Theory of Parton Distributions

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DESY Zeuthen

1. **Our Goals in Physics**
2. **The Challenges for Theory**
3. **Summary**

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– KITP Conference *Collider Physics*, Santa Barbara, January 13, 2004 –

Our Goals in Physics

Structure of the proton

- Structure functions F_2, F_3, F_L in deep-inelastic scattering
  - scaling violations $\longrightarrow$ precision test of perturbative QCD

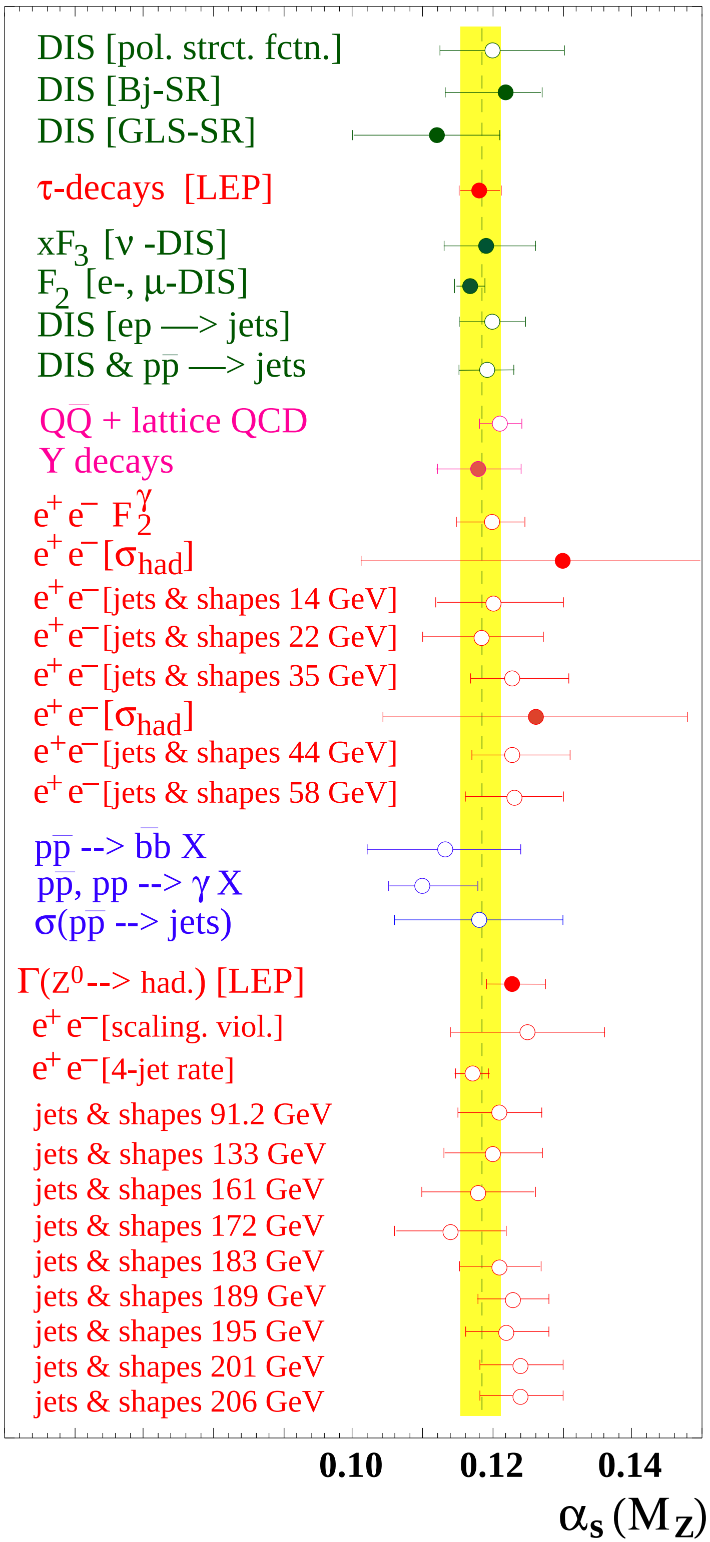
- Parton distributions
  - gluon distribution at small x, quark valence and sea distribution
  - important input for hard scattering reactions at hadron colliders
    $\longrightarrow$ precise parton luminosity at LHC for Higgs or SUSY searches

How well do we know (un)-polarized parton distributions ?

α_s from DIS

- Fundamental parameter of Standard Model
  - determination in inclusive DIS independent of hadronic final state $\longrightarrow$ ideal case

How well do we know α_s ?



Recent determinations of α_s

Bethke `hep-ex/0211012`

– **NLO** QCD analysis of HERA data for $F_2(x, Q^2)$
H1 coll. `hep-ph/0012053`

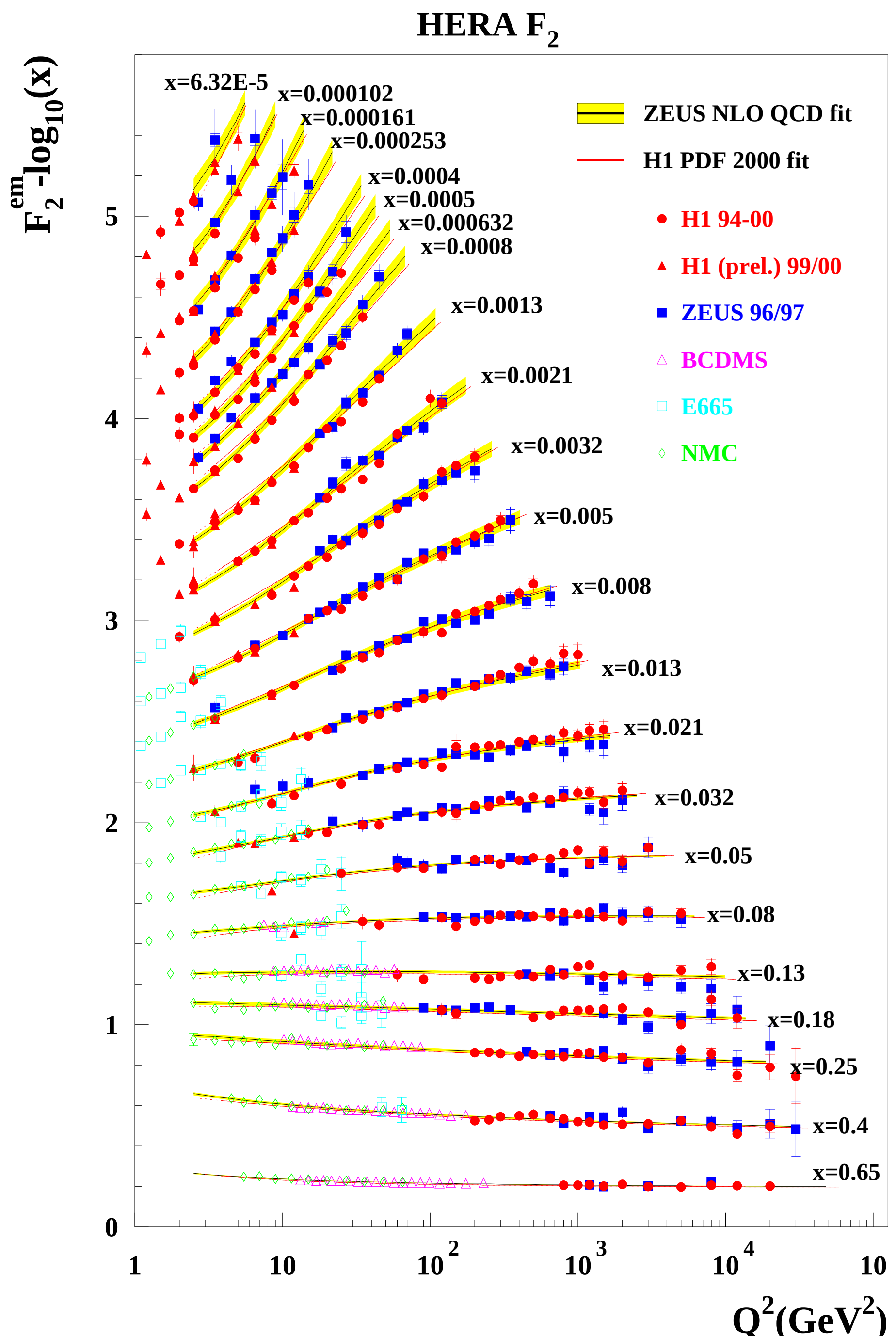
$$\alpha_s(M_Z^2) = 0.115 \pm 0.002(\text{exp}) \pm 0.005(\text{theo})$$

