Mathematical Structure of QCD Wilson Coefficients and Anomalous Dimensions

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DESY and KITP

1. **Introduction**
2. **x Space Representations**
3. **The Mellin Symmetry**
4. **Multiple Zeta Values**
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7. **Deeper Relations**
8. **Evolution**

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HARMONY IN PHYSICS:

AN OLD GRAZ-TOPIC !

J. KEPLER

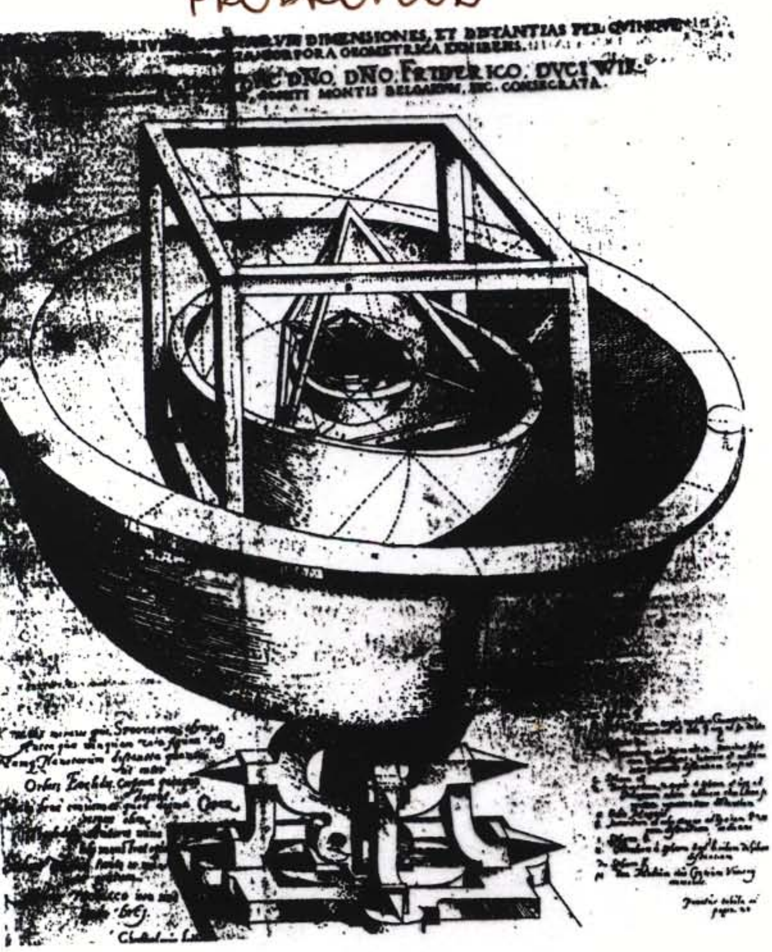
MODERN VERSION:

FEYNMAN DIAGRAMS

$\downarrow$

$\boxed{\text{HARMONIC}}$ SUMS

PRODROMUS



Frontispiece to Johannes Kepler's *Mysterium cosmographicum*

Ut h concordet		In tensione gravissima.	Acutissima.
☿	dø²	379' 20"	
	h ⁷	284 32	295' 56"
	g ⁶	237 4	244 4
	dø⁶	189 40	195 14
♀	dø⁵	94 50	97 37
♁	g⁴	59 16	61 1
	b⁴	35 35	36 37
♂	g³	29 38	30 31
♃	b¹		4 35
♄	b	2 13	
	G	3 51	1 55

Ut c concordet.		In tensione gravissima.	Acutissima.
☿	dø⁷	379' 20"	
	c ⁷	316 5	325' 26"
	g ⁶	237 4	244 4
	dø⁶	189 40	195 14
	c ⁶		162 43
♀	dø⁵	94 50	97 37
♁	g⁴	59 16	61 1
♂	g³	29 38	30 31
♃	c¹	4 56	5 5
♄	G	3 51	1 55

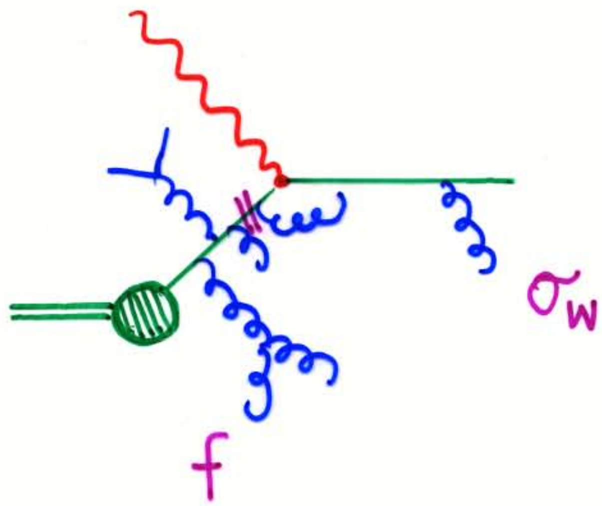
Nam si semidiameter orbis figurae circumscripti, quae est communiter 100000,	Tunc semid. inscripti flat	ex	pro	♄ fiet	Cum sit intervallum ex harmoniis.		
In cubo	8994	♄	57735	5194	Medium	♃	5206
In tetraëdro	4948	♃	33333	1649	aphelium	♂	1661
In dodecaëdro . . .	1384	♂	79465	1100	aphelium	♁	1018
In icosiedro	983	♁	79465	781	aphelium	♀	726
In echino	1384	♂	52573	728	aphelium	♀	726
In octaëdro	716	♀	57735	413	medium	☿	392
In quadrato octaëdri .	716	♀	70711	506	aphelium	☿	476
vel	476	☿	70711	336	perihelium	☿	308

