12 August 2010
C. Teleman

Lie group actions on categories

Relation to TQFT's
"Cobordism Hypothesis" (Lurie)
n-dim'l TQFT (Atiyah, Segal, Witten):

functor from $(\mathrm{Bord}^n_{n-1}, \sqcup) \to (\mathrm{Vect}, \otimes)$

ob: closed $(n-1)$-folds

morphisms: n-bordism.

$(\text{framed } \mathrm{Bord}_0^n) \xrightarrow{Z} (\text{Some } n\text{-category})$

__Thm__ $\quad * \longrightarrow (\text{Some object}) = Z(*)$ Constraints!

(1) Image of a point determines the functor
(2) Constraints on $Z(*)$ known ("fully dualizable objects")
(3) Functors from $\left(\begin{array}{c}G\text{-bordism cat} \\ G \to O(n)\end{array}\right) \longrightarrow$ G-fixed points in the data
f.d. objects

Given a TQFT where G "acts by symmetries", can we "gauge" that action?

"Gauging":

1) promote the theory to a classical gauge theory
domain = (Manifolds + G-bundles)

2) quantum gauge theory: integrate over G-bundles

$G \times SO(n) \to O(n)$.

<u>Need</u> an action of G on $Z(*)$ for classical gauge theory

$$\begin{array}{ccc} G \to P & & \\ \downarrow & & BG \\ * & & \end{array}$$

Quantum theory = "integral of $Z(*)$ over BG"
has two possible interpretations : colimit$(H_*(BG; \ ))$, limit$(H^*(BG; \ ))$

These need to agree if you want to define a TQFT

Examples $n=1$

1 dim'l TQFT

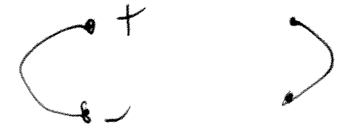
Vect vector spaces

Comp complexes

finite dim

$+\ \bullet \qquad V$

$-\ \bullet \qquad V^\vee$



$$\emptyset \mapsto \mathbb{C} \xrightarrow[\mathrm{Id}]{} V \otimes V^\vee \xrightarrow{ev} \mathbb{C}.$$

$\dim V$

Take trivial action of G on Vect or Comp
and ask for fixed pts: ans: G-~~rep~~ representation.

$$\mathcal{C} \qquad \mathcal{C}^G = \{ x \in \operatorname{Ob}\mathcal{C} \mid \varphi_g : x \xrightarrow{\sim} F_g x \}$$

+ coherence with $\alpha_{g,h}$

G acts $\quad g \in G, \quad F_g : \mathcal{C} \to \mathcal{C}$

$$F_g \circ F_h \xrightarrow{\alpha_{g,h}} F_{gh}$$

Morphisms compatible with data

associativity constraint

<u>Fact</u> Linear actions of G on Vect are classified by

$$H^2(BG;\mathbb{C}^\times)$$
$$\|$$
$$\mathbb{C}^\times$$

Central exts of G by $\mathbb{C}^\times$

$G \to \mathrm{Pic}$

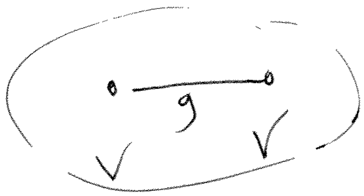
$g \to L_g$

$L_g \otimes L_h = L_{gh}$

= linear action on "unit in linear n-categories" (object) are classified by $H^n(BG;\mathbb{C}^\times)$

Gauged theory. Need a rep of G

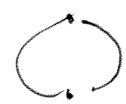
Classically (pt, princ bundle) $\to$ assoc. vector bundle



$\xrightarrow{Z}$ $\varphi_g : V \to V$

Quantum theory: $\bullet \longrightarrow V^G$

$$\|?$$
$$V_G$$



$$\longrightarrow \frac{1}{\#G}\sum_g \mathrm{Tr}_V\, g = \dim(V^G)$$

computation by integrating over bundles

computation from cobordism hypothesis

Compact Lie group

G acting on Vect

— Reps of G

V^G

Vector bundle over BG

" $\int_{BG} V$ " $H^*(BG;V) = V^G$

$H_*(BG;V) = V_G$

Infinitesimally trivialized action

$G/\hat{G}$ $\hat{G} \hookrightarrow G$ formal group at 1.

group stack   normal.

