WCF for motivic DT-invariants

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Mathematical formalism for "DT-type" invariants (BPS states) and their wall-crossing formulas.

Dictionary

BPS states ⟷	semistable objects in a triangulated A_∞-category
BPS degeneracies $\Omega(\gamma)$	# of semistable objects with a given class γ in the K-theory
WCF (change of $\Omega(\gamma)$ as we cross the wall of MS)	Change of # as we cross codim 1 walls in the space of stability structures

3 levels of our formalism: categorical, graded Lie algebras, numerical

1

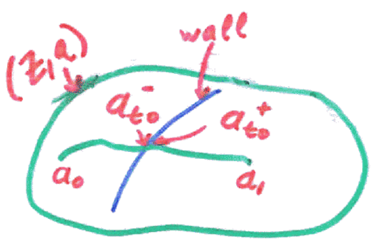
Lie algebras level

T1

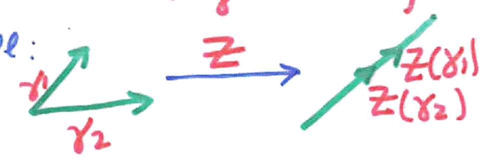
Stability data on $\mathfrak{g} = \bigoplus_{\gamma\in\Gamma} \mathfrak{g}_\gamma$, Γ-graded Lie algebra ($\mathbb{Z}^n$)

$(Z, a=(a(\gamma)))$, $a(\gamma)\in\mathfrak{g}_\gamma$, $Z:\Gamma\to\mathbb{C}$ - central charge

WCF for (Z,a): $\exp\left(\sum_\gamma a^-_{t_0}(\gamma)\right) = \exp\left(\sum_\gamma a^+_{t_0}(\gamma)\right)$



wall:



application

Joyce invariants, polylogs, Bridgeland-Toledano Laredo Stokes factors

$a(\gamma) = \varepsilon_\gamma$ ← series in f_α,
$df_\alpha = -\frac{1}{2}\sum_{\beta+\delta=\gamma}[f_\beta, f_\gamma]\, d\log\frac{Z(\beta)}{Z(\gamma)}$

Lie algebras level: special cases

a) $\mathfrak{g} = \mathfrak{gl}(n)$: Cecotti-Vafa tt^*-geometry, $\Gamma = \{\sum k_i = 0\}$, $Z(\gamma_{ij}) = z_i - z_j$

$a(\gamma_{ij}) = a_{ij} E_{ij}$, $a_{ji} = -a_{ij}$. WCF: $a_{ik} \mapsto a_{ik} + a_{ij} a_{jk}$:

stability

b) Γ carries skew-symmetric form $\langle , \rangle : \Gamma \otimes \Gamma \to \mathbb{Z}$

$\mathfrak{g} = \mathcal{O}(\Gamma \otimes \mathbb{C}^*) \leftarrow$ Lie algebra

$[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle\gamma_1,\gamma_2\rangle} \langle\gamma_1,\gamma_2\rangle e_{\gamma_1+\gamma_2}$

$a(\gamma) =$ = cov(?)

Complex integrable systems, Seiberg-Witten and Hitchin geometry (complex symplectic manifolds)

Hyperkähler metric on the total space (Gaitto-Moore-Neitzke)

Numerical DT-invar. $\Omega(\gamma) \in \mathbb{Z}$. WCF:

$T_{1,0} \cdot T_{0,k} = \prod_{a/b \nearrow} T_{a,kb}^{d(a,b,k)}$

$\Gamma_0 = \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$, $\gamma = a\gamma_1 + b\gamma_2$

$T_{a,b} = \exp\left[-\sum_{n \geq 1} \frac{e_\gamma^n}{n^2}, \cdot\right] = \exp[-Li_2(e_\gamma), \cdot]$

Tropical vertex group, relative GW-invariants (Gross-Pandharipande-Siebert)

$T_{0,1} \cdot T_{1,0} = T_{1,0} \cdot T_{1,1} \cdot T_{0,1}$

Categorical level

Triangulated A_∞-category $\mathcal{C}$ with ind-constructible structure (category of branes)

Hall algebra $H(\mathcal{C})$ associative algebra over the ring of motivic functions on $Ob(\mathcal{C})$

"pre"-DT-invariants : $A_V^{Hall} \in H(\mathcal{C}_V)^\times$

Hausdorff space $Stab(\mathcal{C})$ of stability conditions on $\mathcal{C}$

Categories $\mathcal{C}_V$ for each strict sector V, Hall algebras $H(\mathcal{C}_V)$.

