Keldysh functional integral for open systems & driven criticality

Outline

A) From the Quantum Master eq. to the Keldysh functional integral

→ 3 step derivation

→ simple example & interpretation, correlations vs. responses

B) Applications / Use of open system Keldysh functional integral

→ Keldysh functional RG

→ Driven criticality : classical & quantum

Focus on: driven stationary states!

S. Diehl, TU Dresden & KITP

21.10.2015

A) From the Quantum Master Eq. to the Keldysh functional integral

° Basic idea in 3 steps

1) Schrödinger eq. evolves state vector

$$i\partial_t |\psi\rangle(t) = H|\psi\rangle(t) \;\rightarrow\; |\psi\rangle(t) = \mathcal{U}(t,t_0)|\psi\rangle(t_0)\,;\; \mathcal{U}(t,t_0) = e^{-iH(t-t_0)}$$

differential form integral form

2) Heisenberg–von Neumann eq. evolves state (density) matrix

$$\partial_t \rho(t) = -i[H,\rho(t)] \;\Rightarrow\; \rho(t) = \mathcal{U}(t,t_0)\,\rho(t_0)\,\mathcal{U}^\dagger(t,t_0)$$

[NB: identical for pure states: separability $\rho = |\psi\rangle\langle\psi|$]

3) the same is true for the Master eq.:

$$\partial_t \rho(t) = -i[H,\rho] + \sum_i \kappa_i\left(L_i\rho L_i^\dagger - \tfrac{1}{2}\{L_i^\dagger L_i,\rho\}\right) = \mathcal{L}[\rho]$$

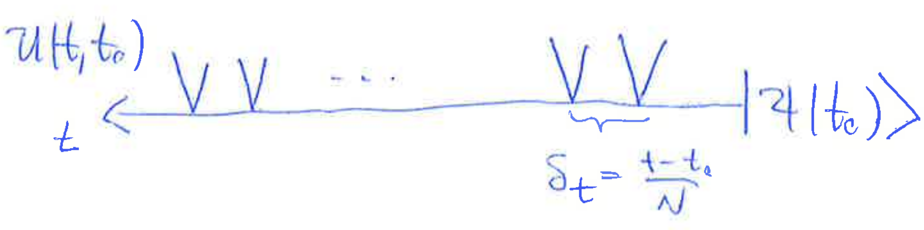
Liouvillian superop.

$$\Rightarrow \rho(t) = e^{(t-t_0)\mathcal{L}}\rho(t_0)$$

Sketch along the three steps:

i) Functional integral idea

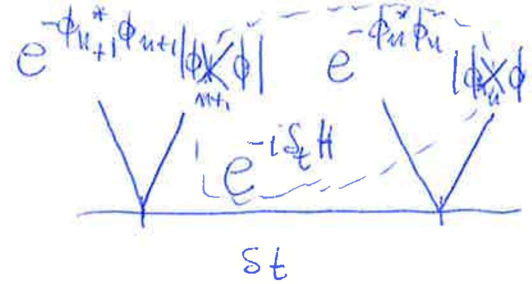
→ "trotterization" & insertion of coherent states



$a|\phi\rangle = \phi|\phi\rangle$

$\langle\phi'|\phi\rangle = e^{\phi'^*\phi}$

$\mathbb{1} = \int \frac{d\phi^* d\phi}{\pi} e^{-\phi^*\phi}|\phi\rangle\langle\phi|$



→ one time step [H normal ordered]

expand

$$\approx e^{-\phi_n^*\phi_n} \langle\phi_{n+1}| 1 - i\delta_t H[a^+, a] |\phi_n\rangle$$

$$= e^{-\phi_n^*\phi_n} e^{\phi_{n+1}^*\phi_n} \left(1 - i\delta_t H[\phi_{n+1}^*, \phi_n]\right)$$

re-expon.

$$\approx e^{i\delta_t \left[-i\frac{(\phi_{n+1}^* - \phi_n^*)}{\delta_t}\phi_n - H[\phi_{n+1}^*, \phi_n]\right)}$$

$$dt \qquad -i\partial_t\phi^* \cdot \phi \qquad \overset{\approx}{\downarrow}\ H[\phi^*(t), \phi(t)]$$