Future

– **NNLO** QCD analysis of HERA data for $F_2(x, Q^2)$ in 2006

$$\alpha_s(M_Z^2) = x \pm 0.001(\text{exp}) \pm 0.001(\text{theo})$$

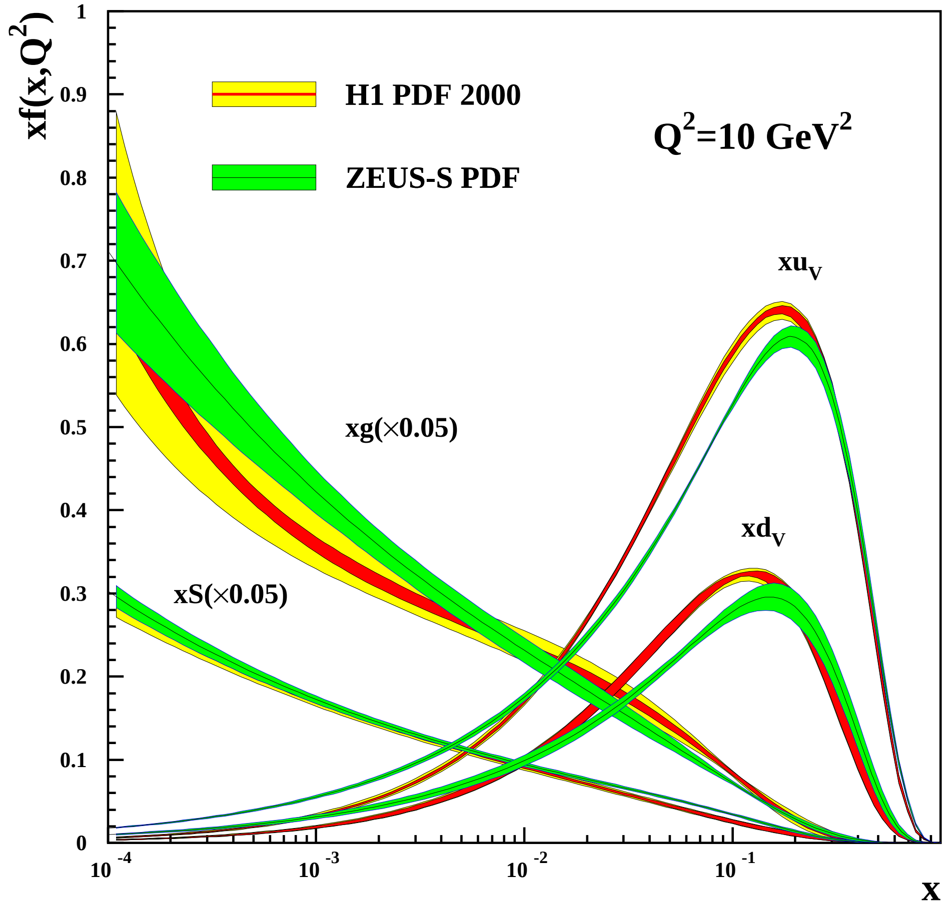
Structure function measurements



- High precision experimental data
  $\longrightarrow$ **NLO** QCD analysis for $F_2(x, Q^2)$

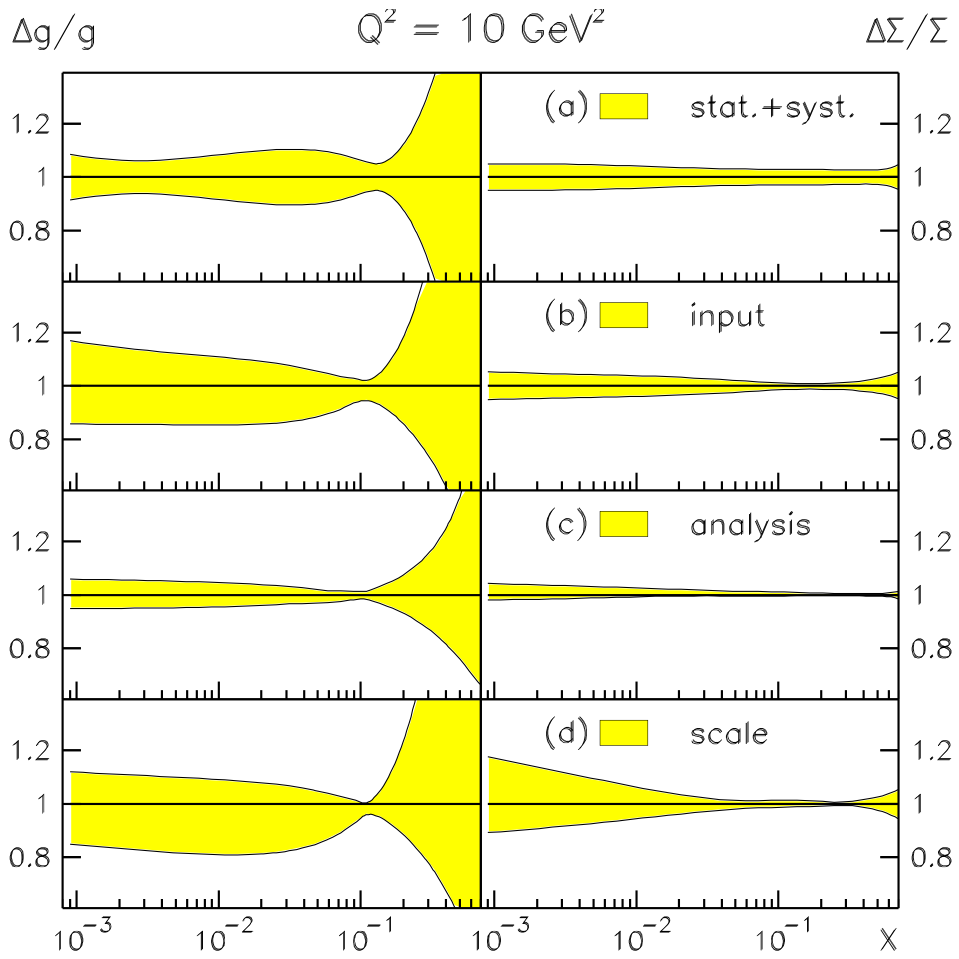
- HERA until 2006
  $\longrightarrow$ data with much higher statistics

PDF uncertainties



- QCD analyses require many choices
  $\longrightarrow$ should be reflected in PDF uncertainties
- Treatment of experimental uncertainties
- Allowed functional form of PDF
  $xf\left(x,Q_0^2\right) = Ax^b(1-x)^c(1+dx+\ldots)$
- Scale dependence
  $\longrightarrow$ renormalization / factorization scale
- Treatment of heavy quarks
- ...

PDF uncertainties (cont'd)



Botje '00

– Similar analysis Barone, Pascaud, Zomer '99

– Analysis of 1999 DIS data with errors
Botje '00

– Scale variation

$$Q/\sqrt{2} \le \mu \le \sqrt{2}Q$$

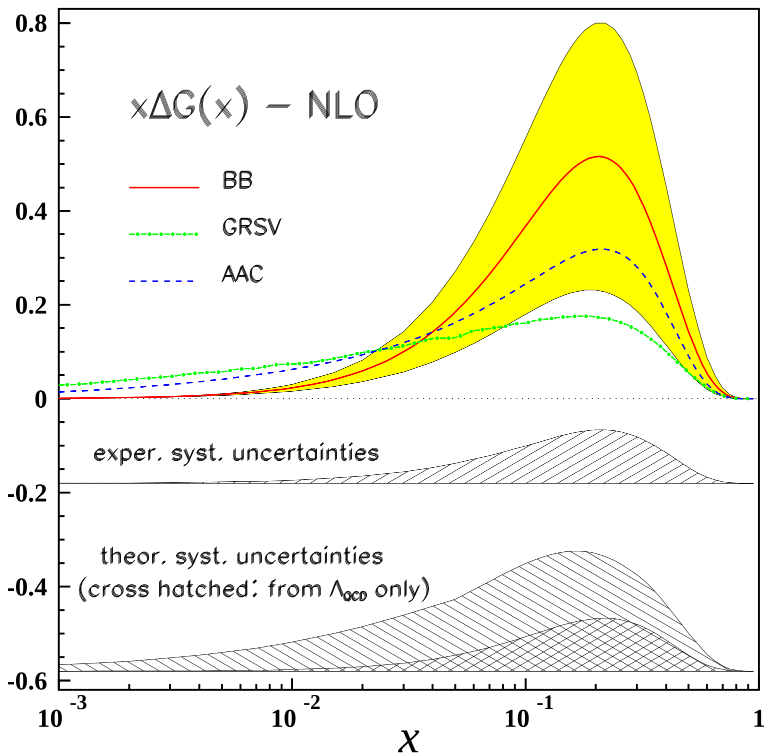
– Gluons (g) :
stat. $\oplus$ syst. $\simeq$ input $\simeq$ scale error

– Quarks (Σ) :
scale error already dominates

Upshot

– **NNLO** improvement of theory needed
$\longrightarrow$ three-loop splitting functions

Polarized deep-inelastic scattering



Blümlein, Böttcher '02

– Precision analysis of polarized DIS data with correlated errors $\longrightarrow$ structure function g_1

Blümlein, Böttcher '02

– Polarized gluon distribution (ΔG) : syst. $\simeq$ scale error from $\Lambda_{\rm QCD}$

