Topological Order: Patterns of Long Range Entanglements of Gapped Quantum States

Xiao-Gang Wen, MIT

KITP; Dec., 2010

arXiv:1004.3835,
arXiv:1008.3745,
arXiv:1010.1517

Xie Chen Z.-C. Gu ZH Wang

New orders beyond symmetry breaking?

- From the chiral spin states and FQH states, we learned that states with the same symmetry can belong to different phases.

New orders beyond symmetry breaking?

- From the chiral spin states and FQH states, we learned that states with the same symmetry can belong to different phases.

- The order in FQH states is a kind of order that
  - cannot be described by symmetry breaking
  - cannot be described by long range correlations
  - cannot be described by local order parameter

New orders beyond symmetry breaking?

- From the chiral spin states and FQH states, we learned that states with the same symmetry can belong to different phases.

- The order in FQH states is a kind of order that
  - cannot be described by symmetry breaking
  - cannot be described by long range correlations
  - cannot be described by local order parameter
- Hard to publish papers by describing what the new order **is not**.

New orders beyond symmetry breaking?

- From the chiral spin states and FQH states, we learned that states with the same symmetry can belong to different phases.

- The order in FQH states is a kind of order that
  - cannot be described by symmetry breaking
  - cannot be described by long range correlations
  - cannot be described by local order parameter
- Hard to publish papers by describing what the new order **is not**.

But how to describe the new order in terms what it is?

The early attempts to characterize the new order

- A gapped state can have a non trivial low energy effective theory even below the gap!

The early attempts to characterize the new order

- A gapped state can have a non trivial low energy effective theory even below the gap!
  Topology-dependent and topologically robust ground state degeneracy can (partially) describe the new order Wen 89; Wen & Niu 90

The early attempts to characterize the new order

- A gapped state can have a non trivial low energy effective theory even below the gap!
  Topology-dependent and topologically robust ground state degeneracy can (partially) describe the new order Wen 89; Wen & Niu 90
  Low energy effective theory is an almost trivial but highly non-trivial theory = **topological field theory** Witten 89
  → motivate us to name the new order as topological order

The early attempts to characterize the new order

- A gapped state can have a non trivial low energy effective theory even below the gap!
  Topology-dependent and topologically robust ground state degeneracy can (partially) describe the new order Wen 89; Wen & Niu 90
  Low energy effective theory is an almost trivial but highly non-trivial theory = **topological field theory** Witten 89
  → motivate us to name the new order as topological order
  Topologically stable ground states can be used as fault tolerant quantum memory. Kitaev 97; Dennis & Kitaev & Landahl & Preskill 02

The early attempts to characterize the new order

- A gapped state can have a non trivial low energy effective theory even below the gap!
  Topology-dependent and topologically robust ground state degeneracy can (partially) describe the new order Wen 89; Wen & Niu 90
  Low energy effective theory is an almost trivial but highly non-trivial theory = **topological field theory** Witten 89
  → motivate us to name the new order as topological order
  Topologically stable ground states can be used as fault tolerant quantum memory. Kitaev 97; Dennis & Kitaev & Landahl & Preskill 02
- Topologically robust non-Abelian Berry's phases of the degenerate ground states from deforming the torus → representation S, T of modular group which can completely (?) describe the topological order. Wen 89

- Topologically robust degeneracy even exists on sphere if we have quasiparticles Wen 91, Moore & Read 91, Nayak & Wilczek

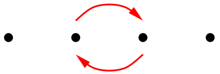
  Topologically robust Non-Abelian Berry's phases from exchanging defects →

  representation of Braid group Wu, 85 → non-Abelian statistics Goldin & Menikoff & Sharp 85

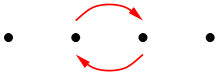
  Can be realized in FQH states Moore & Read 91, Wen, 91

  and lead to topological quantum computation.

  Kitaev 97, Preskill 97, Freedman 00

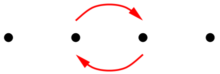


- Topologically robust degeneracy even exists on sphere if we have quasiparticles Wen 91, Moore & Read 91, Nayak & Wilczek
  Topologically robust Non-Abelian Berry's phases from exchanging defects →
  representation of Braid group Wu, 85 → non-Abelian statistics Goldin & Menikoff & Sharp 85
  Can be realized in FQH states Moore & Read 91, Wen, 91
  and lead to topological quantum computation. Kitaev 97, Preskill 97, Freedman 00



- Topologically protected gapless boundary excitations:
  2D bulk → 1D boundary CFT Halperin 82, Wen 90
  4D bulk → 3D boundary chiral fermions (topo. insulator in 4D) Kaplan 92
  Structure of gapless boundary excitation **fully** characterize the structure of bulk topological order Wen 90