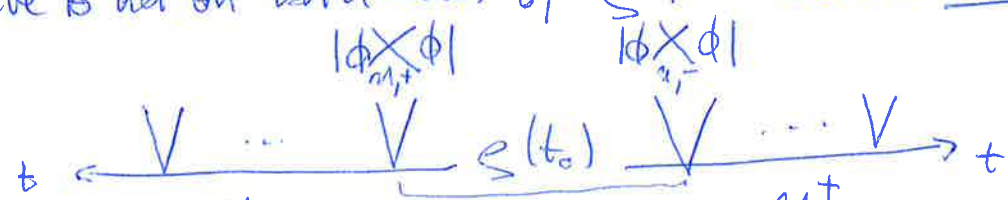
→ many steps:

$$\underbrace{\int \prod_t \frac{d\phi^*(t)\, d\phi(t)}{\pi}}_{=: \mathcal{D}(\phi^*, \phi)} e^{i\int_{t_i}^{t_f} dt \left[-i\partial_t\phi^*(t)\cdot\phi(t) - H[\phi^*(t), \phi(t)]\right]}$$

[NB: • operator $H[a^+, a] \longrightarrow H[\phi^*(t), \phi(t)]$ complex functional
• time evol from overlap of neighbouring states.

2) Generalize to matrix evol. & define "partition function"

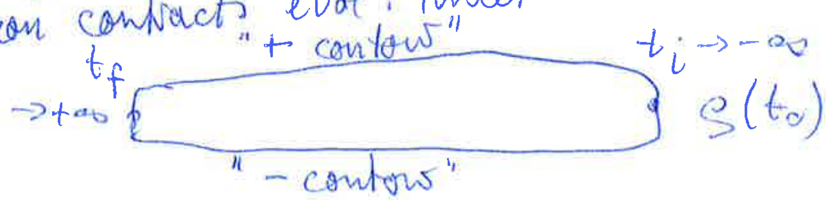
→ have to act on both sides of ρ → needs two sets $|\phi\rangle_{n,\pm}$!



→ partition function

• $Z = \mathrm{tr}\, \rho(t) = \mathrm{tr}\, \mathcal{U}\rho(t_0)\mathcal{U}^+ = \mathrm{tr}\, \rho(t_0) = 1$.

• trace operation contracts evol. times



3) Application to Liouville operator

$$\rho(t) = e^{(t-t_0)\mathcal{L}}\rho(t) \overset{\text{def}}{=} \lim_{N\to\infty}(1+\delta_t\mathcal{L})^N \rho(t_0)$$

→ apply superop on both sides of ρ at each time step.

→ result (eg. lattice system, sites i)

· $Z = \mathrm{tr}\,\rho(t) = \int \mathcal{D}(\Phi_+,\Phi_-)\, e^{iS[\Phi_+,\Phi_-]} = 1$

· $\Phi_\pm = \begin{pmatrix}\phi^*_{\pm,i}(t)\\ \phi_{\pm,i}(t)\end{pmatrix}$; $\mathcal{D}(\Phi_+,\Phi_-) = \prod_{\sigma=\pm}\overset{\text{time}}{\prod_t}\overset{\text{space}}{\prod_i}\int \frac{d\phi^*_{\sigma i}(t)\, d\phi_{\sigma i}(t)}{\pi}$

· $S = \int_{-\infty}^{\infty} dt \sum_i \left[\phi^*_{i+}(t)\, i\partial_t \phi_{i+} - (+\to -) - i\mathcal{L}[\Phi_+,\Phi_-]\right]$

$$\mathcal{L}[\Phi_+,\Phi_-] = \underbrace{-i(H_+ - H_-)}_{\text{Heisenberg commutator}} + \underbrace{\sum_i \kappa_i \left(L_{i+}L^\dagger_{i-} - \tfrac12 L^\dagger_{i+}L_{i+} - \tfrac12 L^\dagger_{i-}L_{i-}\right)}_{\text{Lindblad structure}}$$

$H_\pm = H[\phi_\pm]$ etc.

