantization of Hitchin moduli space, and Liouville theory

(A) Gauge theory

(i) Let's study low energy physics on $\mathcal{G}_C$



$C \to \mathcal{G}_C$

on $\mathbb{R}\times\mathcal{M}$, $\mathcal{M} = S_1 \times \mathbb{R}^2_{\epsilon}$, S^3_{12}, etc.

Remark: Decomp. limit $\rightsquigarrow$ SW theory

Let $\mathcal{H}_0 = \{ |\psi\rangle,\ H|\psi\rangle = 0 \}$

$\quad\quad\quad\quad \uparrow$ Hamiltonian of $\mathcal{G}_C$

Consider Supercharge Q.

$$\mathcal{H}_0 = \mathrm{Ker}\, Q / \mathrm{Im}\, Q \quad \text{(Hodge)}$$

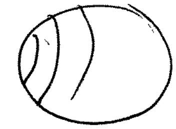
$\Rightarrow$ $\mathcal{H}_0$ may be studied in terms of top. twisted theory

... can play with metric ...

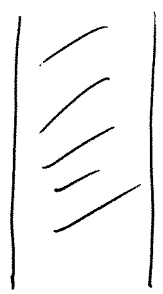
(ii) [Nekrasov, Shatashvili], [Nekrasov, Witten]

Take $M = S^3_{\varepsilon_1 \varepsilon_2} \simeq I \times S^1_{\varepsilon_1} \times S^1_{\varepsilon_2}$

S^2



$\rightsquigarrow$ Effective 2d description:



ε_1 ε_2

open 2d σ-model, target $M_H = M_H(C)$ (for $SU(2)$)

boundary conditions $\longrightarrow$ coisotropic A-branes

$\mathcal{H}_0$ = space of open "strings" $\mathcal{H}^{\sigma}_{\varepsilon_1 \varepsilon_2}$

*) DoF: - zero modes of scalars a_r, $r = 1, \dots, 3g-3+n$.

- monodromy of gauge field

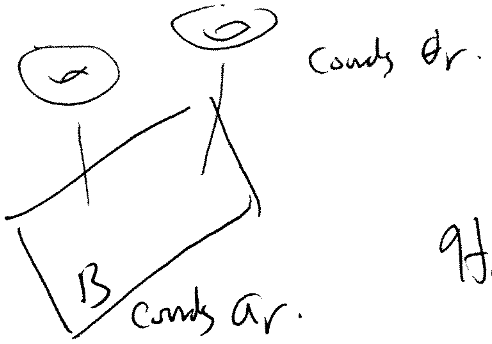
$$e^{i\theta_r} = \exp \int_I A_x^r$$

Collection of (a_r, θ_r)

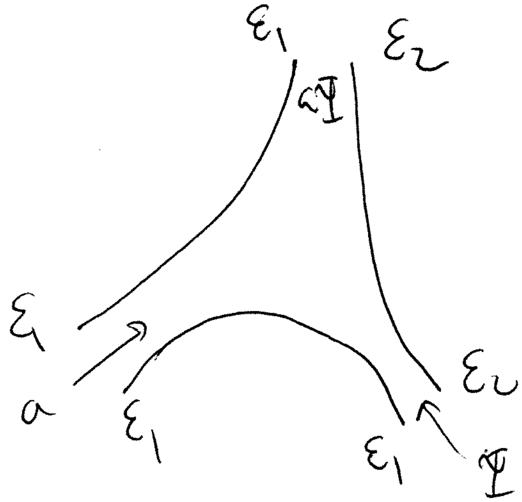
are action-angle variable for (M_H, I).