Each $\mathcal{C}_V$ is generated by semistables with central charges in V

WCF: collection A_V^{Hall} doesn't change if central charges do not cross ∂V $\leftarrow Z(\gamma)$

$\Longleftrightarrow$ control of the change of "motive" of semistable objects in $\mathcal{C}$.

T4

Categorical level: special cases

3d CY category $\mathcal{C}$
+ orientation data

3d CY category $\mathcal{C}$ with cluster spherical generators $(E_i)_{i\in I}$

Quiver with potential

Graded case: dimer models (Mozgovoy, Reineke, Ooguri, Yamazaki, Nagao, Nakajima ...

Quantum cluster variety

Motivic integration map:
$\Phi: H(\mathcal{C}) \to R_{e,q}$
quantum torus
$e_{\gamma_1} e_{\gamma_2} = q^{\frac{1}{2}\langle \gamma_1,\gamma_2\rangle} e_{\gamma_1+\gamma_2}$

$\Phi(A_V^{Hall}) := A_V^{mot}$
motivic DT-invariants.
$A_{V_1 \cup V_2}^{mot} = A_{V_1}^{mot} A_{V_2}^{mot}$
Motivic DT-invariants

$A_V^{mot} = \overrightarrow{\prod} A_{\ell}^{mot}$

T5

Quasi-classical limit $q^{1/2} \to -1$

Quantum cluster variety

Cluster variety structure on $\Gamma \otimes \mathbb{C}^*$

Motivic DT-invariants
$A_V^{\cancel{mot}} \in H(\mathcal{C}_V)^*$

$\lim_{q^{1/2}\to -1} Ad_{A_V^{mot}}$

Stability data on
$\mathfrak{g} = \mathcal{O}(\Gamma \otimes \mathbb{C}^*)$
and $A_V = \overrightarrow{\prod}_{Z(\gamma)\in V} T_\gamma^{\Omega(\gamma)}$

$T_\gamma(e_\mu) = (1-e_\gamma)^{\langle \gamma,\mu\rangle} e_\mu$ -
Poisson automorphisms

Stability function
$a(\gamma) = -\left(\sum_{\substack{n\geq 1\\ \frac{1}{n}\gamma\in\Gamma\setminus\{0\}}} \frac{\Omega(\frac{\gamma}{n})}{n^2}\right)\cdot e_\gamma$

Quasi-classical limit: categorical $\longrightarrow$ Lie algebras

Parameter: $q = \mathbb{L} = [\mathbb{A}^1] \to 1$ (better: $q^{\frac{1}{2}} \to -1$)

Ex. 3CY: $\hat{e}_{\gamma_1} \cdot \hat{e}_{\gamma_2} = q^{\frac{1}{2}\langle \gamma_1, \gamma_2 \rangle} \hat{e}_{\gamma_1+\gamma_2}$ - quantum torus

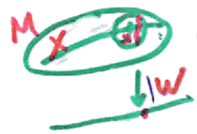
Coefficients are motivic functions on $Ob(\mathcal{C})$.

Why motivic functions?

Kai Behrend's approach to DT-invariants (3d CY case):

$\underset{\substack{\text{moduli}\\\text{space}\\\text{0-dim}}}{X} \subset \underset{\substack{\text{smooth}\\\text{scheme}}}{M}$, $X = \underset{\text{regular}}{Crit(W)} = dW \cap \{\text{0-sect. } T^*M\}$

$\nu_X(x) = (-1)^{\dim M}(1 - \chi(MF_x(W)))$



$dW(x) = 0$

$\nu_X : X \to \mathbb{Z}$ is constructible (e.g. characteristic fnct)

(2)

Invariant: $B(X) = \sum_{n \in \mathbb{Z}} n\,\chi(\{x \mid \nu(x) = n\}) = \int_X \nu_X \, d\chi$ (integral over χ)

X - smooth $\Rightarrow B(X) = (-1)^{\dim X} \chi(X) = \int_{[X]^{vir}} 1 = \int_{[X]} e(\text{obstr. sheaf})$

Local formula $\Rightarrow$ good for non-compact X.

