Probing order beyond the Landau paradigm

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Different phases/Different universality classes

- Crystals
- Ferromagnets
- Superfluids
- Superconductors

Landau theory of ordered phases

Physical characterization	Symmetry breaking Long range order
Low energy effective theory	Ginzburg-Landau field theory
Physical picture	Particle condensation
Mathematical framework	Group theory: Orders , Groups G

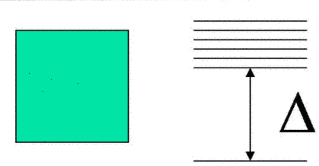
New phases of matter

- No order parameters
- No symmetry breaking
- Beyond the scope of Landau theory
- "Exotic phases"

Topological phases

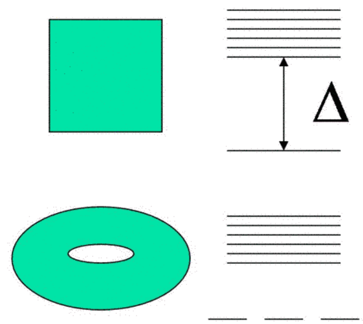
Topological phases

- Gapped



Topological phases

- Gapped
- Degenerate ground state on torus



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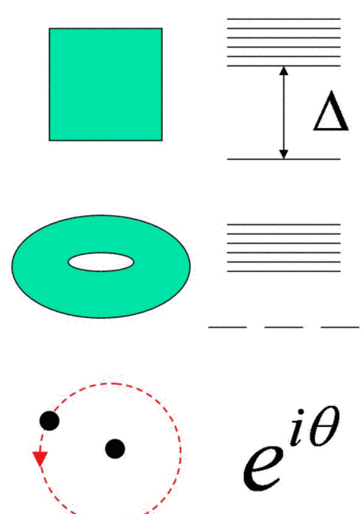
- Fractional statistics



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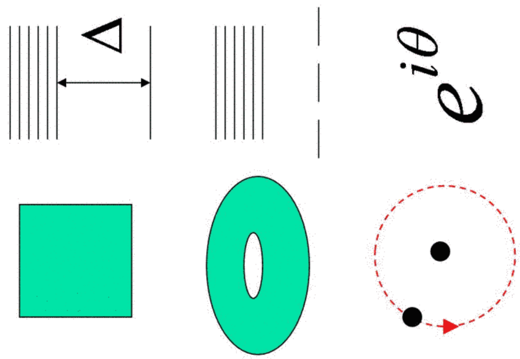
- "Topological order"



Real life examples

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- Hope: Frustrated magnets
  - Many theoretical models
  - A few candidate materials
    - Cs_2CuCl_4
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- Tip of the iceberg?

Theory of topological phases

Physical characterization	
Low energy effective theory	
Physical picture	
Mathematical framework	

Theory of topological phases

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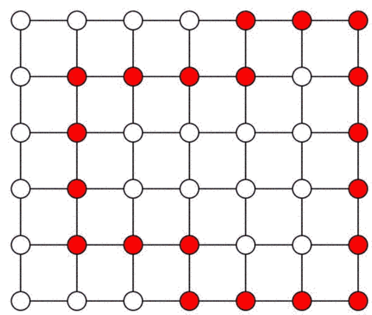

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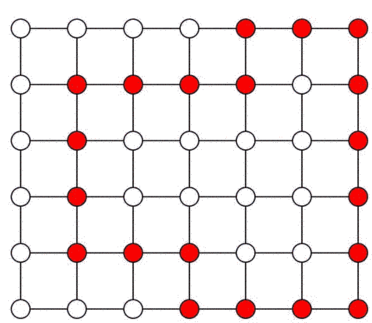

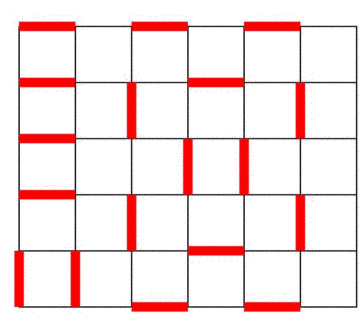

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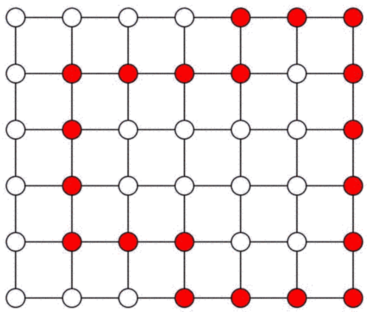