1. Introduction

- STUDY OF MASSLESS FIELD THEORIES

QCD, QED $m_i \to 0$

"SIMPLE" PHASE SPACE(S)



$$\sigma = \sigma_W \otimes f$$

$\uparrow$

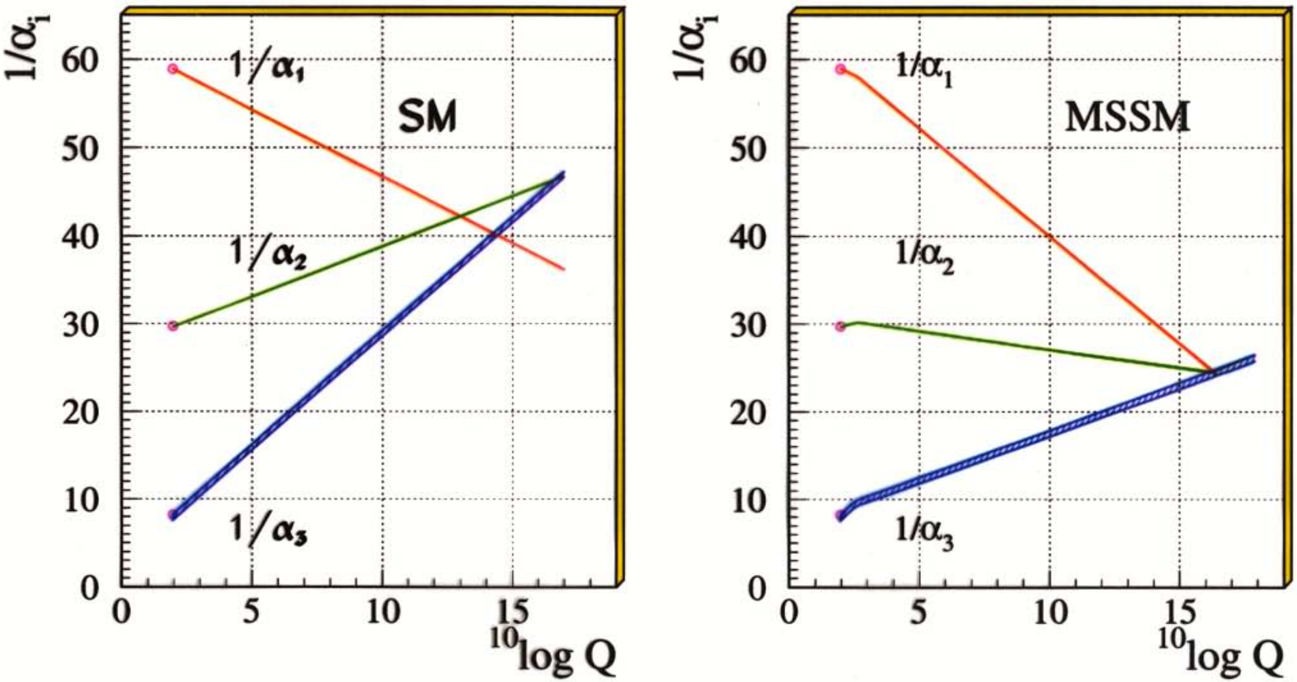
$\boxed{\text{MELLIN CONVOLUTION}}$

σ_W WILSON COEFFICIENT

f PARTON DENSITY

BOTH RENORMALIZED.

Unification of the Coupling Constants in the SM and the minimal MSSM



CURRENTLY : $\Delta\alpha_s(M_Z^2)_{TH} = \pm 5\%$

de Boer '02

WANTED : 1%

$\longrightarrow$ QCD @ 3LOOPS

MAJOR GOALS:

- ANALYZE QCD SCALING VIOLATIONS
- UNFOLD PARTON DENSITIES

FOR UNPOLARIZED & POLARIZED PROTONS

- PRECISION MEASUREMENTS OF α_s & Λ_{QCD}
- SEARCH FOR NEW QCD PHENOMENA

(E.G.: SMALL x ?)

EVOLUTION EQUS.

NS:
$$\frac{\partial \hat{F}_i^{NS}(N, a_s)}{\partial \log Q^2} = P_{NS}^{(i)}(a_s, N) \cdot \hat{F}_i^{NS}(N, a_s)$$

S:
$$\frac{\partial}{\partial \log Q^2} \begin{pmatrix} \Sigma \\ G \end{pmatrix} = \underbrace{\mathbb{P}_S(a_s)}_{\text{MATRIX}} \cdot \begin{pmatrix} \Sigma \\ G \end{pmatrix}$$

$$P^{(i)}(a_s, N) = a_s P_1^{(i)}(N) + a_s^2 P_2^{(i)} + \underbrace{a_s^3 P_3^{(i)}}_{\text{Needed.}} + \ldots$$

2. x–Space Representation

USUAL STARTING POINT OF HIGHER ORDER CALCULATIONS:

→ NIELSEN TYPE INTEGRALS

$$S_{n,p}(x) = \frac{(-1)^{n+p-1}}{(n-1)!\,p!} \int_0^1 \frac{dz}{z} \ln^{n-1}(z) \ln^p(1-zx)$$

OR OUR GENERALIZATION:

$$S_{n,p,q}(x) = \frac{(-1)^{n+p+q-1}}{(n-1)!\,p!\,q!} \int_0^1 \frac{dz}{z} \ln^{n-1}(z) \ln^p(1-zx) \ln^q(1+xz)$$

SPECIAL CASES:

$$\mathrm{Li}_n(x) = S_{n-1,1}(x)$$ WEIGHT n

$$\frac{d\,\mathrm{Li}_2(\pm x)}{d\ln(x)} = -\ln(1 \mp x)$$ WEIGHT 1

$$\mathrm{Li}_0(x) = \frac{x}{1-x}$$ WEIGHT 0

$$\frac{dx}{1 \pm x}, \quad \frac{dx}{x}$$ WEIGHT 1

WHAT IS $C^{(2)}_{2,+}(x)^{-1}$?