- Topologically robust degeneracy even exists on sphere if we have quasiparticles Wen 91, Moore & Read 91, Nayak & Wilczek
  Topologically robust Non-Abelian Berry's phases from exchanging defects $\rightarrow$
  representation of Braid group Wu, 85 $\rightarrow$ non-Abelian statistics Goldin & Menikoff & Sharp 85
  Can be realized in FQH states Moore & Read 91, Wen, 91
  and lead to topological quantum computation.
  Kitaev 97, Preskill 97, Freedman 00



- Topologically protected gapless boundary excitations:
  2D bulk $\rightarrow$ 1D boundary CFT Halperin 82, Wen 90
  4D bulk $\rightarrow$ 3D boundary chiral fermions (topo. insulator in 4D) Kaplan 92
  Structure of gapless boundary excitation **fully** characterize the structure of bulk topological order Wen 90
  The edge-bulk correspondence of topological order can be viewed as the holographic principle in quantum gravity discovered a few years later. Thorn 91, t'Hooft 93, Susskind 94 .
  Is quantum gravity topological?

A modern view of topological order?

- For gapped systems, entanglement entropy has universal constant term: $S_A = \gamma \text{Area} - \gamma_{top}$, topological entanglement entropy, Kitaev & Preskill 06, Levin & Wen 06
  and universal spectrum. Li & Haldane 08
  (Can be probed by quantum noise Klich & Levitov 08)
  Topological order → long range patterns of quantum entanglements. Wen 04

What really is long range of quantum entanglements?
What really is topological order?

What are quantum phases?

What are quantum phases?

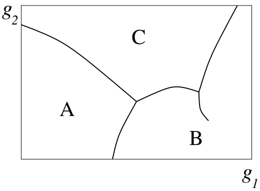
- Phases are defined through phase transitions.

What are quantum phases?

- Phases are defined through phase transitions.

  What are phase transitions?

  As we change a parameter g in Hamiltonian $H(g)$, the ground state energy density $\epsilon_g = E_g/V$ or average of some other local operators $\langle \hat{O} \rangle$ may have a singularity at g_c $\to$ the system has a phase transition at g_c.

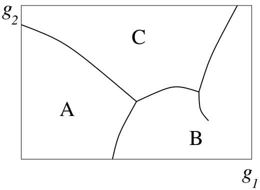


What are quantum phases?

- Phases are defined through phase transitions.

  What are phase transitions?

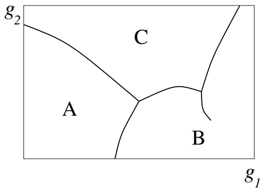
  As we change a parameter g in Hamiltonian $H(g)$, the ground state energy density $\epsilon_g = E_g/V$ or average of some other local operators $\langle \hat{O} \rangle$ may have a singularity at g_c $\rightarrow$ the system has a phase transition at g_c.



The Hamiltonian $H(g)$ is a smooth function of g. How can the ground state energy density ϵ_g be singular at a certain g_c?

What are quantum phases?

- Phases are defined through phase transitions.
  What are phase transitions?
  As we change a parameter g in Hamiltonian $H(g)$, the ground state energy density $\epsilon_g = E_g/V$ or average of some other local operators $\langle \hat{O} \rangle$ may have a singularity at $g_c \to$ the system has a phase transition at g_c.



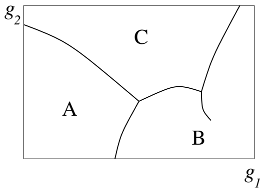
  The Hamiltonian $H(g)$ is a smooth function of g. How can the ground state energy density ϵ_g be singular at a certain g_c?
- Spontaneous symmetry breaking is a mechanism to cause a singularity in ground state energy density ϵ_g.
  $\to$ Spontaneous symmetry breaking causes phase transition.

What are quantum phases?

- Phases are defined through phase transitions.

  What are phase transitions?

  As we change a parameter g in Hamiltonian $H(g)$, the ground state energy density $\epsilon_g = E_g/V$ or average of some other local operators $\langle \hat{O} \rangle$ may have a singularity at g_c $\to$ the system has a phase transition at g_c.



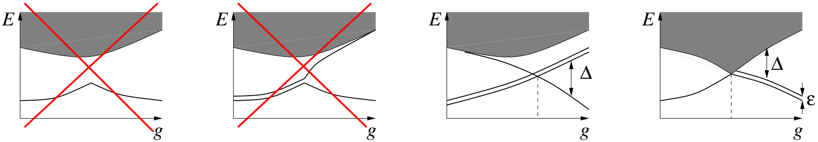
  The Hamiltonian $H(g)$ is a smooth function of g. How can the ground state energy density ϵ_g be singular at a certain g_c?