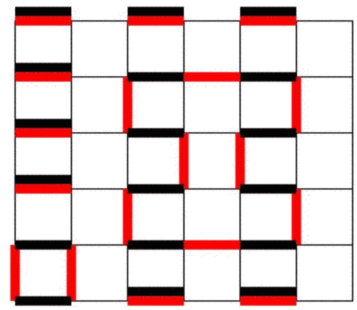

String condensation: two requirements

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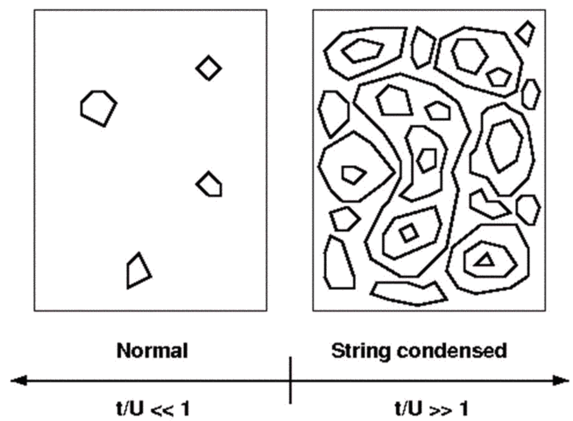
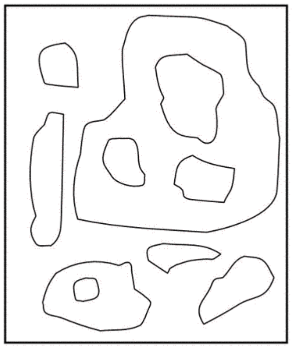


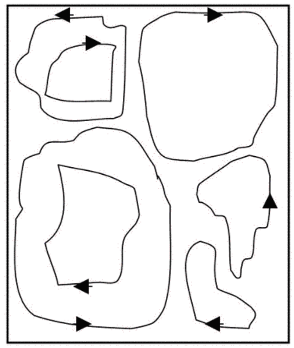
So what?

- String condensed phases ARE topological phases!
- Mechanism for topological phases
- Very general: all (parity invariant) topological phases can be realized

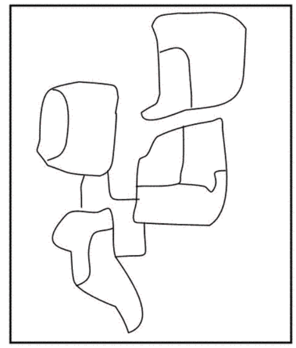
Examples



Z_2 gauge theory



U(1) gauge theory



$SU_3(2)$ £ $SU_3(2)$
Chern-Simons

Tensor categories

A set of tensors ($F^{ijm}_{kln}, d_i, \delta_{ijk}$) satisfying:

$$F^{ijk}_{j^*i^*0} = \sqrt{\frac{d_k}{d_i d_j}}\,\delta_{ijk}$$

$$F^{ijm}_{kln} = F^{lkm^*}_{jin} = F^{jim}_{lkn^*} = \sqrt{\frac{d_m d_n}{d_j d_l}}\,F^{imj}_{k^*nl}$$

$$\sum_{n=0}^{N} F^{mlq}_{kp^*n} F^{jip}_{mns^*} F^{js^*n}_{lkr^*} = F^{jip}_{q^*kr^*} F^{riq^*}_{mls^*}$$

$$(F^{ijm}_{kln})^* = F^{i^*j^*m^*}_{k^*l^*n^*}$$

Tensor categories classify different string-net condensates/topological phases

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Picture is still incomplete

- Normal order can be detected in a wave function
- But we don't know how to detect topological order in a wave function

Many wave functions

- Theoretical wave functions
  - Gutzwiller projected states: $\Psi_{spin} = P\,\Psi_{ferm}$
  - Quantum loop gases: $\Psi_d(X) = d^{N_X}$