VAN NEERVEN, ZIJLSTRA 1992

$$
\begin{aligned}
c^{(2)}_{2,+}(x) \;=\; & C_F^2\Big\{\frac{1+x^2}{1-x}\Big[4\ln^3(1-x)-(14\ln(x)+9)\ln^2(1-x)-\frac{4}{3}\ln^3(x)-\frac{3}{2}\ln^2(x) \\
& -\Big[4\mathrm{Li}_2(1-x)-12\ln^2(x)-12\ln(x)+16\zeta(2)+\frac{27}{2}\Big]\ln(1-x)+48\mathrm{Li}_3(-x) \\
& +\Big[-24\mathrm{Li}_2(-x)+24\zeta(2)+\frac{61}{2}\Big]\ln(x)+12\mathrm{Li}_3(1-x)-12\mathrm{S}_{1,2}(1-x) \\
& +48\mathrm{Li}_3(-x)-6\mathrm{Li}_2(1-x)+32\zeta(3)+18\zeta(2)+\frac{51}{4}\Big] \\
& +(1+x)\Big[2\ln(x)\ln^2(1-x)+4\left[\mathrm{Li}_2(1-x)-\ln^2(x)\right]\ln(1-x)+\frac{5}{3}\ln^3(x) \\
& -4\mathrm{Li}_3(1-x)-4\left[\mathrm{Li}_2(1-x)+\zeta(2)\right]\ln(x)\Big]+\Big(40+8x-48x^2-\frac{72}{5}x^3+\frac{8}{5x^2}\Big) \\
& \times\left[\mathrm{Li}_2(-x)+\ln(x)\ln(1+x)\right]+(5+9x)\ln^2(1-x)+\frac{1}{2}(-91+141x)\ln(1-x) \\
& +(-8+40x)\left[\ln(x)\mathrm{Li}_2(-x)+\mathrm{S}_{1,2}(1-x)-2\mathrm{Li}_3(-x)-\zeta(2)\ln(1-x)\right] \\
& -(28+44x)\ln(x)\ln(1-x)-(14+30x)\mathrm{Li}_2(1-x)+\Big(\frac{29}{2}+\frac{25}{2}x+24x^2+\frac{36}{3}x^3\Big) \\
& \times\ln^2(x)+\frac{1}{10}\Big(13-407x+144x^2-\frac{16}{x}\Big)\ln(x)+\Big(-10+6x-48x^2-\frac{72}{5}x^3\Big)\zeta(2) \\
& +\frac{407}{20}-\frac{1917}{20}x+\frac{72}{5}x^2+\frac{8}{5x}+\Big[6\zeta^2(2)-78\zeta(3)+69\zeta(2)+\frac{331}{8}\Big]\delta(1-x)\Big\} \\
& +C_AC_F\Big\{\frac{1+x^2}{1-x}\Big[-\frac{11}{3}\ln^2(1-x)+\Big[4\mathrm{Li}_2(1-x)+2\ln^2(x)+\frac{44}{3}\ln(x)-4\zeta(2) \\
& +\frac{367}{18}\Big]\ln(1-x)-\ln^3(x)-\frac{55}{6}\ln^2(x)+\Big[4\mathrm{Li}_2(1-x)+12\mathrm{Li}_2(-x) \\
& -\frac{239}{6}\Big]\ln(x)-12\mathrm{Li}_3(1-x)+12\mathrm{S}_{1,2}(1-x)-24\mathrm{Li}_3(-x)+\frac{22}{3}\mathrm{Li}_2(1-x)+2\zeta(3) \\
& +\frac{22}{3}\zeta(2)-\frac{3155}{108}\Big]+4(1+x)\left[\mathrm{Li}_2(1-x)+\ln(x)\ln(1-x)\right] \\
& +\Big(-20-4x+24x^2+\frac{36}{5}x^3-\frac{4}{5x^2}\Big)\left[\mathrm{Li}_2(-x)+\ln(x)\ln(1+x)\right] \\
& +(4-20x)\left[\ln(x)\mathrm{Li}_2(-x)+\mathrm{S}_{1,2}(1-x)-2\mathrm{Li}_3(-x)-\zeta(2)\ln(1-x)\right] \\
& +\Big(\frac{133}{6}-\frac{1113}{18}x\Big)\ln(1-x)+\Big(-2+2x-12x^2-\frac{18}{5}x^3\Big)\ln^2(x) \\
& +\frac{1}{30}\Big(13+1753x-216x^2+\frac{24}{x}\Big)\ln(x)+\Big(-2-10x+24x^2+\frac{36}{5}x^3\Big)\zeta(2) \\
& -\frac{9687}{540}+\frac{59157}{540}-\frac{36}{5}x^2-\frac{4}{5x} \\
& +\Big[\frac{71}{5}\zeta^2(2)+\frac{140}{3}\zeta(3)-\frac{251}{3}\zeta(2)-\frac{5465}{72}\Big]\delta(1-x)\Big\} \\
& +C_FN_F\Big\{\frac{1+x^2}{1-x}\Big[\frac{2}{3}\ln^2(1-x)-\Big(\frac{8}{3}\ln(x)+\frac{29}{9}\Big)\ln(1-x)-\frac{4}{3}\mathrm{Li}_2(1-x)+\frac{5}{3}\ln^2(x) \\
& +\frac{19}{3}\ln(x)-\frac{4}{3}\zeta(2)+\frac{247}{54}\Big]+\frac{1}{3}(1+13x)\ln(1-x)-\frac{1}{3}(7+19x)\ln(x)-\frac{23}{18}-\frac{27}{2}x
\end{aligned}
$$

$$+\left[\frac{4}{3}\zeta(3)+\frac{38}{3}\zeta(2)+\frac{457}{36}\right]\delta(1-x)\bigg\}\,. \tag{1}$$