G conn on Vect, $G/\hat{G}$ actions are trivial

On complexes: G acts on the complexes

$\mathfrak{g}$-action has been homotopy trivialized.

~~I.e.~~ $\xi \in \mathfrak{g}$ ξ $r(\xi) : C^p \to C^{p-1}$

$\mathcal{L}_\xi \circlearrowright C^p$ with $[d, r(\xi)] = \mathcal{L}_\xi$

$\uparrow d$

C^{p-1} $* : \xi \to r(\xi)$ G-invariant

$r(\xi)^2 = 0$

$C^\cdot$ become a coefficient system for $H^*(BG)$

(Derived category of equiv. local system over a point)

$+\quad H^*(BG, C^\cdot)$ do that — not usually fin-dim'l

$-\quad H_*(BG, C^\cdot)$ do that — Didn't quite get a TQFT

eg G acts on X, smooth manifold

$C^\cdot = (\Omega_X, d)$

Then you get just this.

Actions & infinitesimally triv action of groups on linear cats

e.g. $\mathcal{C}$ = A-modules, A some algebra

g, $F_g : A\text{-mod} \to A\text{-mod}$.

$X \longmapsto M_g \otimes_A X$ $\quad M_g$ A-A bimodule.

(all colimit-preserving functors are like this)

What is $(A\text{-mod})^G$?

modules over $G \ltimes A$

Section over G of the bundle of bimodules defining the G action.

$M_g \otimes_A M_h \xrightarrow{\sim} M_{gh}$ specified as part of action

Infinitesimal triv of the action?

Want: G acts as above on A

Infinitesimal action

$$\begin{tikzcd} \mathfrak{g} \arrow[r, mapsto, "Lie"] \arrow[d, phantom, "\wr"] & HZ^1(A) \\ \mathfrak{g} \arrow[r, mapsto, "\iota"'] & HCH^0(A) \arrow[u] \end{tikzcd}$$

Hochschild cocycles, with Gerstenhaber bracket $[\,,]$

should be a dgLa map from $\mathfrak{g}^{-1} \to \mathfrak{g}^{0}$ to

e.g. $(\mathfrak{g} = \mathbb{R})$ $\quad \mathcal{L}_\xi \doteq 0$

$\iota_\xi \in HZ^0(A)$

Gerstenhaber square 0 ! (unobstructedness)

$(\Rightarrow \hbar\, \iota_\xi$ 1-parameter family of formal defs of A!)

<u>Claim</u> The (homotopy) fixed point category for this action on A

$=$ modules over $A[[\hbar]]_{def}$ (the def. family!)

e.g. $G = T$,

assume the stranger said that the action has been trivialized

on $\exp(t)$ $\qquad \pi = \pi_1 T \to \exp(t) \to T$

get a group hom $\Pi \to \mathrm{Aut}(\mathrm{Id}_{\mathcal{C}})$

$$\mathbb{C}[\Pi] \to HH^0(\mathcal{C})$$

$$\mathrm{Spec}(\mathbb{C}[\Pi]) = T^\vee$$

$$\begin{array}{c} \mathcal{C} \\ \downarrow \\ T^\vee \end{array}$$

$\mathcal{C}$: module category over $(\mathrm{Coh}\, T^\vee, \otimes)$

$$\begin{array}{c} \mathcal{P} \\ \downarrow \\ T \times T^\vee \end{array}$$

Action of T on $\mathrm{Coh}(T^\vee)$? answer: Poincaré bundle on $T \times T^\vee$.