– Theoretical uncertainties already dominate at present

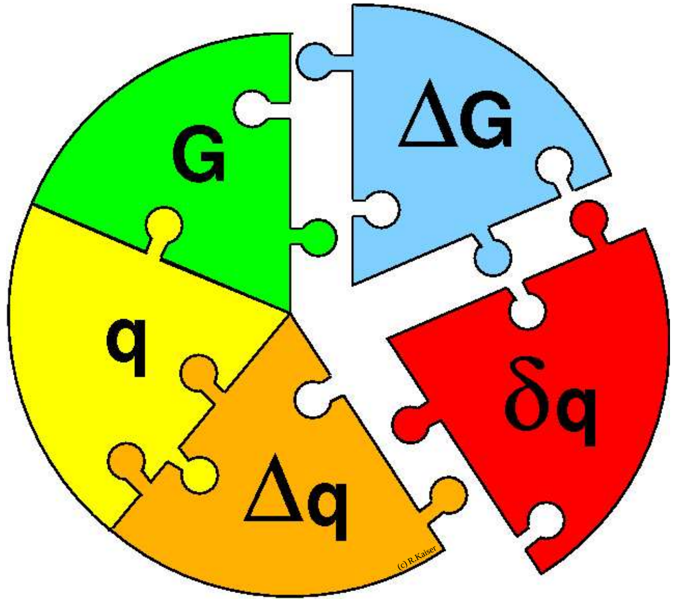
Upshot

– Similar situation as for unpolarized scattering

– **NNLO** improvement of theory needed
$\longrightarrow$ three-loop splitting functions
$\longrightarrow$ $g_1^{\rm c}$ for charm production at NLO

Twist-two parton distributions and the spin puzzle

- Twist-two deep-inelastic structure functions
  - Unpolarized scattering F_2, F_L valence and sea quarks, gluon distribution $q_{\rm val}, q_{\rm sea}, G$
    $\longrightarrow$ H1, ZEUS, ν-experiments
  - Longitudinally polarized scattering g_1, distributions $\Delta q, \Delta G$
    $\longrightarrow$ SLAC, HERMES, COMPASS
  - Transversely polarized scattering h_1, distribution δq $\longrightarrow$ HERMES, RHIC



The Challenges for Theory

What has been done ?

- QCD corrections for DIS structure functions with massless quarks
- LO :
  - anomalous dimensions / splitting functions Gross, Wilczek ‘73 ; Altarelli, Parisi ‘77
- NLO :
  - complete one-loop F_2 and F_L Bardeen, Buras, Duke, Muta ‘78
  - two-loop anomalous dimensions / splitting functions Floratos, Ross, Sachrajda ‘79 ; Gonzalez-Arroyo, Lopez, Ynduráin ‘79 ; Curci, Furmanski, Petronzio ‘80 ; Furmanski, Petronzio ‘80
  - two-loop F_L Duke, Kimel, Sowell ‘82 ; Devoto, Duke, Kimel, Sowell ‘85 ; Kazakov, Kotikov ‘88 ; Kazakov, Kotikov, Parente, Sampayo, Sanchez Guillen ‘90
- NNLO :
  - complete two-loop F_2, F_3 and F_L Zijlstra, van Neerven ‘92 ; S.M., Vermaseren ‘99
  - fixed Mellin moments of F_2, F_3 and F_L at three loops Larin, Nogueira, van Ritbergen, Vermaseren ‘97 ; Retey, Vermaseren ‘00
  - approximate three-loop splitting functions Vogt, van Neerven ‘00

Task $\longrightarrow$ calculate three-loop anomalous dimensions / splitting functions

What has been done ? (cont' d)

– QCD corrections for DIS structure functions with massive quarks

– Neutral current $O(\alpha_s)$:
  – complete structure functions F_2 and F_L Witten '76 ; Glück, Reya '76
– Neutral current $O(\alpha_s^2)$:
  – complete $O(\alpha_s^2)$ structure functions F_2 and F_L Laenen, Riemersma, Smith, van Neerven '92

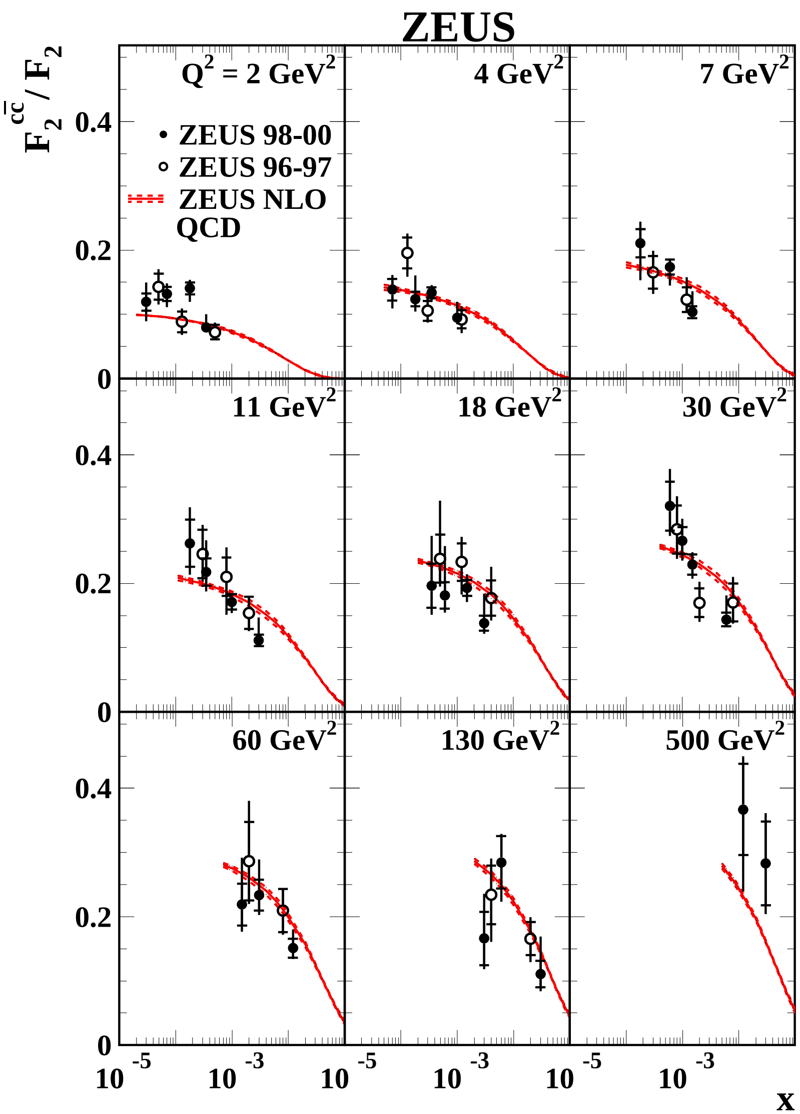
– Charged current $O(\alpha_s)$:
  – complete corrections to νN DIS Gottschalk '81 ; van der Bij, Oldenborgh '91 ; Kramer, Lampe '92
– Charged current $O(\alpha_s^2)$:
  – $O(\alpha_s^2 \ln^n(Q^2/m^2))$ contributions Buza, van Neerven '97

– Resummation of large logarithms $\ln^n(Q^2/m^2)$

Aivazis, Collins, Olness, Tung '94 ; Thorne, Roberts '98 ; Chuvakin, Smith, van Neerven '00

Task $\longrightarrow$ fast NNLO evolution with massive quarks in PDF determinations

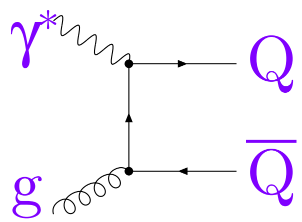
Heavy quark production



ZEUS Collaboration '03

– Data for charm $F_2^c(x,Q)$
$\longrightarrow$ $O(10\%) - O(30\%)$ of total $F_2(x,Q)$

– Heavy quark DIS via boson-gluon fusion
$\longrightarrow$ clean probe of the gluon distribution



– Leading order infrared safe with heavy quark mass m

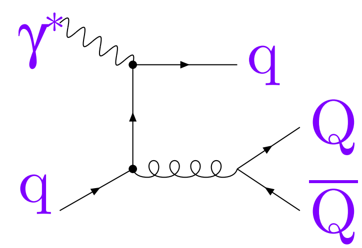
Heavy quark production (cont'd)

– Possible treatments of heavy quarks

– Fixed flavor number scheme
$\longrightarrow$ n_f light flavors + heavy quark of mass m at low scales
$\longrightarrow$ $n_f + 1$ light flavors at high scales

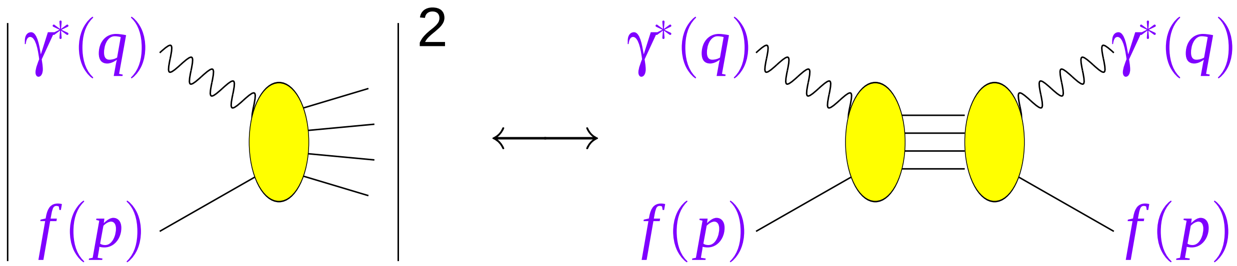
– Variable flavor number schemes $\longrightarrow$ matching of two distinct theories
Aivazis, Collins, Olness, Tung '94 ; Thorne, Roberts '98 ; Chuvakin, Smith, van Neerven '00

