Universality in Fermi Liquids

Petr Hořava

Center for Theoretical Physics, University of California, Berkeley
and
Lawrence Berkeley National Laboratory

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In fact, it is a review of a field that does not quite exist yet!

Stanisław Lem

More seriously . . .

I will present:

- Motivation (from string theory)
- Hard-core results (in condensed matter)
- Speculations (mostly about string theory again)

Outline

- Preliminaries: Strings and D-branes
  D-branes: nonperturbative solitonic defects in string theory
  Goal in string theory: Understand the underlying degrees of freedom.

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  D-branes: nonperturbative solitonic defects in string theory
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- D-branes and K-theory
  Classification of D-branes desired:
  D-brane charges classified by a generalized cohomology theory, known as "K-theory"

- Preliminaries II: Fermi liquids in d dimensions
  Universality classes of nonrelativistic quantum systems described microscopically by $\psi(\mathbf{x}, t)$.
  Goal in condensed matter: Understand universality classes of systems.

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  Systematic classification scheme now available.
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- Tempting speculations: Implications for string theory?

Central observation: Similar patterns emerge in addressing both goals!

D-branes, briefly

D-branes are solitons (hypersurfaces Σ_{p+1} in spacetime?) on which strings can end:

For stable supersymmetric D-branes, the tachyon T^{IJ} is projected out.

Dp-branes couple to Ramond-Ramond (RR) fields $C_{p+1}(x)$.

[Polchinski,. . .]

- C_{p+1} are higher-form analogs of electromagnetism, couple to sources:

$$\int_{\Sigma_{p+1}} C_{p+1}$$

- Field strength $G_{p+2} \sim dC_{p+1}$

- Described by spacetime effective action

$$\int_X d^{10}x \sqrt{g} \left(R - \sum_p \frac{1}{p!} G_{p+2} \wedge \star G_{p+2} + \dots \right)$$

Those RR fields are differential forms, right?

If RR fields were differential forms, charges of D-branes allowed by Dirac quantization should be classified by $H^*(Y, \mathbf{Z})$: the integral refinement of de Rham cohomology $H^*(Y, \mathbf{R})$.

Inside $H^*(Y, \mathbf{R})$, $H^*(Y, \mathbf{Z})$ would define a lattice of charges, perhaps some torsion elements (such as $\mathbf{Z}_n$) would also exist) . . .

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This expectation is wrong!

[Witten, . . .]

D-branes are not just hypersurfaces Σ in spacetime.
They carry extra structure. Charges keep track of this, and as a result, take values in a *generalized cohomology theory* of X, known as K-theory of X.

Brane-antibrane systems

[Sen, Witten, . . .]

Branes are not just naive hypersurfaces Σ. They are places where a $U(N)$ gauge bundle E is supported.

Consider N branes and N antibranes, with gauge bundles E and F. The brane-antibrane pairs can annihilate if their Σ's line up . . .

. . . but only if they carry the same gauge bundle, $E = F$.

Classification of branes up to brane-antibrane annihilation

Define an equivalence relation:

$(E_1, F_1) \sim (E_2, F_2)$ if bundles G, H exist along Σ such that

$$(E_1 \oplus G, F_1 \oplus G) = (E_2 \oplus H, F_2 \oplus H).$$

This coincides with the mathematical definition of the K-theory group $K(\Sigma)$.

If the annihilation is incomplete – i.e., if (E, F) is a non-trivial element of $K(\Sigma)$ – a stable, lower-dimensional brane is left over after annihilation.

Hence, charges of the lower-dimensional branes are classified by equivalence classes of brane-antibrane configurations, or equivalently, elements of $K(X)$! ($\Sigma = X$ for spacetime-filling branes)

K-theory

K-theory is a generalized cohomology theory, deeply connected with the theory of Dirac operators, index theory, Riemannian geometry, topology, . . .

Related to de Rham cohomology:

$$K(X) \otimes \mathbf{R} = \sum_n H^{2n}(X, \mathbf{R})$$

Hence, the naive expectation of D-brane charges being related to conventional cohomology theory is true modulo:

- The precise Dirac quantization of charges,
- The torsion charges.

D-branes as topological solitons

Consider spacetime-filling branes with bundle E and antibranes with F, such that they cannot completely annihilate. This should be a brane along some hypersurfaces Σ.

Gauge symmetry: $U(N) \times U(N)$.

Tachyon Higgs field $T^{IJ}(x)$.

Brane-antibrane annihilation $=$ Higgs mechanism:

$$U(N) \times U(N) \to U(N)$$

When $E \neq F$, the annihilation leaves behind a defect in T. What is the profile of the defect?