- Spontaneous symmetry breaking is a mechanism to cause a singularity in ground state energy density ϵ_g.

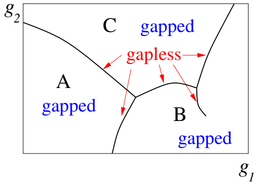
  $\to$ Spontaneous symmetry breaking causes phase transition.

  But symmetry breaking does not describe all the phases.

Mathematical definition of gapped quantum phases



A more general mechanism to cause singularity of ϵ_g for gapped states: gap closing.



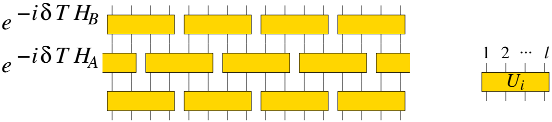
- A precise definition of gapped quantum phases:
  Two gapped states, $|\Psi(0)\rangle$ and $|\Psi(1)\rangle$, are in the same phase iff they are related through a local unitary (LU) evolution

$$|\Psi(1)\rangle = P\left(e^{-\text{i}\int_0^1 dg' \ \tilde{H}(g')}\right)|\Psi(0)\rangle$$

where $\tilde{H}(g) = \sum_{\mathbf{i}} O_{\mathbf{i}}(g)$ and $O_{\mathbf{i}}(g)$ are local hermitian operators.

We can rewrite the LU evolution as

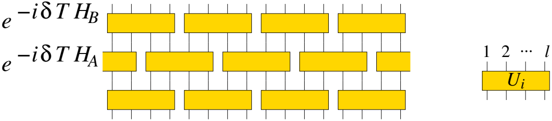
$$|\Psi(1)\rangle = P\left(e^{-\mathrm{i}\,T\int_0^1 dg\ H(g)}\right)|\Psi(0)\rangle$$
$$= (\text{local unitary transformation})|\Psi(0)\rangle$$
$$= (\text{quantum circuit of finite depth})|\Psi(0)\rangle$$



LU evolution and quantum circuit of finite depth

We can rewrite the LU evolution as

$$
\begin{aligned}
|\Psi(1)\rangle &= P\left(e^{-\mathrm{i}\,T\int_0^1 dg\ H(g)}\right)|\Psi(0)\rangle \\
&= (\text{local unitary transformation})|\Psi(0)\rangle \\
&= (\text{quantum circuit of finite depth})|\Psi(0)\rangle
\end{aligned}
$$



- The local unitary transformations define an equivalence relation
  A universality class of a quantum phase is an equivalent class of the LU transformations

Hastings, Wen 05; Bravyi, Hastings, Michalakis 10

Topological order is a pattern of long range entanglement

Two kinds of states if no symmetries:

- The states that are equivalent to product state under LU transformations. All those states belong to the same class (phase) $\rightarrow$ short-range entanglement and trivial topological order.
- The states that are not equivalent to direct-product states. Those states form many different equivalent classes (phases) $\rightarrow$ many patterns of long-range entanglements and many different topological orders.
- In absence of symmetry:

  Quantum phases of matter
  = patterns of long-range entanglement = topological orders
  = equivalence classes of the LU transformations

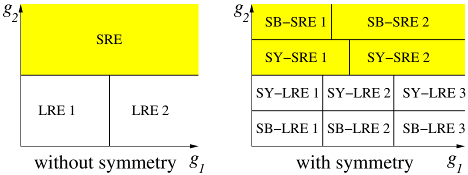
  Examples: FQH states

Symm. breaking orders and symm. protected topo. orders

- If the Hamiltonian H has some symmetries, its phases will correspond to equivalent classes of symmetric LU transformations: $|\Psi\rangle \sim P\left(e^{-\mathrm{i}\int_0^1 dg\ \tilde{H}(g)}\right)|\Psi\rangle$ where $\tilde{H}(g)$ has the same symmetries as H.

Symm. breaking orders and symm. protected topo. orders

- If the Hamiltonian H has some symmetries, its phases will correspond to equivalent classes of symmetric LU transformations: $|\Psi\rangle \sim P\left(e^{-\mathrm{i}\int_0^1 dg\ \tilde H(g)}\right)|\Psi\rangle$ where $\tilde H(g)$ has the same symmetries as H.