– Important aspects of variable flavor number schemes
  – mass factorization to be carried out before resummation
    $\longrightarrow$ mass factorization involves **both heavy and light** component of structure function



  – matching conditions required through NNLO Chuvakin, Smith, van Neerven '00
    $\longrightarrow$ influence on HERA analysis at small $x \leq 10^{-4}$ and low scales $Q^2 \leq 100\mathrm{GeV}^2$
    $\longrightarrow$ α_s large at low scales

Optical theorem and the OPE



– Optical theorem relates hadronic tensor $W_{\mu\nu}$ to imaginary part of Compton amplitude $T_{\mu\nu}$

$$W_{\mu\nu} = e_{\mu\nu}\frac{1}{2x}F_L(x,Q^2) + d_{\mu\nu}\frac{1}{2x}F_2(x,Q^2) + \mathrm{i}\varepsilon_{\mu\nu\alpha\beta}\frac{p^\alpha q^\beta}{p\cdot q}F_3(x,Q^2)$$

– Bjorken limit (x fixed, $Q^2 \to \infty$) allows for OPE of $T_{\mu\nu}$ for short distances $z^2 \simeq 0$

Wilson '69 ; Brandt, Preparata '70 ; Zimmermann '70

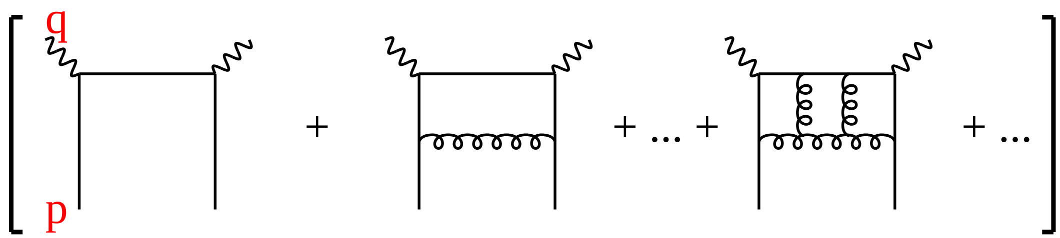
$$T_{\mu\nu} = \mathrm{i}\int d^4z\, \mathrm{e}^{\mathrm{i}q\cdot z}\langle \mathrm{P}|\, T\left(J^\dagger_\mu(\mathrm{z})J_\nu(0)\right)|\mathrm{P}\rangle =$$

$$= \sum_{N,j}\left(\frac{1}{2x}\right)^N \left[e_{\mu\nu}C^N_{L,j}\left(\frac{Q^2}{\mu^2},\alpha_s\right) + d_{\mu\nu}C^N_{2,j}\left(\frac{Q^2}{\mu^2},\alpha_s\right) + \mathrm{i}\varepsilon_{\mu\nu\alpha\beta}\frac{p^\alpha q^\beta}{p\cdot q}C^N_{3,j}\left(\frac{Q^2}{\mu^2},\alpha_s\right)\right] A^j_{\mathrm{P},N}(\mu^2)$$

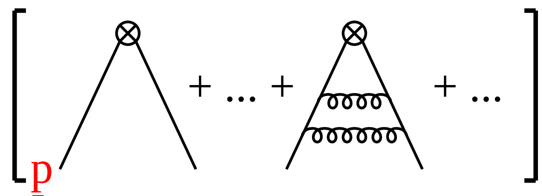
– Coefficient functions $C^N_{2,i}, C^N_{3,i}$ and $C^N_{L,i}$ in Mellin space
– Matrix elements $A^i_{\mathrm{P},N}$ of operators O^i of leading twist

The parton picture

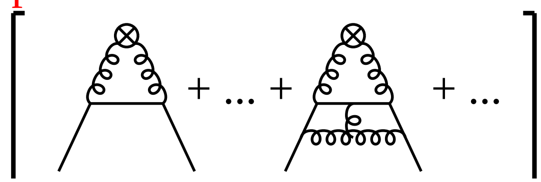
– Apply OPE to parton Green's functions, e.g. external quarks



$$= \sum_N \left(\frac{1}{2x}\right)^N \Big\{ \left(C^N_{i,\mathrm{q}} Z^{\mathrm{qq}} + C^N_{i,\mathrm{g}} Z^{\mathrm{gq}}\right)$$



$$+ \left(C^N_{i,\mathrm{q}} Z^{\mathrm{qg}} + C^N_{i,\mathrm{g}} Z^{\mathrm{gg}}\right)$$



$$\Big\}$$

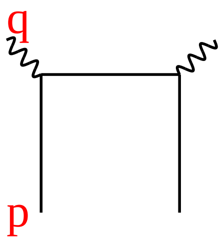
– Apply $\mathcal{P}_N$ to project the N-th moment ; Gorishnii, Larin, Tkachev '83 ; Gorishnii, Larin '87

$$\mathcal{P}_N \equiv \left[\frac{q^{\{\mu_1} \cdots q^{\mu_N\}}}{N!} \frac{\partial^N}{\partial p^{\mu_1} \cdots \partial p^{\mu_N}} \right] \Bigg|_{p=0} \qquad \frac{1}{2x} = -\frac{p \cdot q}{q \cdot q}$$

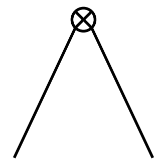
Upshot

– Projection with $\mathcal{P}_N \longrightarrow$ only tree level operator matrix elements survive

$$\mathcal{P}_N \Bigg[$$



$$+ \ldots \Bigg] \; = \; \left(C^N_{i,\mathrm{q}} Z^{\mathrm{qq}} + C^N_{i,\mathrm{g}} Z^{\mathrm{gq}} \right)$$



– **Anomalous dimensions** $\gamma(\alpha_s, N)$ from scale dependence of renormalized operators

$$O^{\mathrm{bare}} = Z\, O^{\mathrm{ren}} \qquad\qquad \frac{d}{d \ln \mu^2} O^{\mathrm{ren}} = -\gamma\, O^{\mathrm{ren}}$$

Structure functions in Mellin space

– Parameters of OPE are directly related to Mellin moments of F_2, F_3 and F_L

$$\int_0^1 dx\, x^{N-2} F_2(x, Q^2) \; = \; \sum_{i=\mathrm{ns,q,g}} C^N_{2,i}\left(\frac{Q^2}{\mu^2}, \alpha_s\right) A^i_{\mathrm{P},N}(\mu^2)$$

The Feynman diagrams

– 625 diagrams with qγ for nonsinglet F_3

– complete table for singlet F_2, F_L

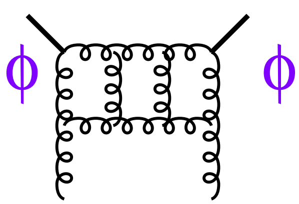
	tree	1-loop	2-loop	3-loop	proj.
qγ	1	3	25	359	2
gγ		2	17	345	2
hγ			2	56	2
qϕ		1	23	696	1
gϕ	1	8	218	6378	1
hϕ		1	33	1184	1
sum	**3**	**20**	**362**	**9778**	

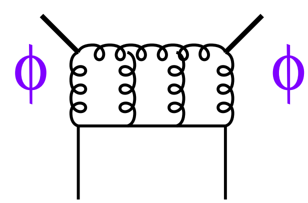
– $P_{\mathrm{qq}}, P_{\mathrm{qg}} \longrightarrow$ DIS with external photon γ^*

– $P_{\mathrm{gq}}, P_{\mathrm{gg}} \longrightarrow$ DIS with external scalar ϕ

– Gluonic currents
$\longrightarrow \phi F^a_{\mu\nu} F^{\mu\nu}_a$ term in QCD Lagrangian
Kluberg-Stern, Zuber '75 ;
Collins, Duncan, Joglekar '77
$\longrightarrow$ different current product, but same parton opertor matrix elements and same OPE

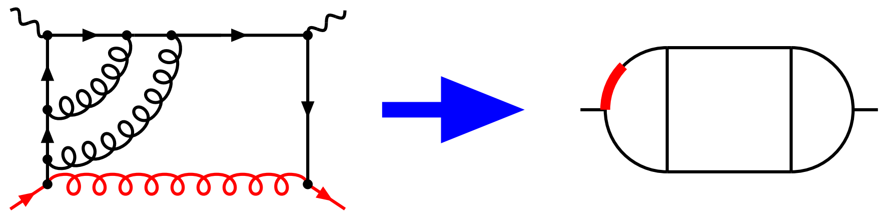
– Gluon polarizations
$\longrightarrow$ diagrams with external ghost h



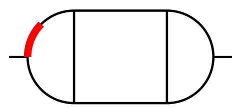


Integrals and how we break them to little pieces

- Reduction scheme for given diagram

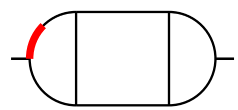


- Scalar diagram with external momenta p and q



$$= \int d^D l_1 d^D l_2 d^D l_3 \, \frac{1}{(p-l_1)^2} \, \frac{1}{l_1^2 \ldots l_8^2}$$