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- Numerical wave functions
  - J_1-J_3 spin-1/2 Heisenberg model:

$$H = J_1 \sum_{n.n} S_i ¢ S_j + J_3 \sum_{3rd\ n.n} S_i ¢ S_j$$

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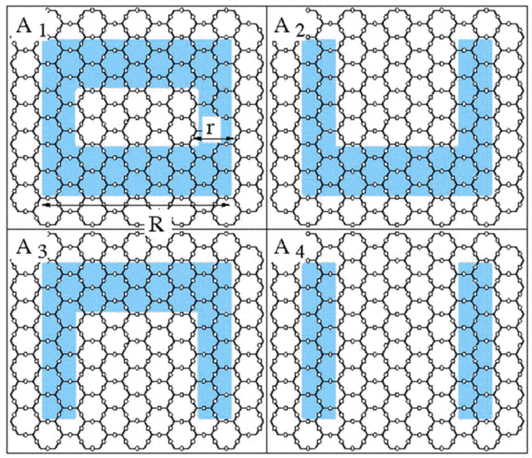
- How do we know if they're topologically ordered?

Topological entropy

Define: $-S_{top} = (S_1 - S_2) - (S_3 - S_4)$



Main Result

- Then: $S_{top}=0$ for normal states, $S_{top} \neq 0$ for topologically ordered states
- S_{top} is universal for each topological phase:

$$S_{top} = \log(D^2)$$

where $D = \sum_i d_i^2$ (e.g. for Z_2 theory, $D = 2$)

- cond-mat/0510613 (Kitaev/Preskill posted similar result 2 weeks earlier!)

Physical picture

- Why does $S_{top} = 0$ for normal phases, $S_{top} \neq 0$ for topological phases?

Physical picture

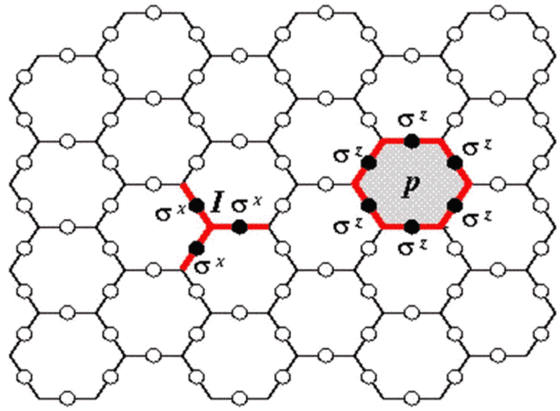
- Why does $S_{top} = 0$ for normal phases, $S_{top} \neq 0$ for topological phases?

Nonlocal entanglement!

- Topologically ordered states have nonlocal entanglement
- S_{top} measures nonlocal entanglement

Topologically ordered states have nonlocal entanglement

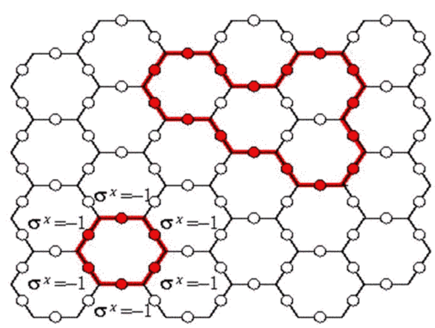
$$H = -V\sum_I \sigma^x_{I1}\sigma^x_{I2}\sigma^x_{I3} - t\sum_p \sigma^z_{p1}\sigma^z_{p2}\sigma^z_{p3}\sigma^z_{p4}\sigma^z_{p5}\sigma^z_{p6}$$



Z_2 topological order

Ground state wave function

- Use string picture:
  σ^x_i =-1 , string on link
  σ^x_i=+1 ,no string on link
- Ψ is uniform superposition of closed string configurations



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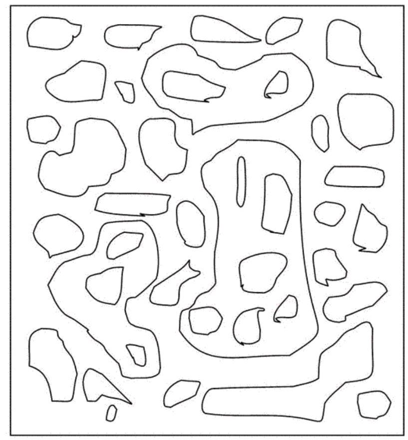