- SRE states with different symmetries
  → Landau's symmetry breaking orders.
- SRE states with the same symmetry can belong to different classes
  → symmetry protected topological orders (symmetry protected trivial orders). Gu & Wen 09, Pollmann & Berg, Turner & Oshikawa 09
  Examples: Haldane phase and $S_z = 0$ phase of spin-1 XXZ chain. Band and topological insulators

Topological order = pattern of long range entanglement
= equivalent class of LU transformations

How to label those equivalent classes?

Labeling and classifying topological orders

Topological order = pattern of long range entanglement
= equivalent class of LU transformations

How to label those equivalent classes?

- We can use the wave function Φ to label the topological orders.

Labeling and classifying topological orders

Topological order = pattern of long range entanglement = equivalent class of LU transformations

How to label those equivalent classes?

- We can use the wave function Φ to label the topological orders. But this is a many-many to one labeling scheme.

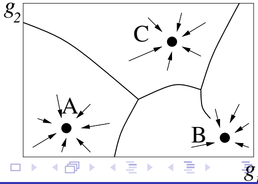
Labeling and classifying topological orders

Topological order = pattern of long range entanglement = equivalent class of LU transformations

How to label those equivalent classes?

- We can use the wave function Φ to label the topological orders. But this is a many-many to one labeling scheme.

  Under the wave function renormalization generated by the LU transformation, Verstraete, Cirac, Latorre, Rico, Wolf 05; Vidal 07; Jordan, Orus, Vidal, Verstraete, Cirac 08; Jiang, Weng, Xiang 09; Gu, Levin, Wen 09 the wave function flows to simpler one within the same equivalent class.
- Use the fixed-point wave function: Φ_{fix} to label topological order. Φ_{fix} may give us a one-to-one labeling of topological order, and a classification of topological order.



Classify 2D topological order

The non-chiral 2D topological orders are classified by the data $N_{ijk}, F^{ijm,\alpha\beta}_{kln,\chi\delta}, P^{kj,\alpha\beta}_{i}, A^i$, that satisfy Levin & Wen 05; Chen & Gu & Wen 10

- $\sum_m N_{jim^*} N_{kml^*} = \sum_n N_{kjn^*} N_{l^* ni},$
- $\sum_{t,\eta,\varphi,\kappa} F^{ijm,\alpha\beta}_{knt,\eta\varphi} F^{itn,\varphi\chi}_{lps,\kappa\gamma} F^{jkt,\eta\kappa}_{lsq,\delta\phi} = \sum_{\epsilon} F^{mkn,\beta\chi}_{lpq,\delta\epsilon} F^{ijm,\alpha\epsilon}_{qps,\phi\gamma},$
- $\sum_{\alpha=1}^{N_{kii^*}} \sum_{\beta=1}^{N_{j^*jk^*}} P^{kj,\alpha\beta}_{i} (P^{kj,\alpha\beta}_{i})^* = 1,$
- $P^{kj,\alpha\beta}_{i} = \sum_{m,\lambda,\gamma,l,\nu,\mu} F^{jj^*k,\beta\alpha}_{i^*i^*m^*,\lambda\gamma} F^{i^*mj^*,\lambda\gamma}_{m^*i^*l,\nu\mu} P^{lm,\mu\nu}_{i^*},$
- $P^{jp,\alpha\eta}_{i} \delta_{im}\delta_{\beta\delta} = \sum_{\chi} F^{ijm,\alpha\beta}_{klk,\chi\delta} P^{jp,\chi\eta}_{k^*}$ for all k, i, l with $N_{kil^*} > 0$.

......

The non-chiral 2D **fermionic** topological orders are (partially?) classified by the data $N_{ijk}, N^f_{ijk}, F^{ijm,\alpha\beta,\pm}_{kln,\gamma\lambda,\pm}, O^{jk,\alpha\beta}_{i,\pm}, A^i$ that satisfy

Gu & Wang & Wen 10

- $$\sum_{m=0}^{N} N_{jim^*} N_{kml^*} = \sum_{n=0}^{N} N_{kjn^*} N_{l^*ni},$$
- $$\sum_{m=0}^{N} (N^b_{jim^*} N^f_{kml^*} + N^f_{jim^*} N^b_{kml^*}) = \sum_{n=0}^{N} (N^b_{kjn^*} N^f_{l^*ni} + N^f_{kjn^*} N^b_{l^*ni}),$$
- $$\sum_{t} \sum_{\eta=1}^{N_{kjt^*}} \sum_{\varphi=1}^{N_{tin^*}} \sum_{\kappa=1}^{N_{lts^*}} F^{ijm,\alpha\beta,+}_{knt,\eta\varphi,-} F^{itn,\varphi\chi,+}_{lps,\kappa\gamma,-} F^{jkt,\eta\kappa,+}_{lsq,\delta\phi,-}$$
$$= (-)^{s_{jim^*}(\alpha) s_{lkq^*}(\delta)} \sum_{\epsilon=1}^{N_{qmp^*}} F^{mkn,\beta\chi,+}_{lpq,\delta\epsilon,-} F^{ijm,\alpha\epsilon,+}_{qps,\phi\gamma,-}$$

......