Translation table

· normal order operators
· then operators left of ρ → + contour
right → − "

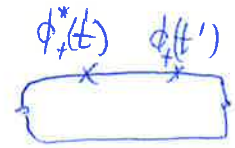
How to extract phys. information? ($Z = 1$)

· introduce sources (cf. stat mech)

$$Z[J_+,J_-] = \int \mathcal{D}(\Phi_+,\Phi_-)\, e^{iS + \sum_\sigma J^\dagger_\sigma \Phi_\sigma} = \langle e^{i\sum_\sigma J^\dagger_\sigma \Phi_\sigma}\rangle$$

$$Z[0,0] = 1$$

· variational derivatives: $\langle \phi^*_+(t')\,\phi_+(t)\rangle = \frac{\delta^2 Z}{\delta j_+(t)\,\delta j^*_+(t)}$



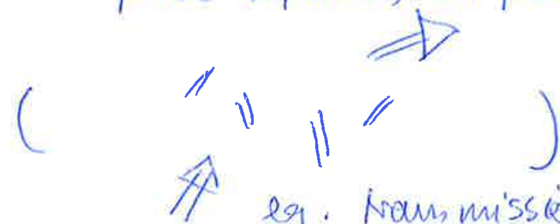
correlation and response functions

- two basic types of exp:

correlation measurement — study w/o disturbing

response measurement — probe w/ (weak) ext fields

photon output, $g^{(2)}(\tau)$



e.g. transmission/abs.

- both kinds are delivered after "Keldysh rotation"

$$\phi_c = \frac{1}{\sqrt{2}}(\phi_+ + \phi_-) \quad \text{"classical field"}$$

$$\phi_q = \frac{1}{\sqrt{2}}(\phi_+ - \phi_-) \quad \text{"quantum field"}$$

[N.B. $\langle\phi_c\rangle \neq 0$ possible, $\langle\phi_q\rangle = 0$ always]

↳ example 1.

discussion:

response: $G^{R/A}$: spectral properties

→ Lorentzian spectral density $A(\omega) = \mathrm{Im}\, G^R(\omega) = \frac{2\kappa}{(\omega-\omega_c)^2+\kappa^2}$

→ retarded decay:

$$G^R(t-t') = \int_\omega e^{-i\omega(t-t')} G^R(\omega) = \theta(t-t')\, e^{-i\omega_c(t-t')} e^{-\kappa(t-t')}$$

correlations: G^K: statistical properties

→ cavity occupation $(t \to \infty)$

$$2\langle \hat{n}(t)\rangle + 1 = \langle\{a^+(t), a(t)\}\rangle = i\, G^K(t-t) = \int_\omega e^{i\omega(t-t)} G^K(\omega)$$

$$= \frac{\gamma_e + \gamma_p}{\gamma_e - \gamma_p} = 2\bar{n} + 1$$

↑ thermal res: $\gamma_e = \bar{n}+1$, $\gamma_p = \bar{n}$.

Example 1 : lossy / pumped cavity (0 + 1 dimensional problem)

$$\underset{\text{space}}{0} + \underset{\text{time}}{1}$$

- master eq.:

$$\partial_t \rho = -i\,[\omega_0 a^\dagger a, \rho] + \gamma_l \left(2 a \rho a^\dagger - \{a^\dagger a, \rho\}\right) + \gamma_p \left(a^\dagger \rho a - \{a a^\dagger, \rho\}\right)$$

- action :

$$S = \int dt\, (a_c^*(t), a_q^*(t)) \begin{pmatrix} 0 & i\partial_t - \omega_0 - i(\gamma_l - \gamma_p) \\ i\partial_t - \omega_0 + i(\gamma_l - \gamma_p) & 2i(\gamma_l + \gamma_p) \end{pmatrix} \begin{pmatrix} a_c(t) \\ a_q(t) \end{pmatrix}$$

$$= \int \frac{d\omega}{2\pi} (a_c^*(\omega), a_q^*(\omega)) \underbrace{\begin{pmatrix} 0 & \overbrace{\omega - \omega_0 - i(\gamma_l - \gamma_p)}^{=:\, P^A(\omega) = P^R(\omega)^\dagger} \\ \underbrace{\omega - \omega_0 + i(\gamma_l - \gamma_p)}_{=:\, P^R(\omega)} & \underbrace{2i(\gamma_l + \gamma_p)}_{=:\, P^K(\omega) = -P^K(\omega)^\dagger} \end{pmatrix}}_{P(\omega) = G^{-1}(\omega)} \begin{pmatrix} a_c(\omega) \\ a_q(\omega) \end{pmatrix}$$