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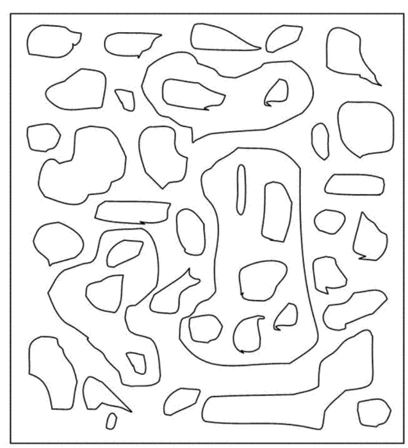

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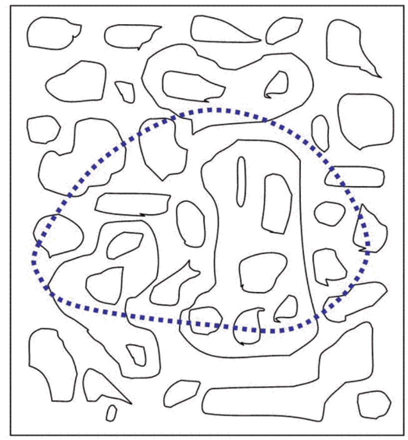
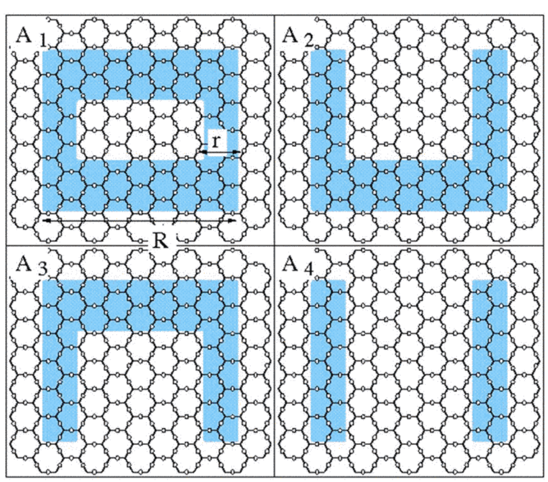
- Ψ is uniform superposition of closed string configurations
- All local correlations $\langle \sigma^x_i \sigma^x_j \rangle$ vanish
- There is a *nonlocal* correlation: $\langle \prod_{i \in C} \sigma^x_i \rangle = 1$



S_{top} measures nonlocal entanglement

$$-S_{top} = (S_1 - S_2) - (S_3 - S_4)$$

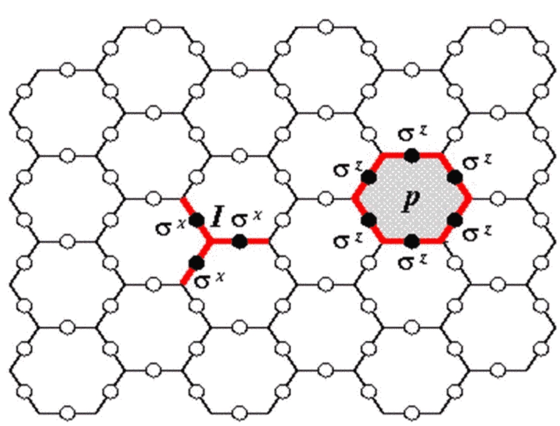


We have argued:

- $S_{top} = 0$ for normal phases
- $S_{top} \neq 0$ for topological phases
- S_{top} is universal
- But why does $S_{top} = \log(D^2)$?