Sen's "$\Gamma \cdot x$" construction

$$T(x^i) = f(r)\Gamma_i x^i.$$

Examples:

- Codimension 1 – a kink in a $U(1) \times U(1)$ theory;
- Codimension 3 – magnetic monopole in $SU(2) \times SU(2)$;
- . . .

This construction (by Sen) can be recognized as a universal construction in K-theory:
the Atiyah-Bott-Shapiro construction!

K-theory versus homotopy theory

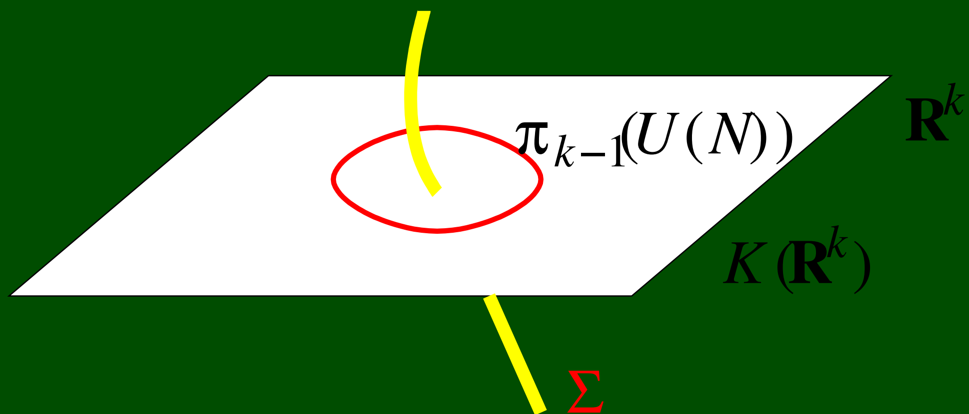
Consider homotopy groups $\pi_p(U(N))$ of $U(N)$ group manifolds. Complicated for low N relative to k, but they simplify in the stable regime of $N > p/2$,

$$K(\mathbf{R}^k) = \pi_{k-1}(U(N)) = \pi_{k-1}(U(N+1)) = \ldots$$

This is another definition of K-theory groups!

To summarize . . .

Stable D-branes = defects in higher-dimensional unstable branes.



Charge measured by homotopy groups defining K-theory.

Bott periodicity and D-brane spectrum

One can calculate:

$$K(\mathbf{R}^{2k}) = \mathbf{Z},$$
$$K(\mathbf{R}^{2k+1}) = 0.$$

This periodicity by two in K-theory predicts a periodicity in the D-brane spectrum.

This is rather boring, well-known in string theory before the K-theory connection.

More interesting: Type I superstrings

K-theory of real (instead of complex) bundles: KO-theory. (Related to homotopy groups of $O(N)$, in the universal regime.)

Bott periodicity: by eight!

$k = 0, 4, 8, \ldots$: $KO(\mathbf{R}^k) = \mathbf{Z}$,

$k = 1, 2$: $KO(\mathbf{R}^k) = \mathbf{Z}_2$

Torsion-charged D0, D7, D8-branes!

. . . Puzzle about RR fields

RR fields couple to D-branes . . .

Are RR fields then also subtle, K-theoretic objects??

If C_{p+1} are no longer differential forms, how do we describe them in the Lagrangian framework of spacetime effective action?

Should we give up any Lagrangian formulation of supergravity when charges are included?

Look for lessons in other areas of physics . . .

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here comes

the condensed matter part!

Fermi Liquids in $d+1$ Spacetime Dimensions

Microscopic field:

$\psi^{i\sigma}(\mathbf{x}, t)$, a complex fermion

Here σ is the spinor index of $SO(d)$, dimension $2^{[d/2]}$.

i is the internal index, say $i = 1, \ldots n$ of $SU(n)$.

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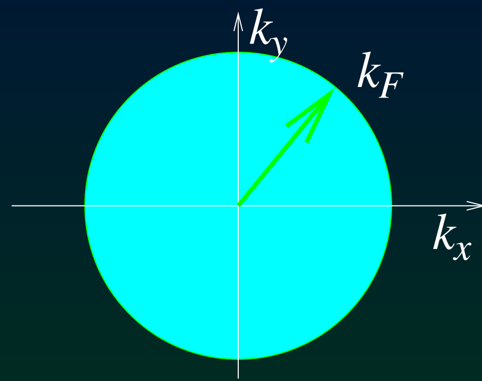
Microscopic Lagrangian:

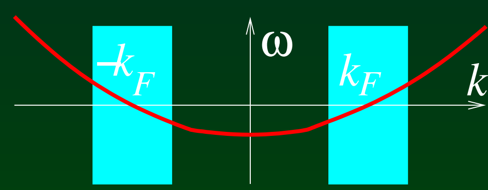
$$\int dt\, d\mathbf{x} \left(i\psi^{\dagger}_{i\sigma}\partial_t\psi^{i\sigma} + \frac{1}{2m}\psi^{\dagger}_{i\sigma}\Delta\psi^{i\sigma} - \mu\,\psi^{\dagger}_{i\sigma}\psi^{i\sigma} + \ldots \right)$$

Landau's theory of Fermi liquids

Elementary example: Free fermions. Ground state:



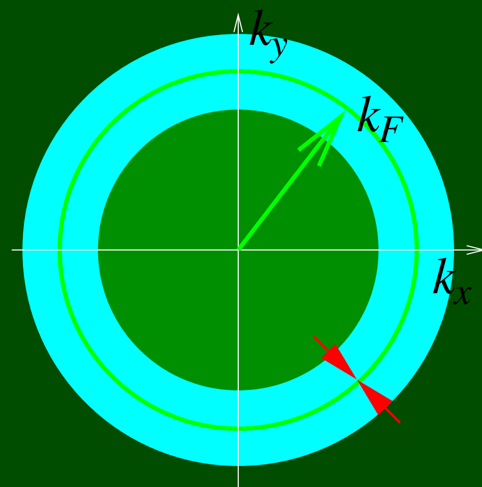
Lowest modes reside in the vicinity of the Fermi surface.
Example: $1+1$ dimensions. Relativistic dispersion relation!



Renormalization group analysis

RG scaling defined towards the Fermi surface:

great reviews: [Shankar, Polchinski, . . .]



Generalized Fermi surface $=$ a submanifold Σ in the $(\mathbf{k}, \omega)$ space where gapless excitations are supported.

Infinite number of (naively) marginal couplings; angles θ are like an internal index; fascinating RG flows . . . *cf.* 2d sigma models!

Patterns of stability of Fermi surfaces

Stable Fermi surface – gapless excitations still there if system slightly perturbed.

Examples of systems in $3+1$ dimensions:

- Free fermions.
  Stable, two-dimensional Fermi surface ($\sim S^2$).
- The A-phase of ^{3}He.
  Interactions leave a stable Fermi point.
- Systems with a stable Fermi line?
  No!

Claim:

- These are first signs of Bott periodicity,
- Stable Fermi surfaces are classified by K-theory!

Stability of Fermi surfaces in $d+1$ dimensions

Look at the inverse propagator (or the 1PI two-point function).

Volovik, . . .

First define

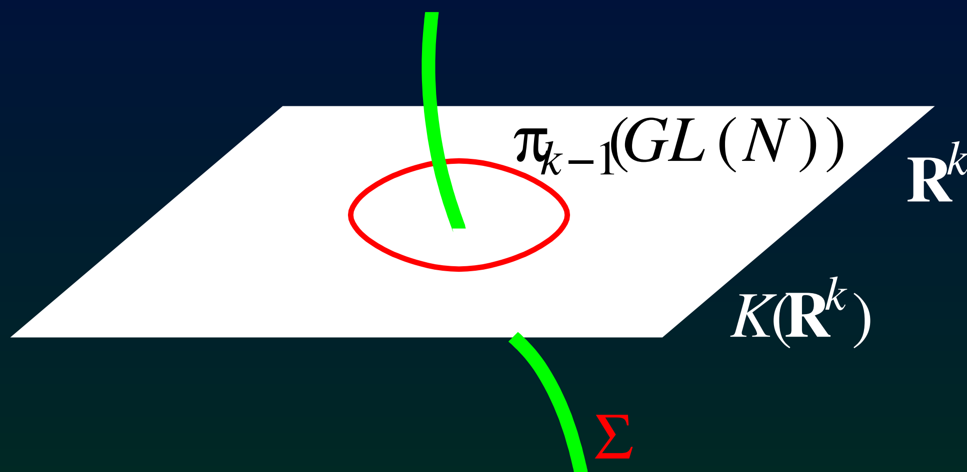
$$G^{i\sigma}{}_{i'\sigma'}(\mathbf{k},\omega) = \langle \psi^{i\sigma}(0,0)\psi^{\dagger}_{i'\sigma'}(\mathbf{k},\omega)\rangle$$

and, introducing a collective index $a=(i,\sigma)$, with $i=1,\ldots N = 2^{[d/2]}\cdot n$, define

$$\mathcal{G}_a{}^{a'} \equiv (G^{-1})_a{}^{a'}(\mathbf{k},\omega)$$

Stability of the Fermi surface: zeros of det $\mathcal{G}$ that cannot be eliminated by a small perturbation.

Hence, they must be protected by a topological "winding number":



- N is always in the stable regime, π_{k-1} define K-theory!
- $\pi_{k-1}(GL(N)) = \pi_{k-1}(U(N))$.