  - N-th moment $\longrightarrow$ coefficient c_N

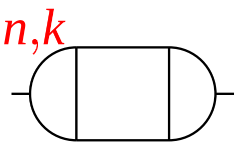


$$= \frac{(2\, p \cdot q)^N}{(q^2)^{N+\alpha}} \, c_N$$

  - Taylor expansion

$$\frac{1}{(p-l_1)^2} = \sum_i \frac{(2p \cdot l_1)^i}{(l_1^2)^{i+1}} \quad \longrightarrow \quad \frac{(2p \cdot l_1)^N}{(l_1^2)^N}$$

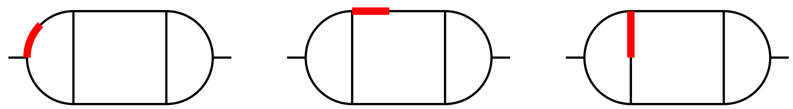
- Two-point functions with symbolic powers



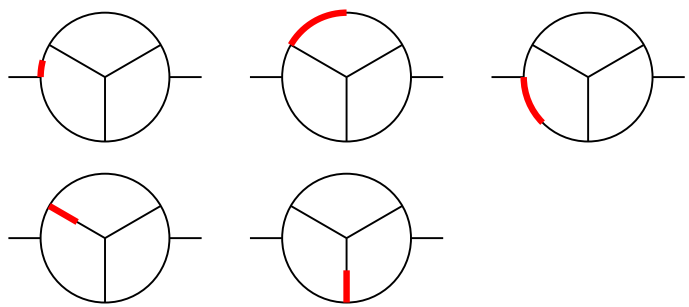
$$= \int d^D l_1 d^D l_2 d^D l_3 \frac{(2p \cdot l_1)^k}{(l_1^2)^n} \frac{1}{l_2^2 \ldots l_8^2}$$

Basic building blocks

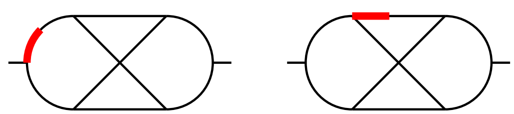
- 10 topologies for basic building blocks
  - ladder



  - benz



  - nonplanar



General strategy of mapping

– Mapping of complicated topologies to simpler topologies
  – hierachy : non-planar $\longrightarrow$ benz $\longrightarrow$ ladder
  – composite building blocks $\longrightarrow$ basic building blocks

– Integration by parts, scaling identies, form factor relations (Passarino–Veltman), ...
’t Hooft, Veltman ‘72 ; Chetyrkin, Tkachov ‘81 ; Larin ‘81 ; S.M., Vermaseren ‘99

$$\int d^Dl_1 d^Dl_2 d^Dl_3 \frac{\partial}{\partial l_i^\mu}\left[l_j^\mu f(l_1,\ldots l_n)\right] = 0 \quad , \qquad q^\mu \frac{\partial}{\partial q^\mu}, \qquad p^\mu \frac{\partial}{\partial q^\mu}, \qquad p^\mu \frac{\partial}{\partial p^\mu}$$

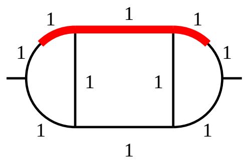
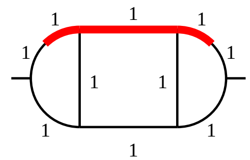
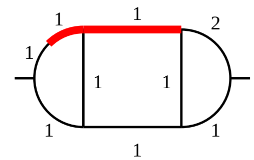
$$\frac{\partial}{\partial q^\mu}\int d^Dl_1 d^Dl_2 d^Dl_3 \left[l_j^\mu f(l_1,\ldots l_n)\right] = \frac{\partial}{\partial q^\mu}\left[q^\mu I^{(q)} + p^\mu I^{(p)}\right]$$

Recursion relations

– Recursion relations $\longrightarrow$ difference equations

$$a_0(N)F(N) - \ldots - a_n(N)F(N-n) - G(N) = 0$$

– Example : single-step difference equation in N

 $= -\dfrac{N+3+3\varepsilon}{N+2}\,\dfrac{2\,p\cdot q}{q^2}$  $+\,\dfrac{2}{N+2}$ 

– Formal solution of single-step difference equations

$$\mathbf{F}(\mathbf{N}) \;=\; \frac{\prod_{j=1}^{N} a_1(j)}{\prod_{j=1}^{N} a_0(j)}\mathbf{F}(\mathbf{0}) + \sum_{\mathbf{i=1}}^{\mathbf{N}} \frac{\prod_{j=i+1}^{N} a_1(j)}{\prod_{j=i}^{N} a_0(j)}\mathbf{G}(\mathbf{i})$$

– Implementation in computer algebra system FORM Vermaseren '89–'03

```
id lafun(n1?,l1?,0,0,l2?,k2?,n3?,l3?,0,<n4?,0,0>,...,<n8?,0,0>,k9?) =
     theta_(N-k2)*sign_(N)* fac_(n1+n3-k2)*invfac_(N+n1+n3-k2)*
     Gamma(-7+N+3*ep+n1+n3+...+n8-k2-k9+l1+l2+l3+1)*
     InvGamma(-7+3*ep+n1+n3+...+n8-k2-k9+l1+l2+l3+1)*
   LA(n1+l1,l2,n3+l3,n4,n5,n6,n7,n8,0,k2,0,0,0,0,0,0,k9)
- sum1(j1,1,N)* theta_(N-j1)*sign_(N-j1)* fac_(j1+n1+n3-k2-1)*invfac_(N+n1+n3-k2)*
     Gamma(-7+N+3*ep+n1+n3+...+n8-k2-k9+l1+l2+l3+1)*
     InvGamma(-7+j1+3*ep+n1+n3+...+n8-k2-k9+l1+l2+l3+1)* (
     -n1*lafun(1+n1,-1+l1,0,0,l2,k2,n3,l3,0,<n4,0,0>,...,<n8,0,0>,k9)
     -n3*lafun(n1,l1,0,0,l2,k2,1+n3,-1+l3,0,<n4,0,0>,...,<n8,0,0>,k9) );
```

Mathematics of harmonic sums

- Harmonic sums $S_j(N)$ Gonzalez-Arroyo, Lopez, Ynduráin ‘79 ; Vermaseren ‘98 ; Blümlein, Kurth ‘98
  - recursive definition $S_{m_1,\ldots,m_k}(N) = \sum_{i=1}^{N} \frac{1}{i^{m_1}} S_{m_2,\ldots,m_k}(i)$
- Algebra of multiplication $S_j(N)S_k(N) \longrightarrow S_{\{j,k\}}(N)$
- Harmonic sums originate from :
  - expansion of Γ-functions in powers of ε
  - solutions of recursion relations
  - sums of type : $\sum_{i=1}^{N}(-1)^i \binom{N}{i} \frac{1}{i^3} = -S_{1,1,1}(N)$
- Multiple polylogarithms in x-space

  Goncharov ‘98 ; Borwein, Bradley, Broadhurst, Lisonek ‘99 ; Remiddi, Vermaseren ‘99

Recursion relations and algebra of harmonic sums $\longrightarrow$ breakthrough in technology

Systematic study of nested sums

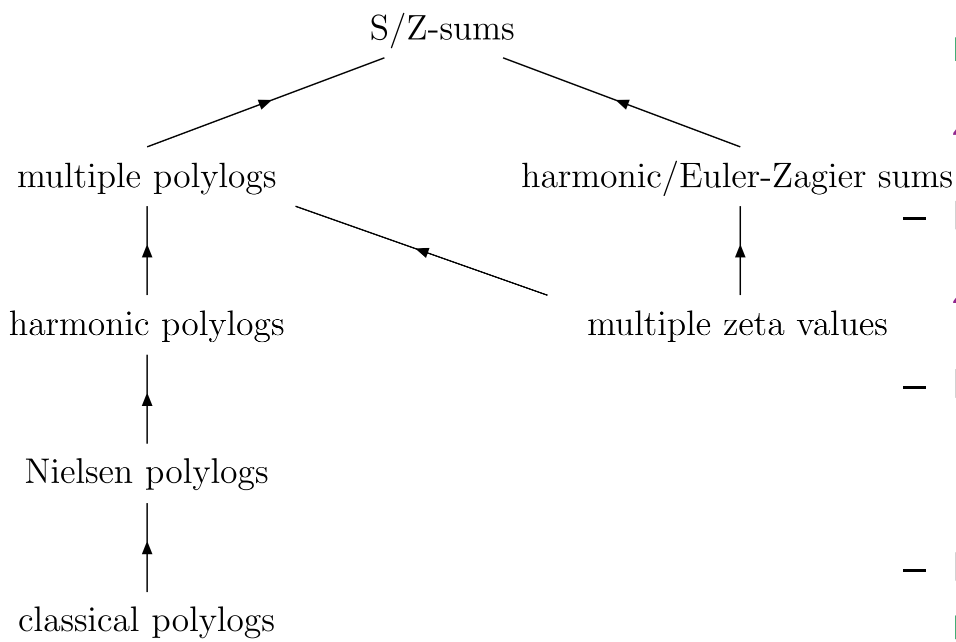
S.M., Uwer, Weinzierl ‘01

– Nested S-sums with multiple scales $x_1, \ldots, x_k$, depth k, weight $w = m_1 + \ldots + m_k$

$$S(n; m_1, \ldots, m_k; x_1, \ldots, x_k) \quad = \quad \sum_{i=1}^{n} \frac{x_1^i}{i^{m_1}} S(i; m_2, \ldots, m_k; x_2, \ldots, x_k)$$