Exactly soluble example

$$H = -V\sum_I \sigma^x_{I1}\sigma^x_{I2}\sigma^x_{I3} - t\sum_p \sigma^z_{p1}\sigma^z_{p2}\sigma^z_{p3}\sigma^z_{p4}\sigma^z_{p5}\sigma^z_{p6}$$



Ground state wave function

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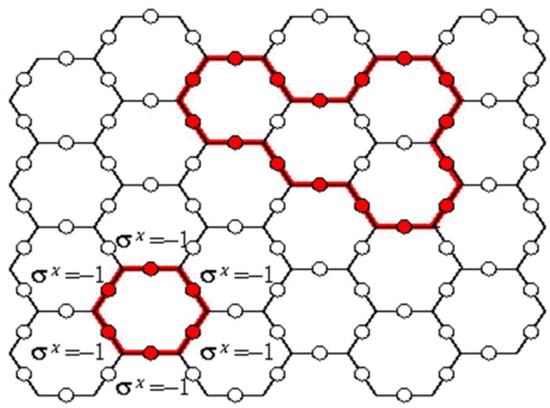
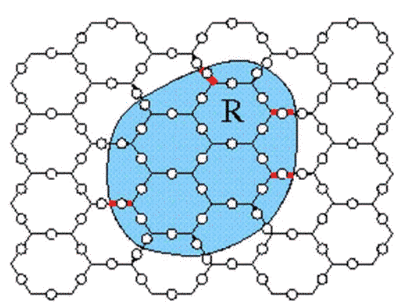


Entanglement entropy



- For any $q_1,\ldots,q_n = 0,1$, $\sum_m q_m$ even, define $\Psi^{in}_{q_1,\ldots q_n}$
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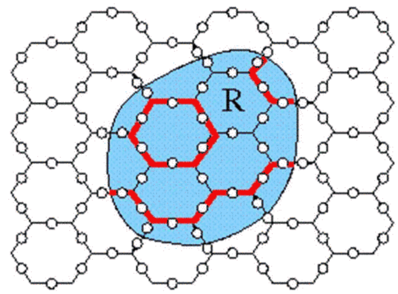

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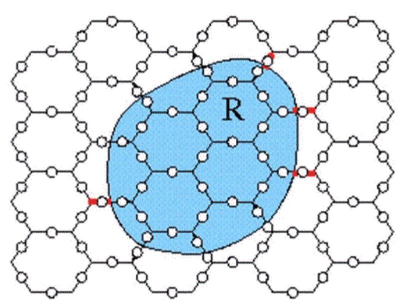

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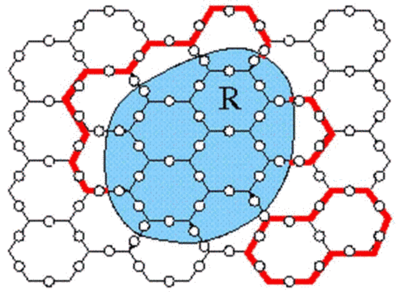

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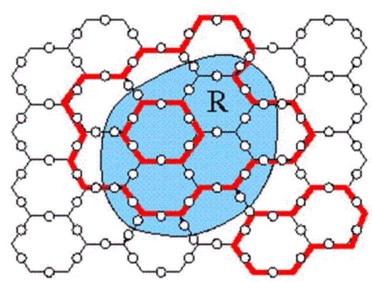
Entanglement entropy

Then: $\Psi = \sum_q \Psi^{in}_q \Psi^{out}_q$



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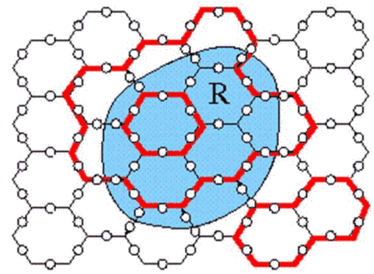

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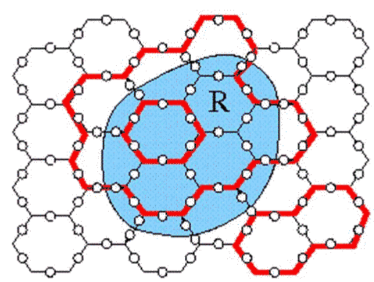