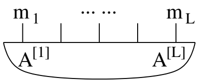
- Those are tensor category theory and super tensor category theory.

Application to 1D: no 1D topological order

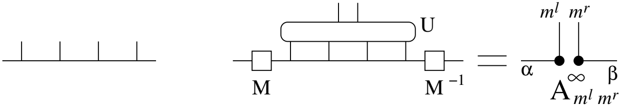
- What are the phases for gapped 1D systems without any symm.?
- What are the phases for short-range correlated (SRC) states without any symmetry? Hastings 04; Hastings, Koma 06
  SRC states: ANY local operator has short range correlation.
- A SRC state can always be represented as a MPS:

Schuch, Wolf, Verstraete, Cirac 08

$$\Psi(m_1,...,m_L) = \mathrm{Tr} A^{[1]}_{m_1} ... A^{[L]}_{m_L}$$



- A sequence of n matrix product can be simplified through the LU transformations if n is large:

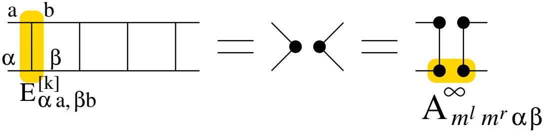


$$A^{\infty}_{m^l\, m^r\, \alpha\beta}$$

Verstraete, Cirac, Latorre, Rico, Wolf, 2005

- Introduce double-tensor $E^{[i]}_{\alpha a,\beta b} = \sum_m A^{[i]}_{m,\alpha\beta}(A^{[i]}_{m,ab})^*$

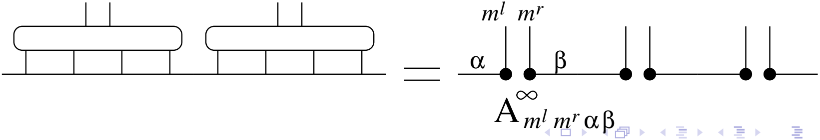
  If $\sum_m A^{[i]}_{m,\alpha\beta}(A^{[i]}_{m,ab})^* = \sum_m B^{[i]}_{m,\alpha\beta}(B^{[i]}_{m,ab})^* \to A^{[i]}_m = \sum_{m'} U_{mm'} B^{[i]}_{m'}$
- One largest eigenvalue dominates:



$$(\prod_k E^{[k]})_{\alpha a,\beta b} = V^{[k]}_{\alpha a} W^{[k]}_{\beta b}$$

- Since $E^{[k]}$ is a completely positive map, one finds, up to a gauge transformation, $V^{[k]}_{\alpha a} = \lambda^{[k]}_{\alpha}\delta_{\alpha a}$, $W^{[k]}_{\alpha a} = \lambda^{[k+1]}_{\beta}\delta_{\beta b}$ and $\lambda_\alpha > 0$.

  So $A_{m^l m^r,\alpha\beta} = \sqrt{\lambda^{[k]}_{\alpha}}\delta_{\alpha m^l}\sqrt{\lambda^{[k+1]}_{\beta}}\delta_{\beta m^r}$
- The fixed point wave function is a product state.



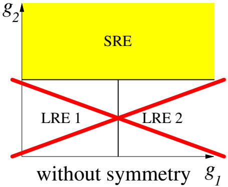
No topological order in 1D, if there are no symmetries

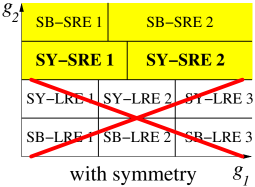
- All product state are linked by LU transformations.
- All SRC MPS are linked by LU transformations.
- All SRC MPS belong to the same quantum phase, if there are no symmetries.

No topological order in 1D, if there are no symmetries

- All product state are linked by LU transformations.
- All SRC MPS are linked by LU transformations.
- All SRC MPS belong to the same quantum phase, if there are no symmetries.