Inheritance chart

S.M., Uwer, Weinzierl ‘01



Special cases

– Multiple polylogarithms for $n = \infty$

Goncharov ‘98 ;

Borwein, Bradley, Broadhurst, Lisonek ‘99

$$Z(\infty; m_1, \ldots, m_k; x_1, \ldots, x_k) = \mathrm{Li}_{m_k, \ldots, m_1}(x_k, \ldots, x_1)$$

– Euler-Zagier sums for $x_1 = \ldots = x_k = 1$

$$Z(n; m_1, \ldots, m_k; 1, \ldots, 1) = Z_{m_1, \ldots, m_k}(n)$$

– Nielsen’s generalized polylogarithms

$$S_{n,p}(x) = \mathrm{Li}_{1,\ldots,1,n+1}(\underbrace{1, \ldots, 1}_{p-1}, x),$$

– Harmonic polylogarithms

Remiddi, Vermaseren ‘99

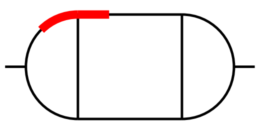
$$H_{m_1, \ldots, m_k}(x) = \mathrm{Li}_{m_k, \ldots, m_1}(\underbrace{1, \ldots, 1}_{k-1}, x)$$

Key aspects of technology

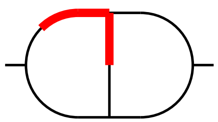
– Solve loop integrals via reductions and difference equations $\longrightarrow$ analytical solutions
– Use algebraic properties of harmonic sums $\longrightarrow$ algorithmic solution of nested sums
– **Check efficienctly** $\longrightarrow$ compute fixed values of N
– Use very flexible and highly optimized FORM programs

Computational complexity

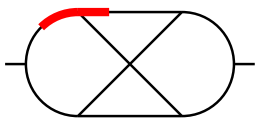
– Complexity with respect to complete two-loop calculation of F_2, F_3 and F_L S.M., Vermaseren '99
$\longrightarrow$ **three orders of magnitude**



– done in 2000 on Pentium III of 700 MHz



– done in 2001 on equivalents of Pentium III of 1.3 GHz with tabulation of integrals



– done in 2003 on Pentium IV of 3.06 GHz with massive tabulation of integrals
$\longrightarrow$ **3 GBytes of tables**

– Sizable extensions of capabilities of computer algebra system FORM Vermaseren '89–'03

Anomalous dimensions in Mellin space

– One-loop : Gross, Wilczek ’73

$$\gamma_{\rm ns}^{(0)}(N) \quad = \quad C_F\big(2(\mathbf{N}_- + \mathbf{N}_+)S_1 - 3\big)$$

– Two-loop : Floratos, Ross, Sachrajda ‘79 ; Gonzalez-Arroyo, Lopez, Ynduráin ‘79

$$\begin{aligned}\gamma_{\rm ns}^{(1)}(N) \;=\; & 4C_AC_F\left(2\,\mathbf{N}_+S_3 - \frac{17}{24} - 2S_{-3} - \frac{28}{3}S_1 + (\mathbf{N}_- + \mathbf{N}_+)\left[\frac{151}{18}S_1 + 2S_{1,-2} - \frac{11}{6}S_2\right]\right) \\ & + 4C_Fn_f\left(\frac{1}{12} + \frac{4}{3}S_1 - (\mathbf{N}_- + \mathbf{N}_+)\left[\frac{11}{9}S_1 - \frac{1}{3}S_2\right]\right) + 4C_F{}^2\left(4S_{-3} + 2S_1 + 2S_2 - \frac{3}{8}\right. \\ & \left. + \mathbf{N}_-\left[S_2 + 2S_3\right] - (\mathbf{N}_- + \mathbf{N}_+)\left[S_1 + 4S_{1,-2} + 2S_{1,2} + 2S_{2,1} + S_3\right]\right)\end{aligned}$$

– Compact notation : $\mathbf{N}_\pm\, f(N) \;=\; f(N \pm 1)\;,\qquad \mathbf{N}_{\pm\mathbf{i}}\, f(N) \;=\; f(N \pm i)$

– Three-loop : fermionic contributions to nonsinglet anomalous dimension

S.M., Vermaseren, Vogt '02

$$\begin{aligned}
\gamma_{\rm ns}^{(2)}(N) \;=\; & 16 C_A C_F n_f \Big(\frac{3}{2}\zeta_3 - \frac{5}{4} + \frac{10}{9}S_{-3} - \frac{10}{9}S_3 + \frac{4}{3}S_{1,-2} - \frac{2}{3}S_{-4} + 2S_{1,1} - \frac{25}{9}S_2 + \frac{257}{27}S_1 \\
& - \frac{2}{3}S_{-3,1} - \mathbf{N}_+\Big[S_{2,1} - \frac{2}{3}S_{3,1} - \frac{2}{3}S_4\Big] + (1-\mathbf{N}_+)\Big[\frac{23}{18}S_3 - S_2\Big] - (\mathbf{N}_- + \mathbf{N}_+)\Big[S_{1,1} + \frac{1237}{216}S_1 \\
& + \frac{11}{18}S_3 - \frac{317}{108}S_2 + \frac{16}{9}S_{1,-2} - \frac{2}{3}S_{1,-2,1} - \frac{1}{3}S_{1,-3} - \frac{1}{2}S_{1,3} - \frac{1}{2}S_{2,1} - \frac{1}{3}S_{2,-2} + S_1\zeta_3 + \frac{1}{2}S_{3,1}\Big]\Big) \\
& + 16 C_F {n_f}^2 \Big(\frac{17}{144} - \frac{13}{27}S_1 + \frac{2}{9}S_2 + (\mathbf{N}_- + \mathbf{N}_+)\Big[\frac{2}{9}S_1 - \frac{11}{54}S_2 + \frac{1}{18}S_3\Big]\Big) + 16 {C_F}^2 n_f \Big(\frac{23}{16} - \frac{3}{2}\zeta_3 \\
& + \frac{4}{3}S_{-3,1} - \frac{59}{36}S_2 + \frac{4}{3}S_{-4} - \frac{20}{9}S_{-3} + \frac{20}{9}S_1 - \frac{8}{3}S_{1,-2} - \frac{8}{3}S_{1,1} - \frac{4}{3}S_{1,2} + \mathbf{N}_+\Big[\frac{25}{9}S_3 - \frac{4}{3}S_{3,1} \\
& - \frac{1}{3}S_4\Big] + (1-\mathbf{N}_+)\Big[\frac{67}{36}S_2 - \frac{4}{3}S_{2,1} + \frac{4}{3}S_3\Big] + (\mathbf{N}_- + \mathbf{N}_+)\Big[S_1\zeta_3 - \frac{325}{144}S_1 - \frac{2}{3}S_{1,-3} + \frac{32}{9}S_{1,-2} \\
& - \frac{4}{3}S_{1,-2,1} + \frac{4}{3}S_{1,1} + \frac{16}{9}S_{1,2} - \frac{4}{3}S_{1,3} + \frac{11}{18}S_2 - \frac{2}{3}S_{2,-2} + \frac{10}{9}S_{2,1} + \frac{1}{2}S_4 - \frac{2}{3}S_{2,2} - \frac{8}{9}S_3\Big]\Big)
\end{aligned}$$