First result: Stable Fermi surfaces in Fermi liquids are classified by K-theory.

Classification of stable Fermi surfaces by K-theory

For complex fermions, Fermi surfaces of codimension $p+1$ in the $(\mathbf{k}, \omega)$-space are stable for p odd and unstable for p even.

Pattern of stability of Fermi surfaces is thus determined by Bott periodicity of K-theory!

Dispersion relation at low energies

Given a stable, generalized Fermi surface Σ, we wish to know:

- What are the lowest-lying modes χ^α,
- What is their low-energy dispersion relation near Σ.

Recall from our discussion of D-branes that K-theory has a universal construction of the non-trivial K-theory class in $K(\mathbf{R}^{2k})$, the Atiyah-Bott-Shapiro construction.

The ABS construction will determine the universal features of the dynamics of χ^α!

Low-energy action for coarse-grained fermions:

$$S = \int d\mu(\omega, \mathbf{k}, \theta) \left(\chi^\dagger_\alpha \mathcal{D}^\alpha{}_\beta \chi^\beta + \ldots \right)$$

The ABS construction implies that near the Fermi surface,

$$\mathcal{D} = \Gamma^\mu p_\mu + \ldots$$

Second result: At low energies, the dispersion relation of the coarse-grained gapless fermions χ^α is governed by the ABS construction.

Third result: χ^α exhibit an emergent relativistic dispersion relation in the dimensions transverse to the Fermi surface. (Spin-statistics nicely reproduced, . . .)

“$\Gamma \cdot x$” versus “$\Gamma \cdot p$”

Traditionally, topology is important for solitons/instantons of interacting field theories . . .

For Fermi liquids, topology of the momentum space classifies free-field fixed points of RG.

We have put the $\Gamma \cdot x$ construction where it belongs: in the momentum space! Indeed, $\Gamma \cdot p$ is just the Dirac operator . . .

Real fermions

More intricate, since now a real K-theory will be relevant.

Torsion classes in K-theory $(\mathbf{Z}_2)$ will start appearing.

Naive guess: $KO(\mathbf{R}^n)$ – incorrect.

A proper reality condition in the $(\mathbf{k}, \omega)$-space leads to KR-theory.

After the dust settles, one finds:

- Stable Fermi surfaces of codimension two, and more generally, of codimension $2 + 4k$;

- Stable Fermi surfaces of codimension three and four (modulo 8), carrying a $\mathbf{Z}_2$ charge in K-theory!

For example, in $2 + 1$ dimensions, we now have a stable Fermi point. Two such points can annihilate each other.

Other K-theories

Only the tip of the iceberg.

One can impose extra symmetries.

Discrete, continuous, gauge.

Corresponding K-theories can be constructed, will classify stable Fermi surfaces.

Topological defects in Fermi liquids

So far, we looked at the vacuum states, with space-time translation invariance. K-theory will naturally extend to defects and their stability.

Simplest case: Consider the semi-classical regime, $\hbar \approx 0$.

Haldane, . . . : Fermi surface now a surface in $(\mathbf{k}, \omega, \mathbf{x}, t)$.

Defects will be again classified by K-theory, low-energy dispersion relations will be determined by the ABS construction.

(in progress @ Berkeley)

The Fermi surface carries an effective tension.

[Fradkin et al., . . .]

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back to string theory . . .

Other appearances of nonrelativistic Fermions in string theory

$1+1$ dimensional nonrelativistic fermions & strings:

- Old matrix models. Matrix quantum mechanics as discretization of two-dimensional worldsheets propagating in $1+1$ dimensions. The fermions emerge as eigenvalues, give rise to the spatial dimension.

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$1+1$ dimensional nonrelativistic fermions & strings:

- Old matrix models. Matrix quantum mechanics as discretization of two-dimensional worldsheets propagating in $1+1$ dimensions. The fermions emerge as eigenvalues, give rise to the spatial dimension.
- New understanding of old matrix models. The fermions are (unstable) D0-branes!

- Half-BPS states in AdS/CFT. CFT reduces to a matrix model, eigenvalues are fermions, correspond to the N original D3-branes.

  Supergravity solutions seeded by semiclassical Fermi droplets in phase space.

  [Jevicki et al., Berenstein, LLM, . . .]

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Conclusions

- Stage set for condensed matter applications . . .
  Systematic classification of universality classes of stable Fermi surfaces.
  Possible applications to topological order [Wen&Zee]

- Intriguing possibilities for string theory?

- Moral for QFT:
  Topology is important even *before* solitons are introduced;
  in fact, topology classifies free-field RG fixed points!

(tell your QFT class!)