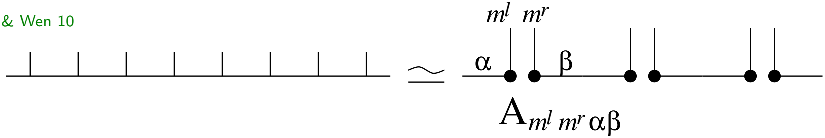
- But for systems with certain symmetries, we can only use the symmetric LU transformations to define states in the same phase.
- In this case symmetric LU transformations cannot links all SRC MPS. SCR MPS can belong to different phases.
  Symmetry protected topological orders





Quantum phases with translation and on-site symmetry

- A translation invariant (TI) SRC state can always be represented as an uniform MPS. Perez-Garcia, Wolf, Sanz, Verstraete, Cirac 08
- A SRC uniform MPS can always be deformed into a “dimer MPS” within the space of SRC uniform MPS. Schuch & Perez-Garcia & Cirac 10; Chen & Gu & Wen 10



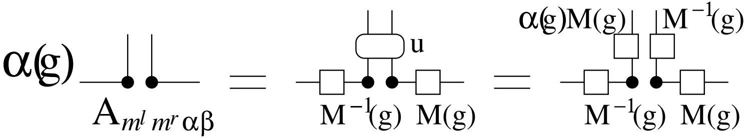
- If the original MPS has a on-site symmetry: $u(g),\ g \in G$,

$$\alpha(g) A_{m^l m^r} = \sum_{k^l k^r} u_{m^l m^r, k^l k^r}(g) M^{-1}(g) A_{k^l k^r} M(g)$$

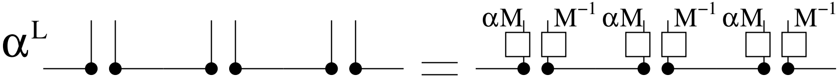
where $u(g)$ is a representation of G,
$\alpha(g)$ is an 1D representation of G,
$M(g)$ is a *projective* representation of G.

- Different quantum phases are classified by the pair $[M(g), \alpha(g)]$, the different *projective* rep. and different 1D rep.

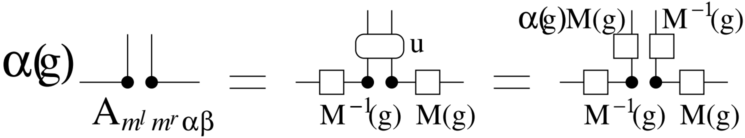
One can show that the representation u always factorize
$u \sim \alpha(g)M(g) \otimes M^{-1}(g)$



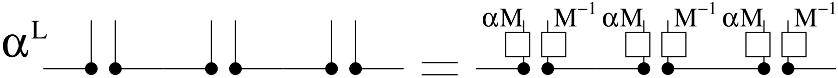
- So the fixed-point state transform as



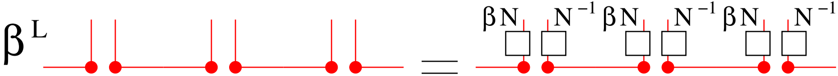
One can show that the representation u always factorize
$u \sim \alpha(g)M(g) \otimes M^{-1}(g)$



- So the fixed-point state transform as



- Consider another fixed-point state that transforms differently



Such a state can be linked to the first fixed-point state via a symmetric LU transformation, iff $M(g)$ and $N(g)$ are the same types of projective rep. and $\alpha = \beta$.

When $\alpha = \beta = 1$, both states are product of "singlet" dimers.
How can the two states belong to two different phases?

Projective representation

The total phase is unphysical $\to$ projective representation

- Matrices $u(g)$ form a projective representation of group G if

$$u(g_1)u(g_2) = \omega(g_1, g_2)u(g_1g_2), \qquad g_1, g_2 \in G.$$

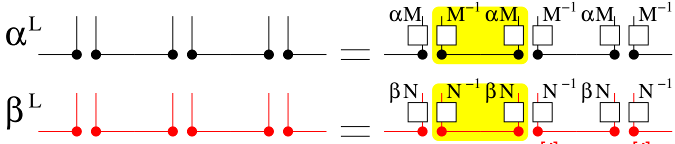
- $[u(g_1)u(g_2)]u(g_3) = u(g_1)[u(g_2)u(g_3)]$ gives rise to the condition

$$\omega(g_2, g_3)\omega(g_1, g_2g_3) = \omega(g_1, g_2)\omega(g_1g_2, g_3).$$

- Adding a phase factor $u'(g) = \beta(g)u(g)$ will lead to a different factor system $\omega'(g_1, g_2) = \frac{\beta(g_1g_2)}{\beta(g_1)\beta(g_2)}\omega(g_1, g_2)$.
  We regard $\omega'(g_1, g_2) \sim \omega(g_1, g_2)$.
  Equivalent classes of the factor systems $\omega(g_1, g_2)$ = $H^2(G, \mathbb{C})$ types of projective representations.
- $u_1(g) \to \omega_1 \in H^2(G, \mathbb{C})$, $u_2(g) \to \omega_2 \in H^2(G, \mathbb{C})$, then $u_1(g) \otimes u_2(g) \to \omega_1 + \omega_2$. $\to$ $H^2(G, \mathbb{C})$ is an Abelian group
- Half-integer spins = projective representation of $SO(3)$
  Integer spins = linear representation of $SO(3)$.
  $\to H^2[SO(3), \mathbb{C}] = \mathbb{Z}_2$

Projective representation and symm. LU trans.