- EoM :

$$\begin{pmatrix} \frac{\delta S}{\delta \phi_c^*(\omega)} \\ \frac{\delta S}{\delta \phi_q^*(\omega)} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & P^A(\omega) \\ P^R(\omega) & P^K(\omega) \end{pmatrix}}_{G^{-1}(\omega)} \underbrace{\begin{pmatrix} \phi_c(\omega) \\ \phi_q(\omega) \end{pmatrix}}_{\Phi(\omega)} = \int_{\omega'} \underbrace{G^{-1}(\omega)\, \delta(\omega - \omega')}_{=:\, \bar{G}^{-1}(\omega, \omega')} \Phi(\omega')$$

$\Rightarrow$ Green's function ($\bar{G} \circ \bar{G}^{-1} = \mathbb{1}$)

$$\bar{G}(\omega,\omega') = \left[\bar{G}^{-1}(\omega, \omega')\right]^{-1} = \begin{pmatrix} G^K(\omega) & G^R(\omega) \\ G^A(\omega) & 0 \end{pmatrix} \delta(\omega - \omega')$$

with $G^{R/A}(\omega) = \left(P^{R/A}\right)^{-1}$

$$G^K(\omega) = - G^R(\omega)\, P(\omega)\, G^A(\omega)$$

B) <u>Use of (Keldysh) functional integrals</u>

- systematic (diagrammatic) perturbation theory
- X- RG, long distance physics (Keldysh: <u>quantum</u> dynamical fieldth.)
- collective behavior, emergent degrees of freedom
- symmetries, eg. vs. noneq.
- time evolution
- fermions, spins (Holstein-Primakoff)
- mean field + fluctuations, semiclass. limit
- nonperturbative effects <u>(eg.</u> vortices)

Keldysh (functional) Renormalization group

- change of variables (schematically)

$$W[J] = -i \log Z[J]$$

Legendre trafo:

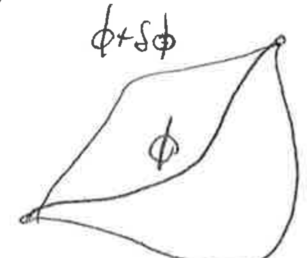
$$\Gamma[\phi] = W[J] - \int J\phi \qquad \text{"effective action"}$$

- functional integral representation:

$$e^{i\Gamma[\phi]} = \int \mathcal{D}\delta\phi \, e^{i\left(S[\phi+\delta\phi] - \frac{\delta\Gamma}{\delta\phi}\delta\phi\right)}$$

full effective action including all fluctuations

microscopic action



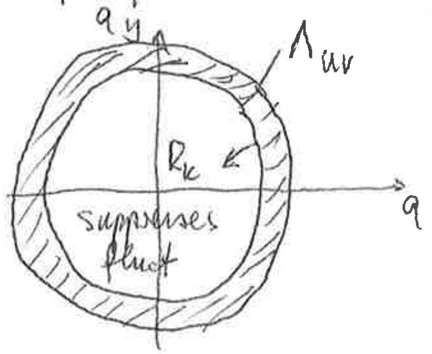
- alternative equivalent! functional <u>differential</u> representation

= Functional RG eq.

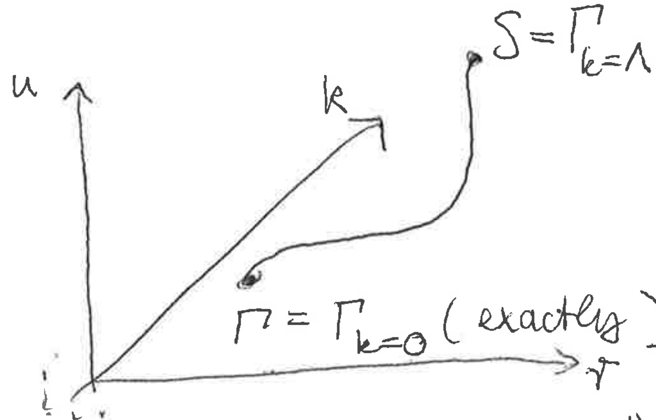
$$\partial_k \Gamma_k[\phi] = \frac{i}{2} \mathrm{Tr} \left(\Gamma_k^{(2)}[\phi] + R_k \right)^{-1} \partial_k R_k$$

change of Γ with scale — second variation — effective mass term: suppresses long wavelength modes but is removed as $k \to 0$