$$\begin{aligned}
c_{2,G}^{(2)}(x) \;=\; & C_F N_F\Big\{8(1+x)^2\left[-4\mathrm{S}_{1,2}(-x)-4\ln(1+x)\mathrm{Li}_2(-x)-2\zeta(2)\ln(1+x)\right.\\
&\left.-2\ln(x)\ln^2(1+x)+\ln^2(x)\ln(1+x)\right]+4(1-x)^2\left\{\frac{5}{6}\ln^3(1-x)\right.\\
&-\left(2\ln(x)+\frac{13}{4}\right)\ln^2(1-x)+\left[2\mathrm{Li}_2(1-x)+2\ln^2(x)+4\ln(x)+\frac{7}{2}\right]\ln(1-x)\\
&-\frac{5}{12}\ln^3(x)+\left[\mathrm{Li}_2(1-x)-4\mathrm{Li}_2(-x)+3\zeta(2)\right]\ln(x)-4\mathrm{Li}_3(1-x)-\mathrm{S}_{1,2}(1-x)\\
&\left.+12\mathrm{Li}_3(-x)+13\zeta(3)+\frac{13}{2}\zeta(2)\right\}+x^2\left\{\frac{10}{3}\ln^3(1-x)-12\ln(x)\ln^2(1-x)\right.\\
&+\left[16\ln^2(x)-16\zeta(2)\right]\ln(1-x)-5\ln^3(x)+\left[12\mathrm{Li}_2(1-x)+20\zeta(2)\right]\ln(x)\\
&\left.-8\mathrm{Li}_3(1-x)+12\mathrm{S}_{1,2}(1-x)\right\}+\left(48+\frac{64}{3}x+\frac{96}{5}x^3+\frac{8}{15x^2}\right)\\
&\times\left[\mathrm{Li}_2(-x)+\ln(x)\ln(1+x)\right]+(14x-23x^2)\ln^2(1-x)-(12x-10x^2)\ln(1-x)\\
&+(-24x+56x^2)\ln(x)\ln(1-x)+64x\mathrm{Li}_3(-x)+(-10+24x)\mathrm{Li}_2(1-x)\\
&+\left(-\frac{3}{2}+\frac{22}{3}x-36x^2-\frac{48}{5}x^3\right)\ln^2(x)+\frac{1}{15}\left(-236+339x-648x^2-\frac{8}{x}\right)\ln(x)\\
&\left.+(64x+36x^2)\zeta(3)+\left(-\frac{20}{3}+46x^2+\frac{96}{5}x^3\right)\zeta(2)-\frac{647}{15}+\frac{239}{5}x-\frac{36}{5}x^2+\frac{8}{15x^2}\right]\\
&+C_A N_F\Big\{4(1+x)^2\left[\mathrm{S}_{1,2}(1-x)-2\mathrm{Li}_3(-x)+4\mathrm{S}_{1,2}(-x)-2\ln(x)\mathrm{Li}_2(1-x)\right.\\
&+4\ln(1+x)\mathrm{Li}_2(-x)+2\ln(x)\mathrm{Li}_2(-x)+2\zeta(2)\ln(1+x)+2\ln(x)\ln^2(1+x)\\
&\left.+\ln^2(x)\ln(1+x)\right]+8(1+2x+2x^2)\left[\mathrm{Li}_3\left(\frac{1-x}{1+x}\right)-\mathrm{Li}_3\left(-\frac{1-x}{1+x}\right)\right.\\
&\left.-\ln(1-x)\mathrm{Li}_2(-x)-\ln(x)\ln(1-x)\ln(1+x)\right]+\left(-24+\frac{80}{3}x^2-\frac{16}{3x}\right)\\
&\times\left[\mathrm{Li}_2(-x)+\ln(x)\ln(1+x)\right]+x^2\left[-4\mathrm{S}_{1,2}(1-x)+16\mathrm{Li}_3(-x)+8\ln(x)\mathrm{Li}_2(1-x)\right.\\
&\left.+8\ln^2(x)\ln(1+x)\right]+\frac{2}{3}(1-2x+2x^2)\ln^3(1-x)+(24x-8x^2)\ln(x)\ln^2(1-x)\\
&+\left(-2+36x-\frac{122}{3}x^2+\frac{8}{3x}\right)\ln^2(1-x)+(-4-32x+8x^2)\ln^2x\ln(1-x)\\
&+(8-144x+148x^2)\ln(x)\ln(1-x)+(4+40x-8x^2)\ln(1-x)\mathrm{Li}_2(1-x)\\
&+(-20+24x-32x^2)\zeta(2)\ln(1-x)+\frac{1}{9}\left(-186-1362x+1570x^2+\frac{104}{x}\right)\ln(1-x)\\
&+(-4-72x+8x^2)\mathrm{Li}_3(1-x)+\frac{1}{3}\left(12-192x+176x^2+\frac{16}{x}\right)\mathrm{Li}_2(1-x)\\
&+\frac{1}{3}(10+28x)\ln^3(x)+\left(-1+88x-\frac{194}{3}x^2\right)\ln^2(x)+(-48x+16x^2)\zeta(2)\ln(x)\\
&+\left(58+\frac{584}{3}x-\frac{2090}{9}x^2\right)\ln(x)-(10+12x+12x^2)\zeta(3)\\
&+\frac{1}{3}\left(12-240x+268x^2-\frac{32}{x}\right)\zeta(2)+\frac{239}{9}+\frac{1072}{9}x-\frac{4493}{27}x^2+\frac{344}{27x}\Big\}
\end{aligned} \tag{2}$$

$$
\begin{aligned}
c^{(2)}_{2,-}(x) \;=\;& C_F\left(C_F - \frac{1}{2}C_A\right)\times \\
& \left\{\frac{1+x^2}{1-x}\left[\left[4\ln^2(x) - 16\ln(x)\ln(1+x) - 16\mathrm{Li}_2(-x) - 8\zeta(2)\right]\ln(1-x)\right.\right. \\
& + \left[-2\ln^2(x) + 20\ln(x)\ln(1+x) - 8\ln^2(1+x) + 8\mathrm{Li}_2(1-x) + 16\mathrm{Li}_2(-x) - 8\right]\ln(x) \\
& -16\ln(1+x)\mathrm{Li}_2(-x) - 8\zeta(2)\ln(1+x) - 16\mathrm{Li}_3\left(-\frac{1-x}{1+x}\right) \\
& \left.+16\mathrm{Li}_3\left(\frac{1-x}{1+x}\right) - 16\mathrm{Li}_3(1-x) + 8\mathrm{S}_{1,2}(1-x) + 8\mathrm{Li}_3(-x) - 16\mathrm{S}_{1,2}(-x) + 8\zeta(3)\right] \\
& +(4+20x)\left[\ln^2(x)\ln(1+x) - 2\ln(x)\ln^2(1+x) - 2\zeta(2)\ln(1+x) - 4\ln(1+x)\mathrm{Li}_2(-x)\right. \\
& \left.+2\mathrm{Li}_3(-x) - 4\mathrm{S}_{1,2}(-x) + 2\zeta(3)\right] + \left(32 + 32x + 48x^2 - \frac{72}{5}x^3 + \frac{8}{5x^2}\right) \\
& \times\left[\mathrm{Li}_2(-x) + \ln(x)\ln(1+x)\right] + 8(1+x)\left[\mathrm{Li}(1-x) + \ln(x)\ln(1-x)\right] + 16(1-x)\ln(1-x) \\
& + \left(-4 - 16x - 24x^2 + \frac{36}{5}x^3\right)\ln^2(x) + \frac{1}{5}\left(-26 - 106x + 72x^2 - \frac{8}{x}\right)\ln(x) \\
& \left. + \left(-4 + 20x + 48x^2 - \frac{72}{5}x^3\right)\zeta(2) + \frac{1}{5}\left(-162 + 82x + 72x^2 + \frac{8}{x}\right)\right\}. \qquad (3)
\end{aligned}
$$

→ 77 FUNCTIONS @ 2 LOOPS.

→ RATHER COMPLICATED ARGUMENTS

→ NOT MANY, IF ANY, RELATIONS

.....

J. BLÜMLEIN, DESY

KEY PROBLEMS:

- 2 LOOP WILSON COEFFICIENTS DEPEND ON ~ 80 FUNCTIONS
- 3 LOOP ANOM. DIMENSIONS $\lesssim$ 240 FUNCTIONS
- 3 LOOP WILSON COEFFICIENTS ~ 730 FUNCTIONS.

CAN THIS BE MADE TRACTABLE ?

→ EVEN MORE INVOLVED: MULTI-JET CROSS SECTIONS.

PRECISION MEASUREMENTS NEED FAST & PRECISE PROGRAMS

→ EXP. SYSTEMATICS

CURRENTLY: 1 CPU YEAR ! (NLO) α_s.

A MUCH DEEPER UNDERSTANDING IS NEEDED BEFORE WE CAN GO TO EVEN MORE LOOPS & LEGS.