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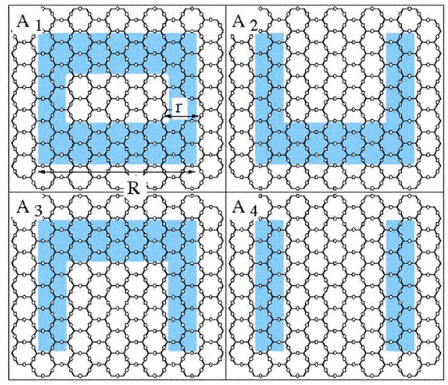
There are 2^{n-1} different Ψ^{in}_q) $S_R = (n-1) \log 2$

Topological entropy

- S_R = (n-k) log 2 where
  k = # boundary curves



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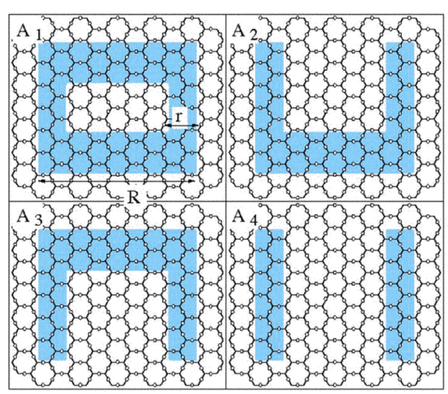

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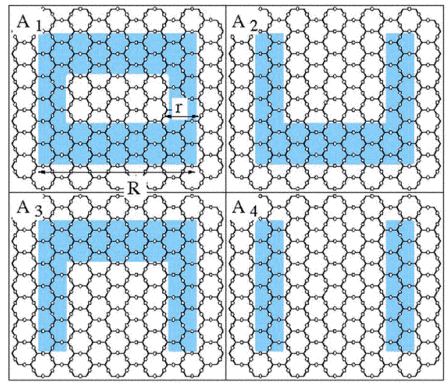

$$-S_{top} = (n_1-n_2-n_3+n_4-2) \log 2 = -2 \log 2 = -\log(2^2)$$

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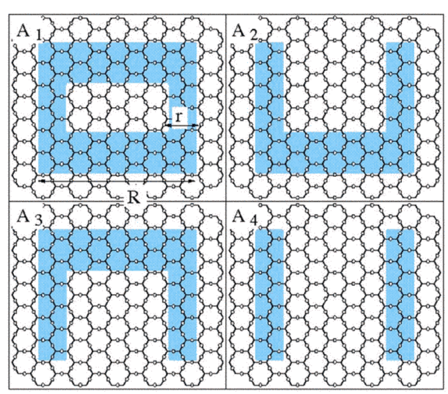

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Right result: D=2 for Z_2 topological order!

Conclusions/New directions

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- Missing something important:

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  Mean field theory!

Conclusions/New directions

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- Missing something important:

  Mean field theory!

- Can string picture provide a mean field theory?

String-net condensation

- String condensed phases ARE topological phases!
- Physical picture for topological phases
- Very general: applies to all (parity invariant) top. phases, if we generalize to string-nets…

A beautiful correspondence

Tensor categories

$$(F^{ijm}_{kln}, d_i, \delta_{ijk})$$

⟷

String-net condensates/ Topological phases

Mathematical framework

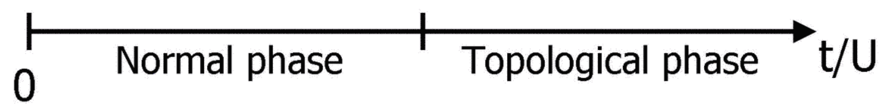
- How can we classify topological phases/string-net condensates?
- Analogous to classification of ordered phases, e.g. crystals
- What is the analogue of symmetry group?
- Tensor category!