Try to link the following two states via symm. LU trans.



- Expand the on-site space of the first state from $V^{[i]}_{\alpha M} \otimes V^{[i]}_{M^{-1}}$ to

$$(V^{[i]}_{\alpha M} + V^{[i]}_{\beta N}) \otimes (V^{[i]}_{M^{-1}} + V^{[i]}_{N^{-1}})$$
$$= V^{[i]}_{\alpha M} \otimes V^{[i]}_{M^{-1}} + V^{[i]}_{\alpha M} \otimes V^{[i]}_{N^{-1}} + V^{[i]}_{\beta N} \otimes V^{[i]}_{M^{-1}} + V^{[i]}_{\beta N} \otimes V^{[i]}_{N^{-1}}$$

- When $\alpha = \beta$, try to rotation the dimer using symm. LU trans.:
$|\psi^{[i]}_{M^{-1}}\rangle|\psi^{[i+1]}_{\alpha M}\rangle \in V^{[i]}_{M^{-1}} \otimes V^{[i+1]}_{\alpha M} \to |\psi^{[i]}_{N^{-1}}\rangle|\psi^{[i+1]}_{\beta N}\rangle \in V^{[i]}_{N^{-1}} \otimes V^{[i+1]}_{\beta N}$
- During the rotation, the following state appears
$|\psi^{[i]}_{\alpha M}\rangle|\psi^{[i]}_{M^{-1}}\rangle + |\psi^{[i]}_{\beta N}\rangle|\psi^{[i]}_{M^{-1}}\rangle + |\psi^{[i]}_{\alpha M}\rangle|\psi^{[i]}_{N^{-1}}\rangle + |\psi^{[i]}_{\beta N}\rangle|\psi^{[i]}_{N^{-1}}\rangle$
Each term correspond to projective rep. $0, \omega_M - \omega_N, \omega_N - \omega_M, 0$
The state form a representation of G only when $\omega_M = \omega_N$.
- The two states are linked via symm. LU trans. iff $\alpha, \omega_M = \beta, \omega_N$.

Symmetry protected topological orders in 1D

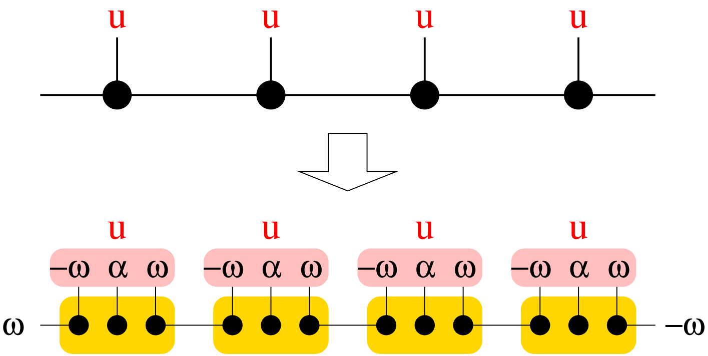
For 1D spin systems with only translation and an on-site symmetry G which is realized by a linear representation, all the phases of gapped states that do not break the two symmetries are classified by a pair (ω, α), where $\omega \in H^2(G, \mathbb{C})$ label different types of projective representations of G and α label different 1D representations of G.

Symmetry protected topological orders in 1D

For 1D spin systems with only translation and an on-site symmetry G which is realized by a linear representation, all the phases of gapped states that do not break the two symmetries are classified by a pair (ω, α), where $\omega \in H^2(G, \mathbb{C})$ label different types of projective representations of G and α label different 1D representations of G.

- $H^2[SO(3), \mathbb{C}] = \mathbb{Z}_2$ and $SO(3)$ has no 1D rep. $\to$ $SO(3)$ spin rotation and translation symmetric integer spin chain has two and only two quantum phases that do not break the two symmetries.
- $H^2[SU(2), \mathbb{C}] = \mathbb{Z}_1$ and $SU(2)$ has no 1D rep. $\to$ $SU(2)$ and translation symmetric integer+half-integer spin chain has only one quantum phases that do not break the two symmetries.
- $H^2(\mathbb{Z}_n, \mathbb{C}) = \mathbb{Z}_1$ and $\mathbb{Z}_n$ has n 1D rep. $\to$ $\mathbb{Z}_n$ and translation symmetric q-dit chain has n and only n quantum phases that do not break the two symmetries.