3. The Mellin Symmetry

COLLINEAR FACTORIZATION $(m_i \to 0)$
IMPLIES THE CONNECTION:

$$\sigma(\hat{s}) = \int_0^1 dx_1 \int_0^1 dx_2 \; \sigma_W(x_1) f(x_2) \, \delta(x - x_1 x_2)$$

$\hat{s} = x s.$

$$\sigma = \sigma_W \otimes f.$$

$$\mathbb{M}[\sigma(x)](N) := \int_0^1 dx \; x^{N-1} \sigma(x).$$

$$\boxed{\mathbb{M}[A \otimes B](N) = \mathbb{M}[A](N) \cdot \mathbb{M}[B](N)}$$

FEYNMAN AMPLITUDES MAY BE SIMPLIFIED BY CONSEQUENT OBSERVATION OF THIS CONNECTION.

WE SHOW THAT :

SPLITTING AND COEFFICIENT FUNCTIONS

UP TO $O(\alpha^2)$ ARE REPRESENTABLE AS

POLYNOMIALS OF FINITE HARMONIC SUMS

IN N-SPACE.

CF. ALSO
MOCH, VERMASEREN
12'99

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2 LOOP QUANTITIES $F_i(x)$ ⟷ HARMONIC SUMS UP TO TRANSCENDENTALITY 4

SINGLE CONTOUR INTEGRAL

SIMPLIFICATION

ALGEBRA

ANALYTIC CONTINUAT.

MELLIN TRANSFORMS $\mathbb{M}[F_i(x)](N)$, $N \in \mathbb{C}$

2 LOOP QUANTITIES $N \in \mathbb{C}$ ⟵

NON-LINEAR MAPPINGS

PHYSICAL ANOMALOUS DIMENSIONS etc.

JB, S.KURTH '98

7 Appendix: Mellin Transforms

No.	$f(z)$	$M[f](N) = \int_0^1 dz z^{N-1} f(z)$
1	$\delta(1-z)$	1
2	z^r	$\frac{1}{N+r}$
3	$\left(\frac{1}{1-z}\right)_+$	$-S_1(N-1)$
4	$\frac{1}{1+z}$	$(-1)^{N-1}[\log(2) - S_1(N-1)]$ $+\frac{1+(-1)^{N-1}}{2} S_1\left(\frac{N-1}{2}\right) - \frac{1-(-1)^{N-1}}{2} S_1\left(\frac{N-2}{2}\right)$
5	$z^r \log^n(z)$	$\frac{(-1)^n}{(N+r)^{n+1}} \Gamma(n+1)$
6	$z^r \log(1-z)$	$-\frac{S_1(N+r)}{N+r}$
7	$z^r \log^2(1-z)$	$\frac{S_1^2(N+r) + S_2(N+r)}{N+r}$
8	$z^r \log^3(1-z)$	$-\frac{S_1^3(N+r) + 3S_1(N+r)S_2(N+r) + 2S_3(N+r)}{N+r}$
9	$\left[\frac{\log(1-z)}{1-z}\right]_+$	$\frac{1}{2}S_1^2(N-1) + \frac{1}{2}S_2(N-1)$
10	$\left[\frac{\log^2(1-z)}{1-z}\right]_+$	$-\left[\frac{1}{3}S_1^3(N-1) + S_1(N-1)S_2(N-1) + \frac{2}{3}S_3(N-1)\right]$
11	$\left[\frac{\log^3(1-z)}{1-z}\right]_+$	$\frac{1}{4}S_1^4(N-1) + \frac{3}{2}S_1^2(N-1)S_2(N-1)$ $+\frac{3}{4}S_2^2(N-1) + 2S_1(N-1)S_3(N-1)$ $+\frac{3}{2}S_4(N-1)$
12	$\frac{\log^n(z)}{1-z}$	$(-1)^{n+1}\Gamma(n+1)\left[S_{n+1}(N-1) - \zeta(n+1)\right]$

Only single sums !

No.	$f(z)$	$M[f](N)$
64	$\frac{\mathrm{Li}_3(-z)}{1+z}$	$(-1)^{N-1}\Big\{S_{3,-1}(N-1)+[S_3(N-1)-S_{-3}(N-1)]\log 2$ $+\frac{1}{2}\zeta(2)S_{-2}(N-1)-\frac{3}{4}\zeta(3)S_{-1}(N-1)$ $+\frac{1}{8}\zeta^2(2)-\frac{3}{4}\zeta(3)\log 2\Big\}$
65	$\mathrm{Li}_3(1-z)$	$\frac{1}{N}[S_1(N)S_2(N)-\zeta(2)S_1(N)+S_3(N)$ $-S_{2,1}(N)+\zeta(3)]$
66	$\frac{\mathrm{Li}_3(1-z)}{1-z}$	$-S_{1,1,2}(N-1)+\frac{1}{2}\zeta(2)S_1^2(N-1)+\frac{1}{2}\zeta(2)S_2(N-1)$ $-\zeta(3)S_1(N-1)+\frac{2}{5}\zeta^2(2)$
67	$\frac{\mathrm{Li}_3(1-z)}{1+z}$	$(-1)^{N-1}\Big[S_{-1,1,2}(N-1)-\zeta(2)S_{-1,1}(N-1)$ $+\zeta(3)S_{-1}(N-1)+\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{9}{20}\zeta^2(2)$ $+\frac{7}{8}\zeta(3)\log 2+\frac{1}{2}\zeta(2)\log^2 2+\frac{1}{24}\log^4 2\Big]$
68	$\mathrm{Li}_3\left(\frac{1-z}{1+z}\right)$ $-\mathrm{Li}_3\left(-\frac{1-z}{1+z}\right)$	$\frac{(-1)^N}{N}\Big[-S_{-1,2}(N)-S_{-2,1}(N)+S_1(N)S_{-2}(N)+S_{-3}(N)$ $+\zeta(2)S_{-1}(N)+\frac{1}{2}\zeta(2)S_1(N)-\frac{7}{8}\zeta(3)+\frac{3}{2}\zeta(2)\log 2\Big]$ $+\frac{1}{N}\Big[-S_{-1,-2}(N)-S_{2,1}(N)+S_1(N)S_2(N)+S_3(N)$ $-\frac{1}{2}\zeta(2)S_{-1}(N)-\zeta(2)S_1(N)+\frac{21}{8}\zeta(3)-\frac{3}{2}\zeta(2)\log 2\Big]$
69	$\frac{1}{1+z}\Big[\mathrm{Li}_3\left(\frac{1-z}{1+z}\right)$ $-\mathrm{Li}_3\left(-\frac{1-z}{1+z}\right)\Big]$	$(-1)^{N-1}\Big\{S_{1,1,-2}(N-1)-S_{1,-1,2}(N-1)+S_{-1,1,2}(N-1)$ $-S_{-1,-1,-2}(N-1)+2\zeta(2)S_{1,-1}(N-1)+\frac{1}{4}\zeta(2)S_1^2(N-1)$ $-\frac{1}{4}\zeta(2)S_{-1}^2(N-1)-\zeta(2)S_1(N-1)S_{-1}(N-1)-\zeta(2)S_{-2}(N-1)$ $-\left[\frac{7}{8}\zeta(3)-\frac{3}{2}\zeta(2)\log 2\right]S_1(N-1)$ $+\left[\frac{21}{8}\zeta(3)-\frac{3}{2}\zeta(2)\log 2\right]S_{-1}(N-1)$ $-2\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{19}{40}\zeta^2(2)+\frac{1}{2}\zeta(2)\log^2 2-\frac{1}{12}\log^4 2\Big\}$