= space of pairs $(\mathcal{E}, \theta)$ mod $SL(2,\mathbb{C})$ transformations

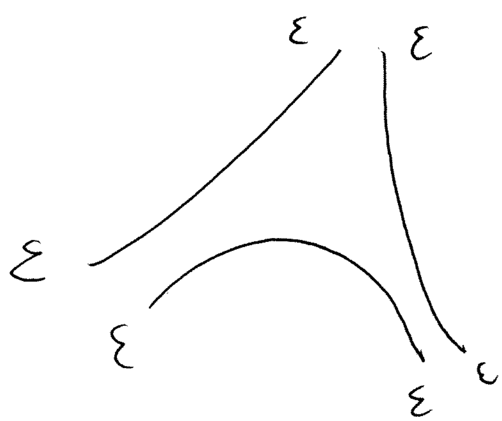
holo. vector bundle $\quad$ $H^0(\mathcal{E}, \mathrm{End}(\mathcal{E}) \otimes K_C$



$\mathcal{H}_0 = \mathcal{H}^{\sigma}_{\varepsilon_1 \varepsilon_2}$ is naturally a module over the space of $\mathcal{H}_{\varepsilon_1 \varepsilon_1}$-strings (or $\mathcal{H}_{\varepsilon_2 \varepsilon_2}$-strings)



Space of $\mathcal{H}_{\varepsilon\varepsilon}$-strings naturally form an algebra $\mathcal{A}_{\varepsilon}$



Claim [NW] $\mathcal{A}_\varepsilon \simeq$ algebra obtained by def. quant. of $(\mathcal{M}_H, J)$, $\varepsilon \sim \hbar$

$(\mathcal{M}_H, J) \simeq$ mod. space of flat connections $d + \mathcal{A}$ in $SL(3\mathbb{C})$

Generators for A_0 $\mathrm{tr}(P\exp \int_\gamma \mathcal{A}) = L_\gamma$

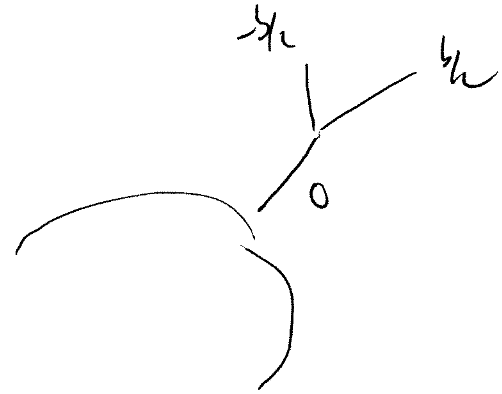
Sympl str. $\Omega_J \rightsquigarrow$ P.B. $\{L_\gamma, L_{\gamma'}\}$ = something known.

Upon quantization, $L_\gamma \to \widehat{L}_\gamma$ op. on $\mathcal{H}^\sigma_{\varepsilon_1 \varepsilon_2}$.

Proposal [NW]

$$\mathcal{H}^\sigma_{\varepsilon_1 \varepsilon_2} \simeq \mathcal{H}^{Liou}_{CB} \quad \text{(conformal blocks)} \qquad b = \varepsilon_1/\varepsilon_2, \quad b^{-1} = \varepsilon_2/\varepsilon_1$$

– both have two commuting actions of $\mathcal{A}_{\varepsilon_1}, \mathcal{A}_{\varepsilon_2}$



Ⓑ Quantization of $\mathcal{M}_H(C, J)$

Flat conns. $d + \mathcal{A} = \partial + A^{0,1} + \bar\partial + A^{1,0}$

$\rightsquigarrow$ loc syst. $(\mathcal{E}, \nabla)$ ← $\partial_y + M(y)$ ← holomorphic

hol. vb.

$\varepsilon \partial_y + M(y)$

- can always be represented by an oper

$$\left(\partial_y^2 + t(y),\ \text{proj. conn.}\right) \text{ with}$$

Same monodromy (in $PSL(2,\mathbb{C})$) than $\partial_y + M(y)$.

but at extra ("apparent") singular points $w_1, \dots, w_d$

$$t(y) \simeq -\frac{3}{4(y-w_k)^2} + \frac{\chi_k}{y-w_k} + \eta_k + \cdots$$

indeed, there exists g s.t.