- picture : smooth interpolation between micro- and macrophysics



$\Longrightarrow$



eg. $S = \int (r|\phi|^2 + K|\nabla\phi|^2 + u|\phi|^4) = \Gamma_{k=\Lambda}$

$\Gamma_k = \qquad r_k \qquad K_k \qquad u_k$

→ Example 2 (blackboard)

→ Application : Driven criticality

quest for noneq. universality classes

$\hat{=}$ how many distinct scaling solutions can we find for Γ ?

2 setups :

1) "classical" (mimics class. (finite T) critical behavior)

$$\gamma_p - \gamma_e \to 0 \quad \Longrightarrow \quad P^R \propto q^2$$

$$\gamma_e + \gamma_p \to \text{const} \quad \Longrightarrow \quad P^K \propto q^0$$

$\Longrightarrow$ $[\phi_c] = \frac{d-2}{2}$, $[\phi_q] = \frac{d+2}{2}$

- massive diagram. simplif.
- semiclass limit (MSR funct. int)

2) "quantum" (mimics q (T=0) critical scaling)

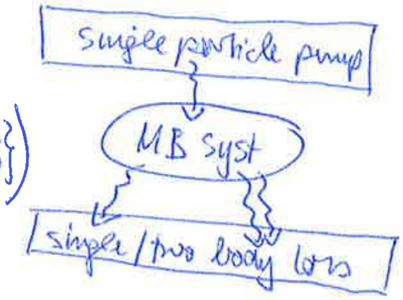
$$\gamma_p - \gamma_e \to 0 \quad \Longrightarrow \quad P^R \propto q^2$$

$$\gamma_e + \gamma_p \to 0 \quad \Longrightarrow \quad P^K \propto q^2$$

(finite scaling regime possible, cf. QCP)

$\Longrightarrow$ $[\phi_c] = [\phi_q] = \frac{d}{2}$ fully necessit. quantum dyn. field theory

Example 2 : nonlinear many-body problem ($d+1$ dimensions)



- master eq :

$$\partial_t \rho = -i[H,\rho] + \sum_i \int_{\vec{x}} \gamma_i \left(L_i(x)\rho L_i^{\dagger}(x) - \tfrac{1}{2}\{L_i^{\dagger}(x)L_i(x),\rho\} \right)$$

$\rightarrow$ $H = \int_x \left(\psi^{\dagger}(x)\left[-\frac{\nabla^2}{2m} - \mu\right]\psi(x) + \frac{g}{2}\psi^{\dagger}(x)^2\psi^2(x)\right)$

$\rightarrow$ Lindblad operators :

single particle loss : $L_1(x) = \psi(x)$; rate $\gamma_1 = \gamma_e$

– " – pump: $L_2(x) = \psi^{\dagger}(x)$; $\gamma_2 = \gamma_p$

two – " – loss : $L_3(x) = \psi^2(x)$; $\gamma_3 = \kappa$

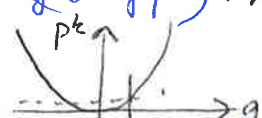
single particle diffusion $L_4(x) = \vec{\nabla}\psi(x)$; " $\gamma_4 = D$

- action :

$$S = \int_{t\,\vec{x}} (\psi_c^* \;\; \psi_q^*) \begin{pmatrix} 0 & P^A \\ P^R & P^K \end{pmatrix} \begin{pmatrix} \psi_c \\ \psi_q \end{pmatrix} - \left\{ \tfrac{1}{2}(g - i\kappa)\left(|\psi_c|^2\psi_c\psi_q^* + |\psi_q|^2\psi_c\psi_q^*\right) + \text{c.c.} + 4i\kappa|\psi_c|^2|\psi_q|^2 \right\}$$