$$
\begin{aligned}
& +\zeta(2)S_{1,-1}(N)+\left[\zeta(2)\log(2)-\frac{5}{8}\zeta(3)\right][S_1(N)-S_{-1}(N)] \\
& -\frac{3}{40}\zeta(2)^2+\frac{5}{8}\zeta(3)\log(2)-\frac{1}{2}\zeta(2)\log^2(2) \\
= & -2S_{-2,1,1}(N)+S_1(N)S_{-2,1}(N)+S_{-2,2}(N)+S_{-3,1}(N) \qquad (122)
\end{aligned}
$$

$$
\begin{aligned}
S_{1,2,-1}(N) = & (-1)^N\mathbf{M}\left\{\frac{1}{1+x}\left[\mathrm{Li}_2(-x)\log(1+x)+2\mathrm{S}_{1,2}(-x)\right]\right\}(N) \\
& -\log(2)\left[\mathrm{S}_{1,2}(N)-S_{1,-2}(N)\right]-\frac{1}{2}\zeta(2)S_{1,-1}(N) \\
& +\left[\frac{1}{4}\zeta(3)-\frac{1}{2}\zeta(2)\log(2)\right][S_1(N)-S_{-1}(N)] \\
& +\frac{6}{5}\zeta(2)^2-3\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{23}{8}\zeta(3)\log(2)+\zeta(2)\log^2(2)-\frac{1}{8}\log^4(2) \qquad (123)
\end{aligned}
$$

$$
\begin{aligned}
S_{1,2,1}(N) = & \mathbf{M}\left\{\left[\frac{1}{x-1}\left(\mathrm{Li}_2(x)\log(1-x)+2\mathrm{S}_{1,2}(x)\right)\right]_+\right\}(N)+\zeta(2)S_{1,1}(N) \\
= & -\mathbf{M}\left[\frac{\mathrm{Li}_3(1-x)}{x-1}\right](N)+\mathbf{M}\left[\left(\frac{1}{x-1}\right)_+\mathrm{S}_{1,2}(x)\right](N) \\
& +S_1(N)S_3(N)+\frac{1}{2}S_1^2(N)S_2(N)+\frac{1}{2}S_2^2(N)-\frac{1}{2}\zeta(2)S_1^2(N) \\
& +S_4(N)-\frac{1}{2}\zeta(2)S_2(N)-\frac{8}{5}\zeta^2(2) \qquad (124)
\end{aligned}
$$

$$
\begin{aligned}
S_{-1,-1,-2}(N) = & (-1)^{N+1}\mathbf{M}\left\{\frac{1}{1+x}\left[\underline{F_1(x)}+\log(1-x)\mathrm{Li}_2(-x)\right]\right\}(N) \\
& +(-1)^{N+1}\mathbf{M}\left\{\frac{1}{1+x}\left[\frac{1}{2}\mathrm{S}_{1,2}(x^2)-\mathrm{S}_{1,2}(x)-\mathrm{S}_{1,2}(-x)\right]\right\}(N) \\
& +\frac{1}{2}\zeta(2)\left[S_{-1,1}(N)-S_{-1,-1}(N)\right]+\left[\frac{9}{8}\zeta(3)-\frac{3}{2}\zeta(2)\log(2)-\frac{1}{6}\log^3(2)\right]S_{-1}(N) \\
& -\frac{1}{10}\zeta(2)^2+\frac{17}{8}\zeta(3)\log(2)-\frac{7}{4}\zeta(2)\log^2(2)-\frac{1}{6}\log^4(2) \\
= & (-1)^{N+1}\mathbf{M}\left\{\frac{1}{1+x}\left[\mathrm{S}_{1,2}(-x)+\mathrm{Li}_2(-x)\log(1+x)+\mathrm{Li}_2(-x)\log(1-x)\right]\right\}(N) \\
& +S_1(N)S_{2,-1}(N)+S_{2,-2}(N)+S_{3,-1}(N)+S_{-1}(N)S_3(N) \\
& +\frac{1}{2}S_2(N)S_{-2}(N)+\frac{1}{2}S_{-1}^2(N)S_{-2}(N) \\
& +[S_1(N)-S_{-1}(N)][S_2(N)-S_{-2}(N)]\log 2+\frac{1}{2}\zeta(2)S_1(N)S_{-1}(N) \\
& +S_{-4}(N)+2\log(2)\left[S_3(N)-S_{-3}(N)\right]+\left[\frac{1}{2}\zeta(2)-\log^2(2)\right]S_2(N) \\
& +S_{-2}(N)\log^2(2)-\left[\frac{1}{4}\zeta(3)-\frac{1}{2}\zeta(2)\log(2)\right]S_1(N) \\
& +\left[\frac{3}{4}\zeta(3)-\frac{1}{2}\zeta(2)\log(2)\right]S_{-1}(N)-4\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{3}{2}\zeta^2(2) \\
& -\frac{5}{2}\zeta(3)\log(2)+\frac{1}{2}\zeta(2)\log^2(2)-\frac{1}{6}\log^4(2) \qquad (125)
\end{aligned}
$$

$(O(\alpha^2))$

$$F_1(x) = S_{1,2}\left(\frac{1-x}{2}\right) + S_{1,2}(1-x) - S_{1,2}\left(\frac{1-x}{1+x}\right)$$

$$+ S_{1,2}\left(\frac{1}{1+x}\right) - \log(2)\, Li_2\left(\frac{1-x}{2}\right)$$

$$+ \frac{1}{2}\log^2 2 \,\log\left(\frac{1+x}{2}\right) - \log(2)\, Li_2\left(\frac{1-x}{1+x}\right).$$

→ THE MELLIN TRANSFORM OF $F_1(x)$ TURNS OUT TO BE A POLYNOM OF MUCH <u>SIMPLER</u> MELLIN TRANSFORMS.