Splitting functions in x-space

$$
\begin{aligned}
P_{\rm ns}^{(2)}(x) \;=\; & 16 C_A C_F n_f \Big(\mathbf{p_{qq}(x)} \Big[\mathbf{\tfrac{5}{9}\zeta_2 - \tfrac{209}{216} - \tfrac{3}{2}\zeta_3 + \tfrac{1}{3}{\rm Li}_3(x)} - \frac{167}{108}\ln(x) + \frac{1}{3}\ln(x)\zeta_2 - \frac{1}{4}\ln^2(x)\ln(1-x) \\
& - \frac{7}{12}\ln^2(x) - \frac{1}{18}\ln^3(x) - \frac{1}{2}\ln(x){\rm Li}_2(x) \Big] + p_{qq}(-x) \Big[\frac{1}{2}\zeta_3 - \frac{5}{9}\zeta_2 - \frac{2}{3}\ln(1+x)\zeta_2 + \frac{1}{6}\ln(x)\zeta_2 - \frac{10}{9}\ln(x)\ln(1+x) \\
& + \frac{5}{18}\ln^2(x) - \frac{1}{6}\ln^2(x)\ln(1+x) + \frac{1}{18}\ln^3(x) - \frac{10}{9}{\rm Li}_2(-x) - \frac{1}{3}{\rm Li}_3(-x) - \frac{1}{3}{\rm Li}_3(x) + \frac{2}{3}{\rm H}_{-1,0,1}(x) \Big] \\
& + (1+x)\Big[\frac{1}{6}\zeta_2 + \frac{1}{2}\ln(x) - \frac{1}{2}{\rm Li}_2(x) - \frac{2}{3}{\rm Li}_2(-x) - \frac{2}{3}\ln(x)\ln(1+x) + \frac{1}{24}\ln^2(x) \Big] + (1-x)\Big[\frac{1}{3}\zeta_2 - \frac{257}{54} \\
& + \ln(1-x) - \frac{17}{9}\ln(x) - \frac{1}{24}\ln^2(x) \Big] + \delta(1-x)\Big[\frac{5}{4} - \frac{167}{54}\zeta_2 + \frac{1}{20}{\zeta_2}^2 + \frac{25}{18}\zeta_3 \Big] \Big) + 16 C_F {n_f}^2 \Big(\mathbf{p_{qq}(x)} \Big[\mathbf{-\tfrac{1}{54}} \\
& + \frac{5}{54}\ln(x) + \frac{1}{36}\ln^2(x) \Big] + (1-x)\Big[\frac{13}{54} + \frac{1}{9}\ln(x) \Big] - \delta(1-x)\Big[\frac{17}{144} - \frac{5}{27}\zeta_2 + \frac{1}{9}\zeta_3 \Big] \Big) + 16 {C_F}^2 n_f \Big(\mathbf{p_{qq}(x)} \Big[\mathbf{\tfrac{5}{3}\zeta_3 - \tfrac{55}{48}} \\
& \mathbf{- \tfrac{2}{3}{\rm Li}_3(x)} + \frac{5}{24}\ln(x) + \frac{1}{3}\ln(x)\zeta_2 + \frac{10}{9}\ln(x)\ln(1-x) + \frac{1}{4}\ln^2(x) + \frac{2}{3}\ln^2(x)\ln(1-x) + \frac{2}{3}\ln(x){\rm Li}_2(x) - \frac{1}{18}\ln^3(x) \Big] \\
& + p_{qq}(-x)\Big[\frac{10}{9}\zeta_2 - \zeta_3 + \frac{4}{3}\ln(1+x)\zeta_2 - \frac{1}{3}\ln(x)\zeta_2 - \frac{5}{9}\ln^2(x) + \frac{20}{9}\ln(x)\ln(1+x) - \frac{1}{9}\ln^3(x) + \frac{1}{3}\ln^2(x)\ln(1+x) \\
& + \frac{20}{9}{\rm Li}_2(-x) + \frac{2}{3}{\rm Li}_3(-x) + \frac{2}{3}{\rm Li}_3(x) - \frac{4}{3}{\rm H}_{-1,0,1}(x) \Big] + (1+x)\Big[\frac{7}{36}\ln^2(x) - \frac{67}{72}\ln(x) + \frac{4}{3}\ln(x)\ln(1+x) \\
& + \frac{1}{12}\ln^3(x) + \frac{2}{3}{\rm Li}_2(x) + \frac{4}{3}{\rm Li}_2(-x) \Big] + (1-x)\Big[\frac{1}{9}\ln(x) - \frac{10}{9} - \frac{4}{3}\ln(1-x) + \frac{2}{3}\ln(x)\ln(1-x) - \frac{1}{3}\ln^2(x) \Big] \\
& - \delta(1-x)\Big[\frac{23}{16} - \frac{5}{12}\zeta_2 - \frac{29}{30}{\zeta_2}^2 + \frac{17}{6}\zeta_3 \Big] \Big)
\end{aligned}
$$

Easy-to-use parametrization

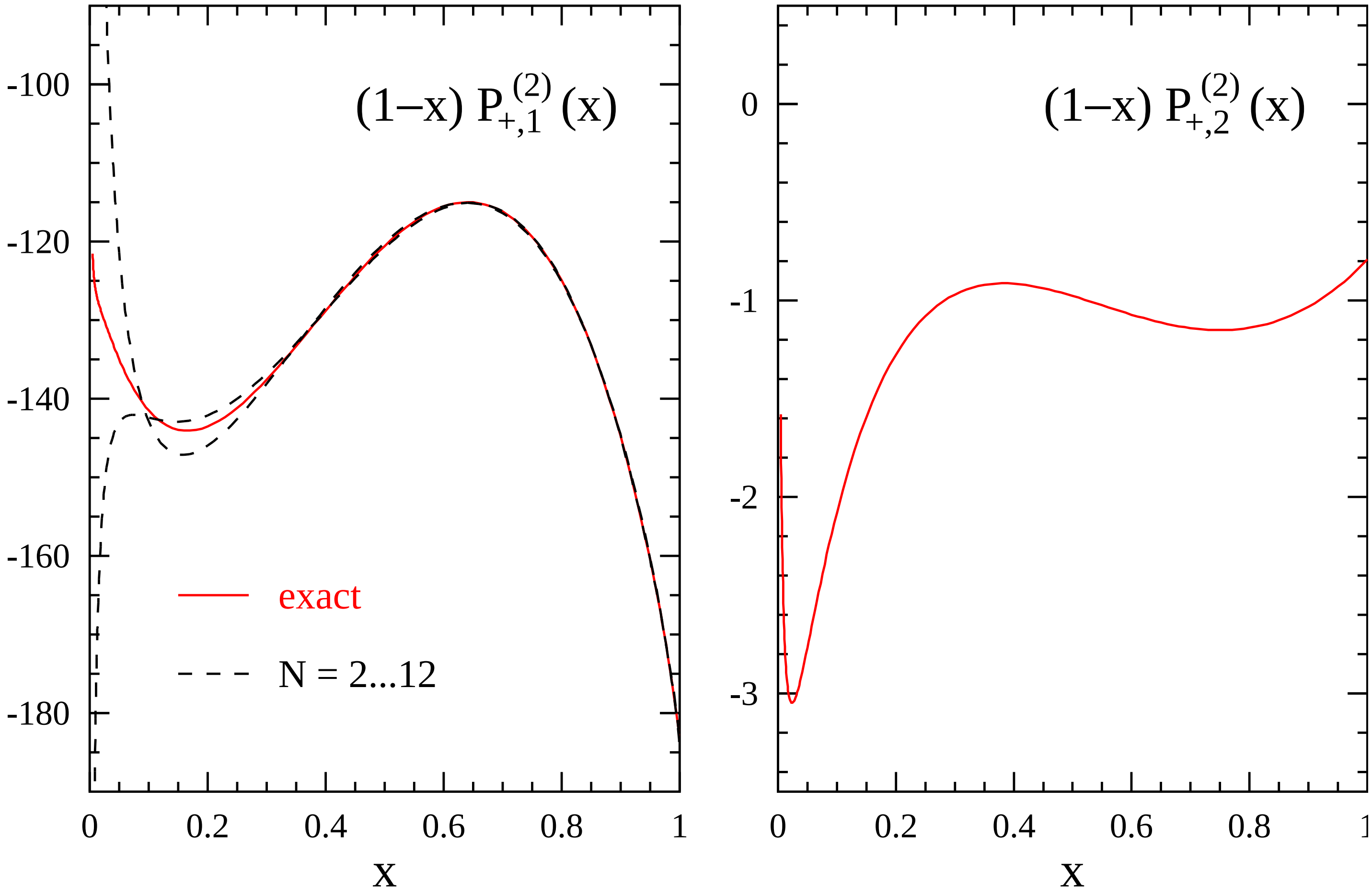
– Combine exact limits for $x \longrightarrow 0$ and $x \longrightarrow 1$ with smooth fit for intermediate x

– Notation : end-point logarithms $L_0 = \ln(x)$, $L_1 = \ln(1-x)$, +-distributions $\mathcal{D}_i = \left[\frac{\ln^i(1-x)}{1-x}\right]_+$

$$
\begin{aligned}
P_{\rm ns}^{(2)}(x) \quad \cong \quad & n_f \Big(-183.187\, \mathcal{D}_0 - 173.927\, \delta(1-x) - 5120/81\, L_1 - 197.0 + 381.1\, x + 72.94\, x^2 \\
& \quad + 44.79\, x^3 - 1.497\, x L_0^3 - 56.66\, L_0 L_1 - 152.6\, L_0 - 2608/81\, L_0^2 - 64/27\, L_0^3 \Big) \\
+ \quad & n_f^2 \Big(- \mathcal{D}_0 - (51/16 + 3\, \zeta_3 - 5\, \zeta_2)\, \delta(1-x) + x\, (1-x)^{-1} L_0\, (3/2\, L_0 + 5) + 1 \\
& \quad + (1-x)\, (6 + 11/2\, L_0 + 3/4\, L_0^2) \Big)\; 64/81
\end{aligned}
$$

Comparison with estimates from fixed moments

van Neerven, Vogt ‘00



Three-loop coefficient functions

– OPE and optical theorem
$\longrightarrow$ obtain simultaneously anomalous dimension $\gamma_{\rm ns}(N)$ and coefficient function $C^N_{2,\rm ns}$ or $C^N_{L,\rm ns}$

– Coefficient function $C^N_{2,\rm ns}$ at three loops
$\longrightarrow$ numerically the most relevant part of **NNNLO** corrections

– **Longitudinal** coefficient function $C^N_{L,\rm ns}$ at three loops
$\longrightarrow$ needed for **NNLO** analysis of $R = \sigma_L/\sigma_T$