String picture

- $\sigma^x = -1$, string
  $\sigma^x = 1$, no string

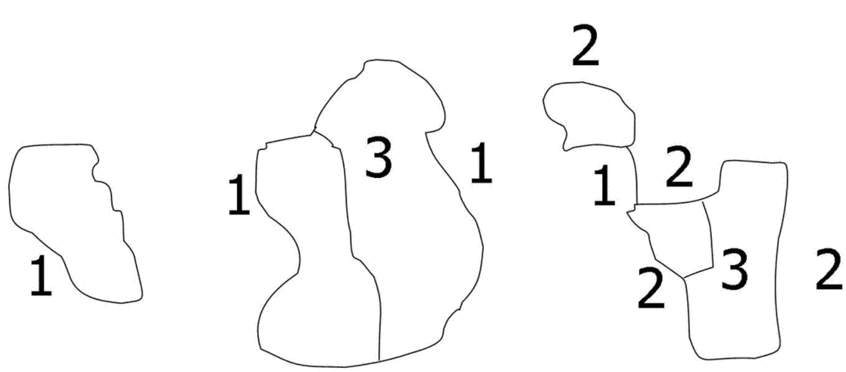
$$H = \underbrace{-V\sum_I \sigma^x_{I1}\sigma^x_{I2}\sigma^x_{I3}}_{\text{constraint}} \underbrace{-U\sum_i \sigma^x_i}_{\text{string tension}} \underbrace{-t\sum_p \sigma^z_{p1}\sigma^z_{p2}\sigma^z_{p3}\sigma^z_{p4}\sigma^z_{p5}\sigma^z_{p6}}_{\text{string kinetic energy}}$$

String picture



String-net models

Low energy Hilbert space:

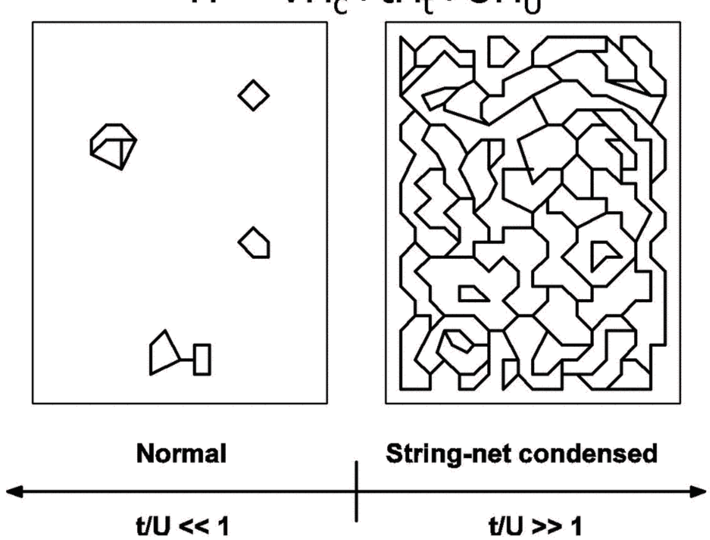


String-net Hamiltonian

$$H = \underbrace{V H_c}_{\text{constraint}} + \underbrace{t H_t}_{\text{String kinetic energy}} + \underbrace{U H_U}_{\text{String tension}}$$

Two phases

$$H = VH_c + tH_t + UH_U$$



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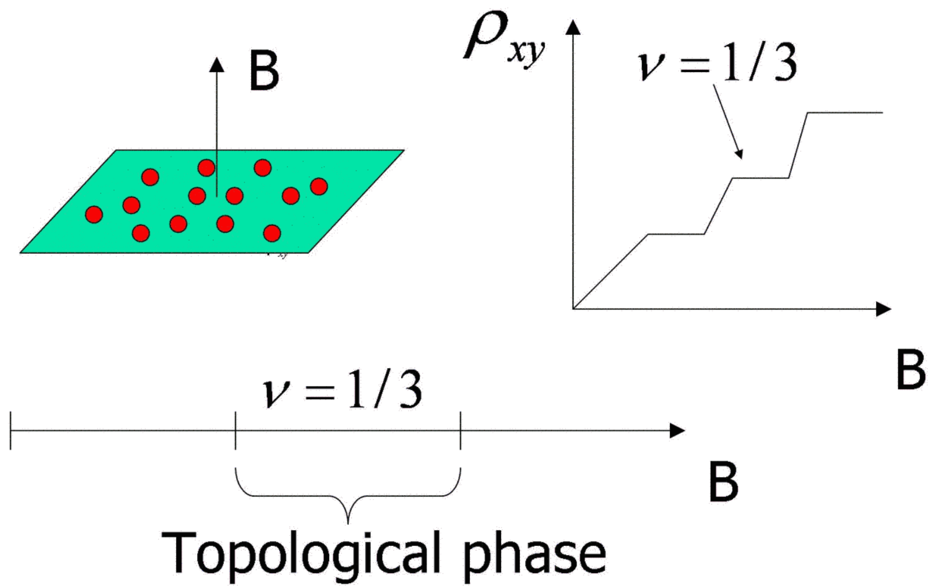
Example: FQH states