$$t \in T, \quad \mathcal{P}_t, \quad \mathcal{P}_t \otimes \mathcal{P}_{t'} = \mathcal{P}_{tt'}$$

In symplectic geometry, if T acts in Hamiltonian manner on X

get $T/\exp(\mathfrak{t})$ action on $\mathrm{Fukaya}(X)$

$$\begin{tikzcd} X^\vee \arrow[dr] & \mathcal{C} \arrow[d] \\ & T^\vee \end{tikzcd}$$

<u>Claim</u> $\mathcal{C}^{T/\exp(\mathfrak{t})}$ = the fiber over $1 \in T^\vee$.

$\mathrm{Hom}_{\mathrm{Coh}(T^\vee), \otimes}(\mathrm{Vect}_1, \mathcal{C})$?

Can also study $\mathrm{Hom}_{(\mathrm{Coh}(T^\vee), \otimes)}(\mathrm{Vect}_x, \mathcal{C})$ $x \in T^\vee$? → other rep of $T/\exp(t)$

<u>Exercise</u>: Action of $T/\exp(t)$ on Vect are classified by points of $T^\vee$ (long exact seq)

<u>Fact</u> action of $T/\exp(t)$ on Gr Vect

$$T^\vee \times \mathrm{Sym}^? t^\vee$$

Def of Vect as module over $(\mathrm{Coh}(T^\vee), \otimes)$

$H^2(B(T/\exp(t)), \mathcal{O}^\times)$

$(B^2\pi ; \mathbb{C}^*)$

$$\pi^\vee \to t^\vee \to H^2(BT; \mathbb{C}^*) \to 0$$

$$H^2(BT; \mathbb{C}^*) = T^\vee$$

$B(G/\hat{G})$

Vect_1 $\quad$ $\mathrm{Coh}(T^\vee)$ module

Ψ

$\mathbb{C}$ $\quad$ $\mathcal{F} \otimes \mathbb{C} = \mathcal{F} \overset{L}{\otimes}_{T^\vee} \mathbb{C}_1$

$$\mathrm{Ext}_{(\mathrm{Coh}(T^\vee),\otimes)}(\mathrm{Vect}_z, \mathrm{Vect}_1) = \begin{cases} 0 & \text{if } z \neq 1 \\ \underbrace{\mathrm{Sym}\, t^\vee}_{\deg 2}\text{-mod} & \text{if } z = 1 \end{cases}$$

$$\mathrm{Ext}_{T^\vee}(\mathbb{C}_1, \mathbb{C}_1) = \underbrace{\Lambda t^\vee}_{\deg ?}$$

Fixed pt cats will be cats of modules over an algebra

Obvious model: "$G/\hat{G} \ltimes A$"

$$C^\infty(G, \Lambda \mathfrak{g}) \ltimes A$$

$\partial =$ Lie alg. diff

Have enough info to make into an algebra.

$$[\xi, a] = \rho(\xi)(a).$$

Koszul dual models:

$G \curvearrowright (A \otimes \mathrm{Sym}\, \mathfrak{g}^*)$ $\quad d = d_A + \xi^a r_a$

$W = \xi_a (\delta_a) \otimes \xi^a$

derivative of fn on $\mathfrak{g}^*$

This is a curved algebra and has a well def dg cat of modules

Can define action on Vect by $(\mathrm{Sym}\, \mathfrak{g}^*)^G$ $\ni \sum \xi^a \xi^a = H$ inv. fns

$\longrightarrow$ curvatures (= LG potentials)

Thm adding H semisimplifies the cat of modules $\cong \mathrm{Rep}(G)$!

top YM in dim 2

Thm There is a loop group analogue.

(G action on tensor cats)

$d^2 = [W, \cdot\,]$

$(\mathrm{Vect}^{G/G}_{\mathrm{def}})$ does not generate a TQFT