$P^R = i\partial_t - \left(-\frac{\vec{\nabla}^2}{2m} + iD\vec{\nabla}^2 - \mu - i(\gamma_e - \gamma_p)\right)$; $P^K = i(\gamma_e + \gamma_p) + iD\vec{\nabla}^2$

$\xrightarrow{\text{Fourier}}$ $\omega - \left(\frac{\vec{q}^2}{2m} - iD\vec{q}^2 - \mu - i(\gamma_e - \gamma_p)\right)$ $\xrightarrow{\text{Fourier}}$ $i(\gamma_e + \gamma_p) + iD\vec{q}^2$

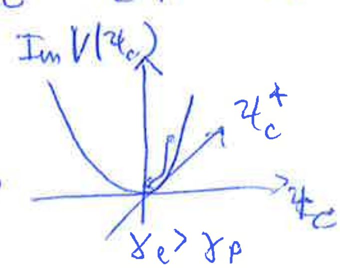


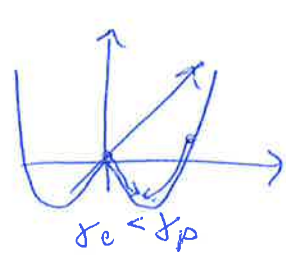
- orientation : EOM w/ approximations : • $\psi_q = 0$ (noiseless/deterministic)
Mean field theory • $\psi_c(t,\vec{x}) = \psi_{c,0}(t)$ (single mode) homog.

$\frac{\delta S}{\delta \psi_q^*} \stackrel{!}{=} 0 \rightarrow$

$\partial_t \psi_c = \left[i\mu + (\gamma_p - \gamma_e) - (ig + \kappa)|\psi_c|^2\right]\psi_c$ $\rightarrow$ overdamped motion in pot. landscape

$\psi_{c,0} = 0$

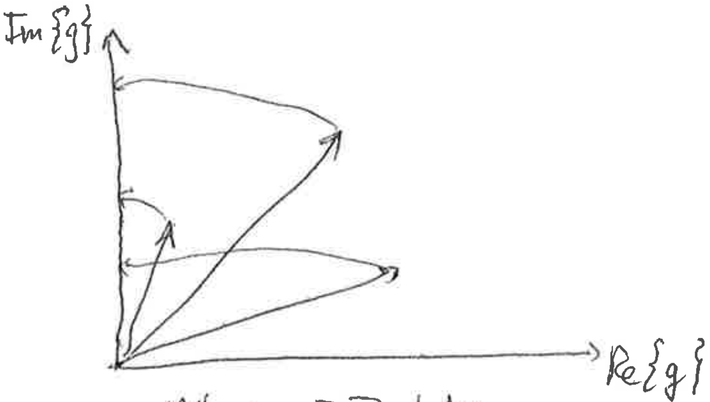




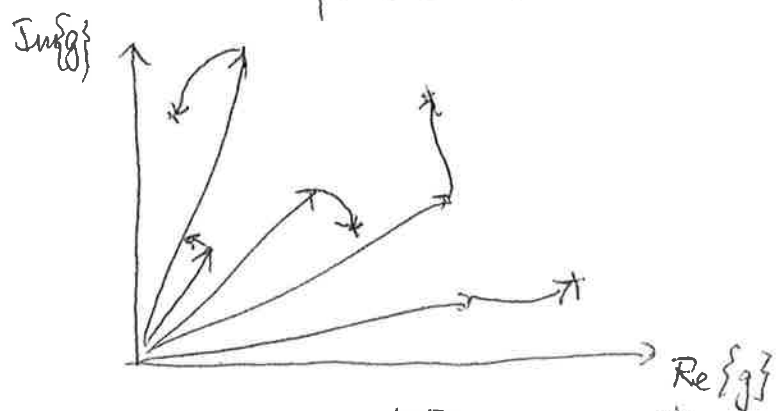
$\psi_{c0} \neq 0$

Summary of findings

classical



quantum



- equilibrium FP stable
- universal decoherence, new exp. η_c } exposes
- eq. vs. noneq. finestructure in η_c }
- asympt. ~~deco~~ thermalization of correlations

- new noneq. FP + universality class
- no decoherence
- no thermalization
- RG limit cycle in spectral dens.