Breakdown of Landau theory

- Fractional quantum hall effect (1983)
- High-Tc superconductors (?) (1987)
- Frustrated magnets

$$-\kappa-(BEDT-TTF)_2Cu_2(CN)_3$$

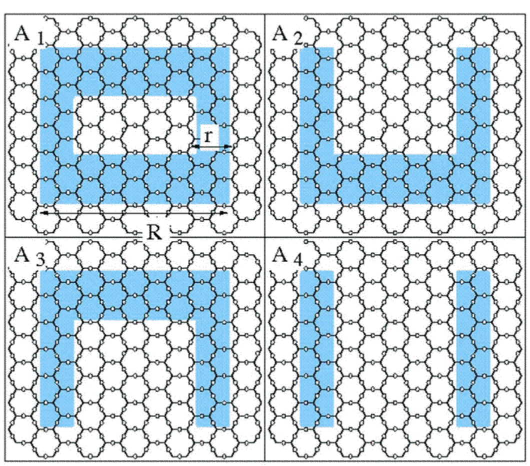
Landau theory

	Normal order	Topological order
Physical characterization	Symmetry breaking Long range order	Degenerate gd. states Anyonic quasiparticles
Low energy effective theory	Ginzburg-Landau field theory	Topological quantum field theory
Physical picture	Particle condensation	String-net Condensation
Mathematical framework	Group theory	Tensor category theory

Main Result

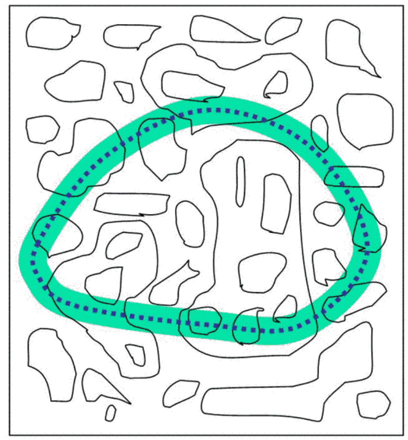
$$(S_1 - S_2) - (S_3 - S_4) = -\mathrm{Log}(D^2)$$

"Topological entropy" $-S_{top}$



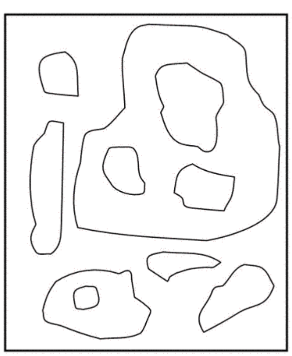
Ground state wave function

- Use string picture:
  σ^x_i =-1 , string on link
  σ^x_i=+1 ,no string on link
- Ψ is uniform superposition of closed string configurations
- All local correlations h$\sigma^x_i \sigma^x_j$i vanish
- There is a *nonlocal* correlation: h $\prod_{i \, 2 \, C} \sigma^x_i$ i =1

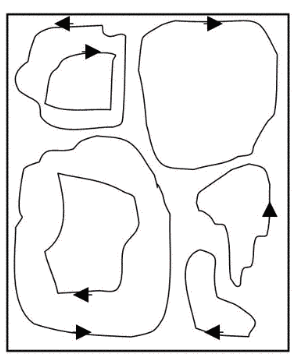


Examples

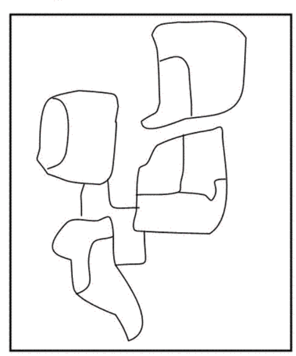
Non-Abelian anyons!
Universal quantum
computer!



Z_2 gauge theory



U(1) gauge theory



$SU_3(2)$ £ $SU_3(2)$
Chern-Simons