– Easy-to-use parametrizations

$$
\begin{aligned}
c^{(3)}_{L,\rm ns}(x) \quad \cong \quad & n_f \,\big(1024/81\, L_1^3 - 112.4\, L_1^2 + 340.3\, L_1 + 409 - 210\, x - 762.6\, x^2 - 1792/81\, x L_0^3 \\
& \quad + L_0 L_1\, (969.2 + 304.8\, L_0 - 288.2\, L_1) + 200.8\, L_0 + 64/3\, L_0^2 + 0.046\, \delta(1-x)\big) \\
+ \quad & n_f^2 \,\big(3x\, L_1^2 + (6 - 25\, x)\, L_1 - 19 + (317/6 - 12\, \zeta_2)\, x - 6x\, L_0 L_1 + 6x\, \mathrm{Li}_2(x) \\
& \quad + 9x\, L_0^2 - (6 - 50\, x)\, L_0\,\big)\; 64/81
\end{aligned}
$$

$$
\begin{aligned}
c^{(3)}_{2,\mathrm{ns}}(x) \quad\cong\quad & n_f\,\Big(640/81\,\mathcal{D}_4 - 6592/81\,\mathcal{D}_3 + 220.573\,\mathcal{D}_2 + 294.906\,\mathcal{D}_1 - 729.359\,\mathcal{D}_0 \\
& \quad + 2572.597\,\delta(1-x) - 640/81\,L_1^4 + 167.2\,L_1^3 - 315.3\,L_1^2 + 4742\,L_1 \\
& \quad + 762.1 + 7020\,x + 989.4\,x^2 + L_0L_1\,(326.6 + 65.93\,L_0 + 1923\,L_1) \\
& \quad + 260.1\,L_0 + 186.5\,L_0^2 + 12224/243\,L_0^3 + 728/243\,L_0^4\Big) \\
+ \quad & n_f^2\,\Big(64/81\,\mathcal{D}_3 - 464/81\,\mathcal{D}_2 + 7.67505\,\mathcal{D}_1 + 1.00830\,\mathcal{D}_0 - 103.2655\,\delta(1-x) \\
& \quad - 64/81\,L_1^3 + 15.46\,L_1^2 - 51.71\,L_1 + 59.00\,x + 70.66\,x^2 + L_0L_1\,(-80.05 \\
& \quad - 10.49\,L_0 + 41.67\,L_1) - 8.050\,L_0 - 1984/243\,L_0^2 - 368/243\,L_0^3\Big)
\end{aligned}
$$

Applications

– Coefficient function $C_{2,\mathrm{ns}}$ for $N \longrightarrow \infty$ cf. $x \longrightarrow 1$
$\longrightarrow$ large double logarithmic corrections from soft and collinear regions in Feynman diagrams

$$\alpha_s^l \left[\frac{\ln^{2l-1}(1-x)}{1-x}\right]_+ \qquad \longleftrightarrow \qquad \alpha_s^l \ln^{2l}(N)$$

– Use **factorization** properties in soft/collinear limit $\longrightarrow$ **resummation** of large logarithms
Collins, Soper '81 ; Sterman '87 ; Catani, Trentadue '89 ; Magnea, Sterman '90 ; Catani, Webber '98 ; etc. . . .

$$C_{2,\mathrm{ns}}(\alpha_s, N) = g(Q^2)\exp[G_{\mathrm{DIS}}(\alpha_s, N)] =$$

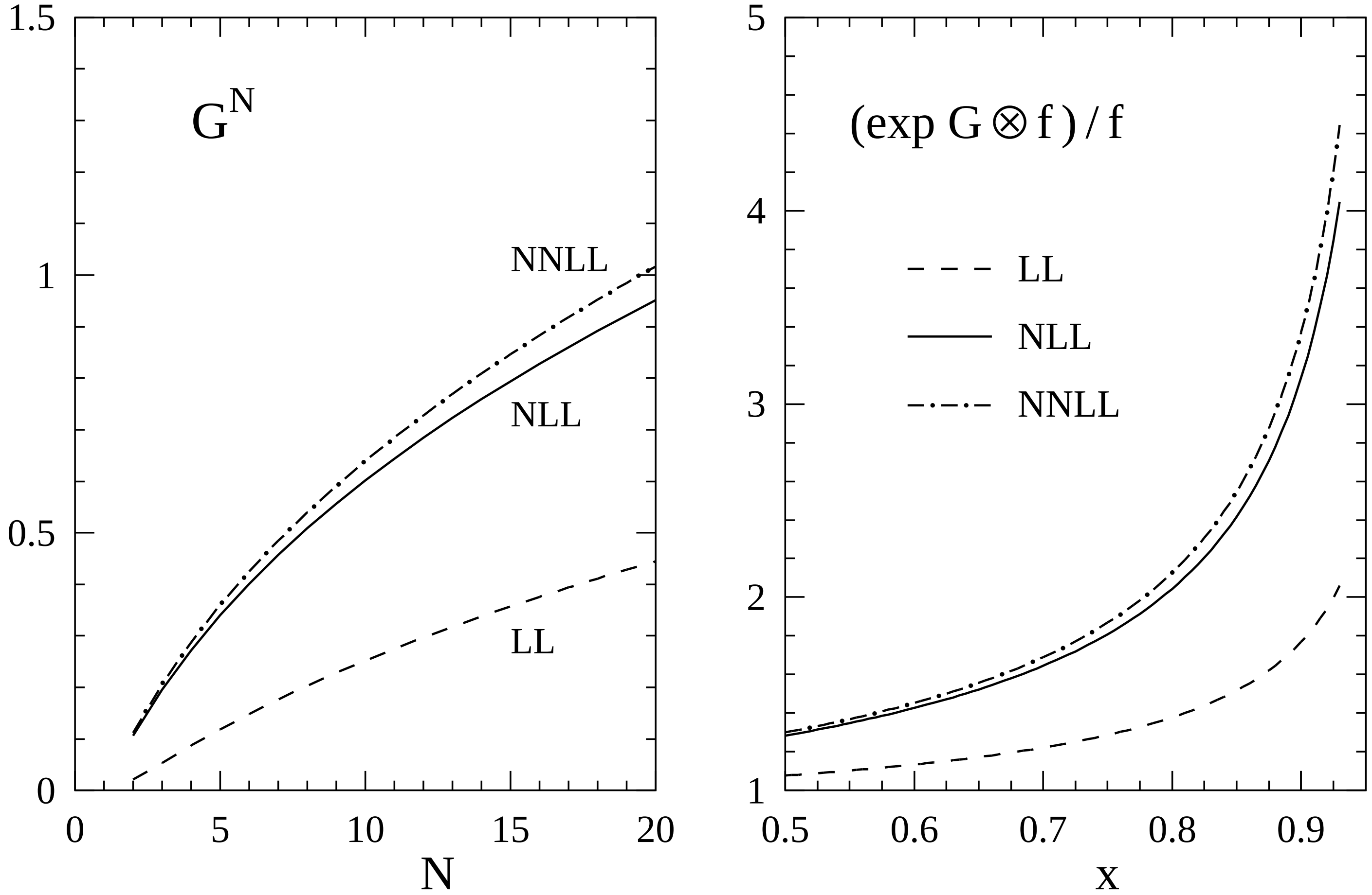
$$(1+\alpha_s g_{01} + \alpha_s^2 g_{02} + \ldots)\exp\left[\ln(N)\, g_1(\alpha_s \ln(N)) + g_2(\alpha_s \ln(N)) + \alpha_s g_3(\alpha_s \ln(N)) + \ldots\right]$$

– Resumming to NNLL accuracy requires $g_3 \longrightarrow$ new coefficients A_3, B_2 and D_2^{DIS}
  – matching : n_f-terms in A_3 from $\gamma_{\mathrm{ns}}^{(2)}$ and B_2, D_2^{DIS} from $c_{2,\mathrm{ns}}^{(3)}$
  – independent check on A_3 Berger '02

$$A_3 = (1178.8 \pm 11.5) - 183.18743 n_f - 0.79012 n_f^2$$

$$B_2 = 36.26570 + 6.34888 n_f \qquad\qquad D_2^{\mathrm{DIS}} = 0$$

Numerical analysis



– **LL, NLL, NNLL resummed** exponent $G^N(Q^2)$ for $\mu = Q$, $n_f = 4$ and $\alpha_s = 0.2$

– $G^N(Q^2)$ convoluted with typical input shape $xf = x^{1/2}(1-x)^3$

Upshot

– Large perturbative corrections to structure functions for $x \longrightarrow 1$

– Use **NNLO + NNLL resummed** perturbative QCD

– Investigate implications for higher twist
  – common phenomenological ansatz

$$F_2^{\mathrm{DATA}} = F_2^{\mathrm{QCD}} \left(1 + \frac{ht(x)}{Q^2}\right)$$

  – ansatz mixes $1/Q^2$-corrections with leading twist perturbative QCD corrections

Summary

What do we want ?

- **NNLO** analysis of deep-inelastic structure functions $F_2, F_3 \longrightarrow$ **high precision**
  - match experimental accuracy in final HERA data
  - provide precise parton distributions for LHC

What do we learn ?

- Mellin moments and nested sums $\longrightarrow$ **powerful technology**
  - apply innovative and efficient method to solve multi-loop integrals

- Formalism with wide range of applications
  - get photon structure function at order $O(\alpha\alpha_s^2)$ and other distributions at three loops

What do we get (in the end) ?

- Calculation of deep-inelastic structure functions F_2, F_3 and F_L at **NNLO** (and beyond)
  - nonsinglet n_f-terms done S.M., Vermaseren, Vogt '02
  - more results on disk and completion of calculation under way

- Phenomenology for deep-inelastic scattering and hard hadronic interactions
  - reach **new level of precision**