Integration over χ is good: $\nu_{X \sqcup Y} = \nu_X + \nu_Y$, $\nu_{X \times Y} = \nu_X \cdot \nu_Y$

Idea: use invariants with similar "motivic" behavior, e.g. $P_X(q)$ (Poincare polynom.), $\# X(\mathbb{F}_q)$ (over finite field)

$\Rightarrow$ Motivic fncts, motivic integration (Kontsevich, Denef-Loeser)

generalizations of char. fncts — generalization of integration over χ.

(3)

Constructible sets = Boolean algebra generated by schemes.

$Mot(X) = \{[S \xrightarrow{\pi} X]$, π - morphism of constr. sets $\}$

$[S_1 \sqcup S_2 \to X] = [S_1 \to X] + [S_2 \to X]$

$\Rightarrow$ $Mot(X)$ is a commutative ring (via fiber product)

Several "realizations", e.g. $[\pi : S \to X] \rightsquigarrow \chi(\pi) =$ Constructible $\mathbb{Z}$-valued fnct: $= \chi(\pi^{-1}(x))$

Or: $[\pi : S \to X] \rightsquigarrow (x \mapsto \# \{ s \in X(\mathbb{F}_{q^n}) \mid \pi(s) = x \})$

Advantage: can define fractions $\frac{[X]}{[GL(n)]}$ $\chi(-)=0$

But $P_{GL(1)}(q) = 1 - q \neq 0$

Distinguished element $q = \mathbb{L} = [\mathbb{A}^1]$ - invert it

(later: add $q^{1/2}$, invert $[GL(n)]$)

$\Rightarrow$ **Idea:** $\mathcal{C}$ - 3CY category (A_∞ always) $\Rightarrow \exists$ W - superpotential

Assume: $Ob(\mathcal{C})$ = countable union of constructible sets.

$\Rightarrow$ can define $Mot(Ob(\mathcal{C}))$. **Important:** $\exists$ motivic $MF(W)$.

4

Crash course on 3CY-categories (e.g. with one object)

- Non-commutative formal symplectic ~~supermanifold~~ ($\mathbb{Z}$-graded) (X, ω) with marked pt $x_0 \in X$ (object),
- vector field Q, $\deg Q = +1$, $[Q, Q] = 0$, $Q(x_0) = 0$, s.t. $Lie_Q(\omega) = 0$. $\Rightarrow \exists$ W s.t. $i_Q \omega = dW$, $\{W, W\} = 0$.

In general: objects of the category = crit. pts of W (CS-functional)

$Tr\left(\frac{\partial A \cdot A}{2} + \frac{A^3}{3}\right)$

Can translate to categories: $Q = \sum_n Q_n \Rightarrow A_\infty$ - structure:

$m_n : \bigotimes_{i=0}^{n-1} Hom(E_i, E_{i+1}) \to Hom(E_0, E_n)[2-n]$

ω has degree (dimension of CY). For $d = 3$:

$\exists$ non-degenerate $(\cdot,\cdot) : Hom(E,F) \otimes Hom(F,E) \to k[-3]$ $\mathbb{C}$

Potential $W_E(\alpha) = \sum_{n \geq 1} \frac{(m_n(\alpha, \dots, \alpha), \alpha)}{n+1}$ $\left(CS = \frac{(m_1(\alpha), \alpha)}{2} + \frac{(\alpha \cdot \alpha, \alpha)}{3} \right)$

$\alpha \in Ext^1(E, E)$ $\alpha \cdot \alpha = m_2(\alpha, \alpha)$

Also: $\mathcal{C}$ is triangulated (e.g. $\exists E[n]$)

5

Ind-constructible 3CY category $\mathcal{C}$:

$Ob(\mathcal{C}) = \bigsqcup_{i \in I\text{-count}} X_i \leftarrow$ constructible sets (ind-constructible category)

m_n, W, etc : constructible maps (require: locally regular)

Examples: $D^b(X)$ ($\uparrow$ 3CY, proj.), $D^b_Z(X)$ (compact subset; $\uparrow$ local 3CY),

probably all categories of branes

Stability conditions

Bridgeland: triangulated categories. Can modify for ind-constr.