→ $\otimes$-PRODUCT REDUCIBLE.

No.	$f(z)$	$M[f](N)$
70	$z^r \mathrm{S}_{1,2}(z)$	$\frac{\zeta(3)}{N+r} - \frac{S_1^2(N+r)+S_2(N+r)}{2(N+r)^2}$
71	$\left(\frac{1}{1-z}\right)_+ \mathrm{S}_{1,2}(z)$	$S_{2,1,1}(N-1) - \zeta(3)S_1(N-1) - \frac{6}{5}\zeta^2(2)$
72	$\frac{\mathrm{S}_{1,2}(z)}{1+z}$	$(-1)^N\Big[S_{-2,1,1}(N-1) - \zeta(3)S_{-1}(N-1) + \mathrm{Li}_4\left(\frac{1}{2}\right)$ $-\frac{1}{8}\zeta^2(2) - \frac{1}{8}\zeta(3)\log 2 - \frac{1}{4}\zeta(2)\log^2 2 + \frac{1}{24}\log^4 2\Big]$
73	$\mathrm{S}_{1,2}(-z)$	$\frac{(-1)^N}{N^2}\Big[S_{-1,1}(N) + S_1^2(N) + S_2(N) - 2S_1(N)\log 2\Big]$ $-\frac{1+(-1)^N}{2N^2}\Big[S_1(N)S_1\left(\frac{N}{2}\right) - S_1\left(\frac{N}{2}\right)\log 2 + \frac{1}{2}S_2\left(\frac{N}{2}\right)\Big]$ $+\frac{1-(-1)^N}{2N^2}\Big[S_1(N)S_1\left(\frac{N-1}{2}\right) - S_1\left(\frac{N-1}{2}\right)\log 2$ $+\frac{1}{2}S_2\left(\frac{N-1}{2}\right) - \log^2 2\Big] + \frac{\zeta(3)}{8N}$
74	$\frac{\mathrm{S}_{1,2}(-z)}{1+z}$	$(-1)^N\Big\{S_{2,1,-1}(N-1) + [S_{2,1}(N-1) - S_{2,-1}(N-1)]\log 2$ $-\frac{1}{2}[S_2(N-1) - S_{-2}(N-1)]\log^2 2$ $-\frac{1}{8}\zeta(3)S_{-1}(N-1) - 3\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{6}{5}\zeta^2(2)$ $-\frac{11}{4}\zeta(3)\log 2 + \frac{3}{4}\zeta(2)\log^2 2 - \frac{1}{8}\log^4 2\Big\}$
75	$\mathrm{S}_{1,2}(1-z)$	$-\frac{1}{N}\Big[S_3(N) - \zeta(3)\Big]$
76	$\frac{\mathrm{S}_{1,2}(1-z)}{1-z}$	$S_1(N-1)S_3(N-1) - \zeta(3)S_1(N-1) + S_4(N-1)$ $-S_{3,1}(N-1) + \frac{\zeta(2)^2}{10}$
77	$\frac{\mathrm{S}_{1,2}(1-z)}{1+z}$	$(-1)^N\Big[S_{-1,3}(N-1) - \zeta(3)S_{-1}(N-1)$ $+\frac{19}{40}\zeta^2(2) - \frac{7}{4}\zeta(3)\log 2\Big]$

4. Multiple Zeta Values

CONSIDER THE MELLIN TRANSFORM OF NIELSEN INTEGRALS IN THE LIMIT

$$N \to \infty.$$

- TOTAL CROSS SECTIONS
- BETA FUNCTION
- CREWTHER RELATION AND ASSOC. QUANTIT.
- INDIVIDUAL INTEGER MOMENTS

....

$$\beta_3 = C_A^4\left(\frac{150653}{486} - \frac{44}{9}\underline{\zeta_3}\right) + C_A^3 C_F^2 T_F n_f \left(-\frac{4204}{27} + \frac{352}{9}\underline{\zeta_3}\right)$$

$$\cdots + n_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_A}\left(-\frac{704}{9} + \frac{512}{3}\underline{\zeta_3}\right)$$

$$\gamma_{ij}^3(N) \longrightarrow \zeta_3$$

$$C_k^3(N) \longrightarrow \zeta_3, \zeta_5$$

$$\zeta_i = \sum_{k=1}^{\infty} \frac{1}{k^i}, \quad i \geq 2$$

SINGLE ZETA VALUES.

MULTIPLE ζ-VALUES: (MZV)

$$\zeta_{a,b,c,\ldots n} = \sum_{k_1=1}^{\infty} \frac{1}{k_1^a} \sum_{k_2=1}^{k_1} \frac{1}{k_2^b} \sum \cdots \sum_{k_n=1}^{k_{n-1}} \frac{1}{k_n^n}$$

COLORED MZV'S: $\pm 1 = \pm\sqrt{1}$ CASE

$$\zeta_{a,b,c\ldots} = \sum_{k_1=1}^{\infty} \frac{[\mathrm{sign}(a)]^{k_1}}{k_1^{|a|}} \sum_{k_2=1}^{k_1} \frac{[\mathrm{sign}(b)]^{k_2}}{k_2^{|b|}} \sum \cdots$$

MUCH SIMPLER OBJECTS THAN HARMONIC SUMS.

- SHUFFLE RELATION
- STUFFLE RELATIONS
- CONJUGATION.

EXAMPLES:

PETITOT et al.