Canonical fixed point wave function



u: linear representation of G

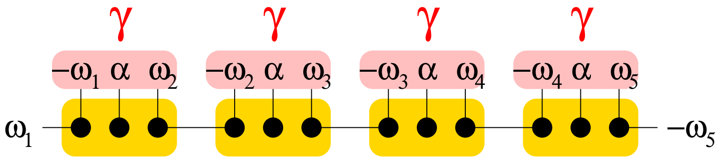
α :1D linear representation of G

$\omega, -\omega$: projective representations of G

- The boundary states form ω or $-\omega$ projective representations of G

Generalizing Lieb-Schultz-Mattis theorem

For an 1D spin system with translation and an on-site symmetry G which is realized by a non-trivial projective representation, the system must gapless if it does break the two symmetries.

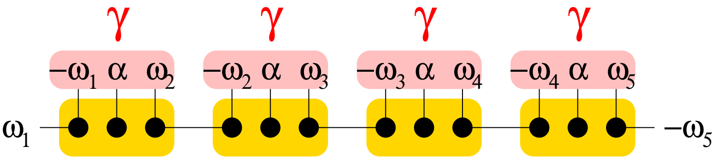


$\omega_{i+1} - \omega_i = \gamma$

Gapless states I

Generalizing Lieb-Schultz-Mattis theorem

For an 1D spin system with translation and an on-site symmetry G which is realized by a non-trivial projective representation, the system must gapless if it does break the two symmetries.



$\omega_{i+1} - \omega_i = \gamma$

- $SO(3)$ spin rotation and translation symmetric half-integer spin chain is gapless if it does not break the two symmetries.

Gapless states II

Hastings 03

In general, a symmetric state of L-sites satisfies

$$u(g) \otimes ... \otimes u(g)|\phi_L\rangle = \alpha_L(g)|\phi_L\rangle$$

Localization of 1D representation

For 1D spin systems of L sites with translation and an on-site symmetry G which is realized by a linear representation, a gapped state that do not break the two symmetries must transform as $u(g) \otimes ... \otimes u(g)|\phi_L\rangle = [\alpha(g)]^L|\phi_L\rangle$ for all large L.

Gapless states II

Hastings 03

In general, a symmetric state of L-sites satisfies

$$u(g) \otimes \dots \otimes u(g)|\phi_L\rangle = \alpha_L(g)|\phi_L\rangle$$

Localization of 1D representation

For 1D spin systems of L sites with translation and an on-site symmetry G which is realized by a linear representation, a gapped state that do not break the two symmetries must transform as $u(g) \otimes \dots \otimes u(g)|\phi_L\rangle = [\alpha(g)]^L|\phi_L\rangle$ for all large L.

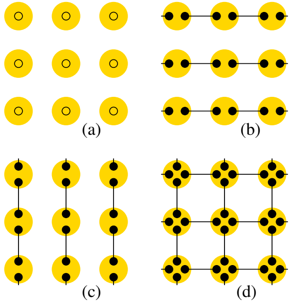
- a 1D state of conserved bosons with fractional bosons per site must be gapless, if the state does not break the translation symmetry. Only integer m boson per site $\to$ on-site 1D rep. $\alpha(\theta) = e^{i m\theta}$.

A simple result in higher dimensions

For d-dimensional spin systems with only translation and an on-site symmetry G which is realized linearly, the object $(\alpha, \omega_1, \omega_2, ..., \omega_d)$ label distinct gapped quantum phases that do not break the two symmetries. Here α labels the different 1D representations of G and $\omega_i \in H^2(G, \mathbb{C})$ label the different types of projective representations of G.

$(\omega_1 = 0, 1; \omega_2 = 0, 1)$ label four distinct states in integer spin systems with translation and $SO(3)$ spin rotation symmetries:

(a) $(\omega_1, \omega_2) = (0, 0)$,
(b) $(\omega_1, \omega_2) = (0, 1)$,
(c) $(\omega_1, \omega_2) = (1, 0)$,
(d) $(\omega_1, \omega_2) = (1, 1)$.



Topological order and entanglement – a rich world

- We classify all 1D symmetric quantum phases using symmetric LU transformation, MPS, and projective representation.
- One can also partially classify 2D quantum phases using LU transformation, string-nets, and TPS.

Topological order and entanglement – a rich world

- We classify all 1D symmetric quantum phases using symmetric LU transformation, MPS, and projective representation.
- One can also partially classify 2D quantum phases using LU transformation, string-nets, and TPS.