Assume: $cl: K_0(\mathcal{C}) \to \Gamma \simeq \mathbb{Z}^n$ (chern character, class map)

In 3CY category case: $\langle , \rangle : \Gamma \otimes \Gamma \to \mathbb{Z}$, Assume: cl agrees with

Euler form $\chi(E,F) = \sum_i (-1)^i \dim Ext^i(E,F)$ and $\langle , \rangle$

stability condition includes:

- $Z: \Gamma \to \mathbb{C}$ - central charge "$Z(cl(E))$"
- $\mathcal{C}^{ss} \subset Ob(\mathcal{C})$ - semistable objects, $Z(E) \neq 0$
- Choice of $\operatorname{Log} Z(E) := \operatorname{Log} Z(cl(E))$, $E \in \mathcal{C}^{ss} \Rightarrow \exists \operatorname{Arg} Z(E)$

6

Axioms include:

- Each $E \in Ob(\mathcal{C})$ admits a filtration (in triangulated sense) with quotients $F_1, F_2, \dots, F_n \in \mathcal{C}^{ss}$ s.t. $Arg(F_1) > Arg(F_2) > \dots$
- $\forall \gamma \neq 0$ $\mathcal{C}^{ss}_\gamma = \{E \in \mathcal{C}^{ss} \mid cl(E) = \gamma,\ Arg(E)\text{ - fixed}\}$ is a constructible set (details in our Sect. 3.4)

$\Rightarrow$ The space $Stab(\mathcal{C}, cl)$ is Hausdorff. Projection to $Z \in Hom(\Gamma, \mathbb{C})$ is a local homeomorphism. (covering map)

$\Gamma_{\mathbb{R}} = \Gamma \otimes \mathbb{R}$, $\|\cdot\|$ $\|E\| \leq C|Z(E)|$

Example (cluster collection)

$\mathcal{C} = \langle E_1, E_2, \dots, E_n \rangle$, $Ext^\bullet(E_k, E_k) = H^\bullet(S^3)$ (spherical object)

$Ext^m(E_i, E_j) \neq 0$ for $m=1$ or $m=2$

Fact: such 3d CY categories $\leftrightarrow$ (Q, W): quiver with potential

stability: choice of $z_i \in \mathcal{H} =$ z_i

$Z(E_i) = z_i$, $1 \leq i \leq n$

7

Hall algebra $H(\mathcal{C})$

$q = [\mathbb{A}^1]$, $E_{1,2} \in Ob(\mathcal{C})$

Contains δ-functions of objects: δ_E. Product:

$$\delta_{E_1} \cdot \delta_{E_2} = q^{-(E_2,E_1)_{\le 0}} [Ext^1(E_2,E_1) \to Ob\,\mathcal{C}] := q^{-(E_2,E_1)_{\le 0}} \int_{\alpha \in Ext^1(E_2,E_1)} \delta_{E_\alpha}$$

"motive of extensions"

$$(E_2,E_1)_{\le 0} = \sum_{i \le 0} (-1)^i \dim Ext^i(E_2,E_1), \quad q = [\mathbb{A}^1]$$

Th. The product is associative ($\Rightarrow H(\mathcal{C}) = \bigoplus_{\gamma \in \Gamma} H(\mathcal{C})_\gamma$ is a graded algebra)

For any $V = \angle$ - strict sector $\Rightarrow \mathcal{C}_V =$ subcategory, generated by semisimples E, $Z(E) \in V$, and their extensions (and 0)

$\Rightarrow$ have $\widehat{H}(\mathcal{C}_V) \Rightarrow$ define $A_V^{Hall} = 1 + \sum_{[E] \in Iso\,\mathcal{C}_V} \frac{[E]}{[Aut(E)]}$ ← motive of gp. of autom.

Analog for finite fields:
$A_V^{Hall} = 1 + \sum_{[E]} \frac{[E]}{\# Aut(E)}$ $\quad q^{-1}, \frac{1}{[GL(n)]}, n \ge 1$

Th. If $\angle^{V_1}_{V_2}$ then $A^{Hall}_{V_1 \cup V_2} = A^{Hall}_{V_1} \cdot A^{Hall}_{V_2} \Rightarrow A_V^{Hall} = \overrightarrow{\prod_{\ell \subset V}} A_\ell^{Hall}$

8

WCF ("abstract"):

Collection (A_V^{Hall}) does not change if we change stability condition s.t. $\nexists\, \gamma \in \Gamma \setminus \{0\}$ with $Z(\gamma) \cap \partial V \ne \emptyset$. ($V$ generic)

$\mathcal{C}$ - 3CY category $\Rightarrow (\Gamma, \langle\,,\rangle)$ ← skew-symmetric