$$g\left(\partial_y + \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) g = \partial_y + \begin{pmatrix} 0 & t(y) \\ 1 & 0 \end{pmatrix}$$

away from the zeros w_k of $\gamma(y)$

No monodromy (in $PSL(2,\mathbb{C})$) around w_k $\Leftrightarrow$

$$\chi_k^2 + \eta_k = 0 \qquad (*).$$

Example $g=0$

$$t(y) = \sum_{r=1}^{n} \frac{\delta_r}{(y-z_r)^2} + \frac{H_r}{(y-z_r)} + \sum_{k=1}^{d} \left(\frac{-3}{4(y-w_k)^2} + \frac{\chi_k}{(y-w_k)} \right)$$

$$(*) \iff 0 = \kappa_k^2 + \sum_{r=1}^{n} \left(\frac{\delta_r}{(w_k - z_r)^2} + \frac{H_r}{(w_k - z_r)} \right) + \sum_{\substack{k'=1 \\ k' \neq k}}^{d} \left(\frac{-3}{4(w_k - w_{k'})^2} + \frac{\kappa_{k'}}{w_k - w_{k'}} \right)$$

Linear eqns for H_r!

If $d = 3g-3+n$ then $(w_k, \kappa_k)_{k=1,\dots,3g-3+n}$

are local coords on $\mathcal{M}_H(C)$.

s.t. $\{w_k, w_{k'}\} = 0$ $\quad \{\kappa_{k'}, w_k\} = i\,\delta_{kk'}$!

$\{\kappa_k, \kappa_{k'}\} = 0$

Furthermore, can solve $(*)$ to get $H_r = H_r(\kappa, w)$.

$$\left.\begin{aligned} \frac{\partial w_k}{\partial z_r} &= \frac{\partial H_r}{\partial \kappa_k} \\ \frac{\partial \kappa_k}{\partial z_r} &= -\frac{\partial H_r}{\partial w_k} \end{aligned}\right\}$$

monodromy of $\partial_y^2 + t(y)$ stays constant under variations of C.

<u>Garnier system</u> in $g=0$.

Quantization

$W_k \to \hat{W}_k$

$K_k \to \hat{K}_k$

$[\hat{K}_k, \hat{W}_{k'}] = b^2 \delta_{kk'}$ etc.

param. for cx. structure on C

Realized on holo. functions of w, $\Psi(w, z)$

q - Hamiltonians from quantization of (*):

$$\mathcal{D}_k^{BPZ} = b^4 \frac{\partial^2}{\partial w_k} + \sum \left(\frac{\delta_r}{(w_r - z_r)^2} + \cancel{\frac{H_r}{w_r - z}} \right) + \sum_{k' \neq k} \left(\frac{-3}{4(\hat{w}_k - \hat{w}_{k'})} + \left\{ \frac{1}{(\hat{w}_k - \hat{w}_{k'})} \frac{\partial}{\partial w} \right\} \right)$$

$$\frac{b^2}{w_r - z_r} \frac{\partial}{\partial z_r}$$

Operators H_r: Gens of deformation of C

$$H_r \Psi(w, z) = b^2 \frac{\partial}{\partial z_r} \Psi(w, r)$$

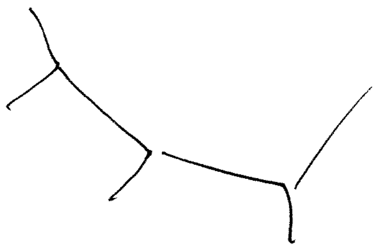
$\mathcal{D}_k^{BPZ} \Psi = 0 \Longleftrightarrow$ BPZ null-vector decoupling eqns.

Key observation: Conformal blocks of Liouville theory*

$\rightsquigarrow$ Complete set of solutions to $\mathcal{D}_k^{BPZ} \Psi = 0$

* holo parts of $\left\langle \prod_{r=1}^{n} V_{\alpha_r}(z_r, \bar{z}_r) \prod_{k=1} V_{-1/2b}(w_k, \bar{w}_k) \right\rangle_C$

$$\frac{3b^{-2}}{4} + \frac{1}{2} = \Delta_{-1/2b}$$



$$\begin{array}{l} A \\ \theta \end{array} \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$$\downarrow$$

$w \qquad \alpha(w_k) = K_k$

$$\begin{tikzcd}
 & \text{cl-Hitchin} \arrow[dl, dash] \arrow[dr, dash] & \\
\begin{array}{c}(M_H, J) \\ \text{flat}\end{array} \arrow[dr, dash] & & q\text{-Hitchin} \arrow[dl, dash] \\
 & \text{Liouville} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & q\text{-Hitchin} \\
\text{WZW} \arrow[ur, dash] &
\end{tikzcd}$$