Define motivic quantum torus $R_{\Gamma,q}$:

$$\hat{e}_{\gamma_1} \cdot \hat{e}_{\gamma_2} = q^{\frac{\langle \gamma_1,\gamma_2 \rangle}{2}} \hat{e}_{\gamma_1+\gamma_2}$$

Th. $\exists$ homomorphism of algebras $\Phi: H(\mathcal{C}) \to R_{\Gamma,q}$ given explicitely in terms of $MF_x(W)$ - motivic Milnor fiber

$\Rightarrow$ define motivic DT-invariants $A_V^{mot} = \Phi(A_V^{Hall})$. $\boxed{(A_V^{mot})}$

Ex. $\mathcal{C} = \langle E \rangle$ ← spherical. Stability: $Z(E) = z \in$ ← stable

$$\Rightarrow A_V^{mot} = \sum_{n \ge 0} \frac{q^{n^2/2}\, x^n}{(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})} := E(x) \leftarrow Li_{2,q^{1/2}}(x)$$

$x = e_{cl(E)}$

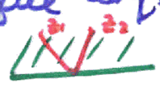
9

WCF: we can interpret $R_{\mathcal{C},q}$ (or $H(\mathcal{C})$) as a Γ-graded Lie algebra.

Observation: to have a stability data $(Z, (a(\gamma))$ on a graded Lie algebra $\mathfrak{g} = \bigoplus_\gamma \mathfrak{g}_\gamma$ is equivalent to: $\forall$ strict sector V to have group element $A_V = \exp(\sum_{Z(\gamma)\in V} a(\gamma))$ s.t. $A_{V_1 \sqcup V_2} = A_{V_1} \cdot A_{V_2}$. But we have A_V^{mot}!

Example of WCF. $\mathcal{C} = \langle E_1, E_2 \rangle$ s.t. E_i - spherical, $Ext^1(E_2, E_1)$ - 1-dim.

i.e. $\exists$ unique extension $E_1 \to E_{12} \to E_2$.

Stability:  $z_i = Z(E_i)$, $i = 1, 2$, both stable.

(from $\operatorname{Arg} z_1 > \operatorname{Arg} z_2$ to $\operatorname{Arg} z_2 > \operatorname{Arg} z_1$)

$$\Rightarrow \quad E(x_1)E(x_2) = E(x_2)E(x_{12})E(x_1) \text{ in the algebra } x_1x_2 = q\, x_2x_1, \quad x_{12} := q^{1/2} x_2 x_1$$

5-term identity for $Li_{2,q}$.

10

Conjecturally: $\exists$ quasi-classical limit of $Ad_{A_V^{mot}} := A_V$. It gives known formulas, e.g. $\overrightarrow{\prod_{a/b \nearrow}} T_{a,b}^{c_{a,b}} = \overleftarrow{\prod_{a/b \searrow}} T_{a,b}^{d_{a,b}}$.

Example of emerging cluster transformations. $\Gamma \simeq \mathbb{Z}^n$

$\mathcal{C} = \langle E_1, \dots, E_n \rangle$ - cluster generators.

In the quasi-classical limit: $A_V : (\mathbb{C}^*)^{2n} \to (\mathbb{C}^*)^{2n}$ - formal symplectomorphisms of the bracket like

$\{x_i, x_j\} = a_{ij} x_i x_j$, $x_i = e_{cl(E_i)}$.

Categorical mutations (tilting) $\Rightarrow$ x_i's get changed but collection (A_V) do not (invariant of Δ-category not t-structure)

Change of coordinates is cluster transformations (Fock - Goncharov, Fomin - Zelevinsky).

E.g. $x_i \mapsto x_i' (1 - x_0')^{a_{0i}}$.

11

Open problems (some).

- Invariants w.r.t. deformations of $\mathcal{C}$.
- Relation to 4d Donaldson
- Complex integrable systems for general $\mathcal{C}$
  (e.g. for (Q,W) - quiver w/potential).
- Relation to "abstract" GW-theory for CY-categories
  (M.K.-Y.S., Costello)
- "Quaternionic" Fukaya category ?
  (quaternionic curves in hyperkahler mfld)
  (cf. AG/0406564)
- Stability conditions in dimer theory.
- Foundational questions: existence $\lim\limits_{q^{1/2}\to -1}$
- Orientation data (need for existence Φ)
  Roughly: superline bundle $\sqrt{\mathrm{sdet}\,\mathrm{Ext}^{\bullet}(E,E)} \to \mathrm{Ob}\,\mathcal{C}$.

12