NORMAL RANK 12

▶ $\zeta(2,1,1,5,1,2) = -\frac{19}{4}\zeta(3)^4 + \frac{511}{4}\zeta(2)\zeta(5)^2 - \frac{26907}{16}\zeta(5)\zeta(7) + \frac{6639743}{63000}\zeta(2)^6 + \frac{1377}{20}\zeta(5)\zeta(3)\zeta(2)^2 - \frac{3740}{3}\zeta(3)\zeta(9) + \frac{7723}{210}\zeta(2)^3\zeta(3)^2 + \frac{1943}{8}\zeta(7)\zeta(2)\zeta(3) + \frac{1107}{8}\zeta(2)\zeta(8,2) + \frac{1943}{40}\zeta(2)^2\zeta(6,2) + \frac{123}{2}\zeta(8,2,1,1) - \frac{7045}{32}\zeta(10,2)$

COLORED : ± 1 RANK 7

▶ $\zeta(1,1,1,1,1,2;-1,1,1,1,-1,1) = \frac{295}{256}\zeta(2)\zeta(5) + \frac{1469}{13440}\zeta(3)\zeta(2)^2 + \frac{3}{8}\zeta(1;-1)\zeta(5,1;-1,1) - \frac{23}{192}\zeta(1;-1)^4\zeta(3) + \frac{85}{84}\zeta(5,1,1;1,1,-1) - \frac{1}{2}\zeta(1;-1)\zeta(3)^2 + \frac{93}{560}\zeta(1;-1)\zeta(2)^3 + \frac{1}{120}\zeta(1;-1)^5\zeta(2) + \frac{5091}{1792}\zeta(1;-1)^2\zeta(5) - \frac{6721}{3072}\zeta(7) - \frac{73}{336}\zeta(3)\zeta(3,1;-1,1) - \frac{1}{3}\zeta(3,1,1,1,1;1,1,1,1,-1) - \frac{313}{192}\zeta(2)\zeta(1;-1)^2\zeta(3) - \frac{5}{6}\zeta(2)\zeta(3,1,1;1,1,-1) + \frac{13}{14}\zeta(5,1,1;-1,1,1) - \frac{1}{120}\zeta(1;-1)^3\zeta(2)^2 - \frac{1}{8}\zeta(2)\zeta(1;-1)\zeta(3,1;-1,1) - \frac{5}{6}\zeta(1;-1)\zeta(3,1,1,1;1,1,1,-1) - \frac{5}{12}\zeta(1;-1)^2\zeta(3,1,1;1,1,-1)$

WHAT DO WE KNOW ?

MZV's:

BROADHURST	'99	RANK 9
MINH, PETITOT	2000	RANK 10
BIGOTTE et al.	98	RANK 12
VERMASEREN	2000	RANK 9

COLORED MZV's:

GASTMANS, TROOST (JB, KURTH COMPLETED)	'81	RANK 4
BIGOTTE et al.	2002	RANK 7
VERMASEREN	2000	RANK 9

THE BASIS

$$S_1(\infty),\ \ln(2),\ \zeta_2,\ \zeta_3,\ \mathrm{Li}_4\left(\tfrac{1}{2}\right),\ \zeta_5,\ \mathrm{Li}_5\left(\tfrac{1}{2}\right)$$

$$\mathrm{Li}_6\left(\tfrac{1}{2}\right),\ S_{-5,-1}(\infty),\ \zeta_7,\ \mathrm{Li}_7\left(\tfrac{1}{2}\right),\ S_{-5,1,1}(\infty),$$

$$S_{5,-1,-1}(\infty),\ \ldots$$

2-LOOP LEVEL: $\ldots - \mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{9}{20}\zeta_2^2 - \frac{7}{8}\zeta_3 \ln(2) - \frac{1}{2}\zeta_2 \ln^2(2) - \frac{1}{24}\ln^4(2)$

THESE TYPES OF CONSTANTS CANCEL !

COUNT THE BASIS ELEMENTS :

$$\boxed{N(w) = \frac{1}{w}\sum_{d|w} \mu\left(\frac{w}{d}\right) P_d}$$

BROADHURST-KREIMER CONJECTURE '96

$k_i > 0$

$P_1 = 0,\ P_2 = 2,\ P_3 = 3,\quad P_d = P_{d-2} + P_{d-3},\quad d \geq 3$

PERRIN NUMBERS (1899) OK FOR MZV TO O(12).

5. Multiple Harmonic Sums

THE SIMPLEST EXAMPLE:

$$P_{qq}(x) = \left(\frac{1+x^2}{1-x}\right)_+ = \frac{2}{(1-x)_+} + \dots$$

$$\int_0^1 dx \frac{x^{N-1}}{(1-x)_+} = -\sum_{k=0}^{N-2} \int_0^1 dx\, x^k = -\sum_{k=1}^{N-1} \frac{1}{k} = \underline{-S_1(N-1)}.$$

ALTERNATING SUMS (→ COLORED ζ-VALUES)

$$S_{-1}(N) = (-1)^N M\left[\frac{1}{1+x}\right](N) - \ln(2).$$

$$= \sum_{k=1}^{N} \frac{(-1)^k}{k} \qquad \text{(FINITE FOR } N \to \infty\text{)}.$$

GENERAL CASE:

$$S_{a_1 \dots a_\ell}(N) = \sum_{k_1=1}^{N} \frac{(\mathrm{sign}(a_1))^{k_1}}{k_1^{|a_1|}} \sum_{k_2=1}^{k_1} \frac{(\mathrm{sign}(a_2))^{k_2}}{k_2^{|a_2|}} \dots .$$

ALL MELLIN TRANSFORMS IN MASSLESS FIELD THEORIES FOR 1-PAR. QUANTITIES ARE REPRESENTED BY HARMONIC SUMS.

(KNOWN TO $O(\alpha^3)$). OR SIMILAR TYPE.

EXAMPLES FOR n-FOLD SUMS:

$$\underbrace{S_{-1,\ldots,-1}}_{k} = \frac{1}{k!}\left|\begin{array}{ccccc} S_{-1}(N) & 1 & 0 & \ldots & 0 \\ -S_2(N) & S_{-1}(N) & 2 & \ldots & 0 \\ S_{-3}(N) & -S_2(N) & S_{-1}(N) & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^{k+1}S_{(-1)^k k}(N) & (-1)^k S_{(-1)^{k-1}(k-1)}(N) & (-1)^{k-1}S_{(-1)^{k-2}(k-2)}(N) & \ldots & S_{-1}(N) \end{array}\right| \quad (56)$$

Similarly the corresponding expressions for $S(1,1,\ldots,1)$ are

$$\underbrace{S_{1,\ldots,1}}_{k} = \frac{1}{k!}\left|\begin{array}{cccccc} S_1(N) & 1 & 0 & 0 & \ldots & 0 \\ -S_2(N) & S_1(N) & 2 & 0 & \ldots & 0 \\ S_3(N) & -S_2(N) & S_1(N) & 3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ (-1)^{k+1}S_k(N) & (-1)^k S_{k-1}(N) & (-1)^{k-1}S_{k-2}(N) & (-1)^{k-2}S_{k-3}(N) & \ldots & S_1(N) \end{array}\right|. \quad (57)$$

Further sums of this type are

$$\begin{aligned} \underbrace{S_{-1,\ldots,-1}}_{5} &= \frac{1}{120}S_{-1}^5 + \frac{1}{12}S_2 S_{-1}^3 + \frac{1}{6}S_{-3}S_{-1}^2 + \frac{1}{4}S_4 S_{-1} + \frac{1}{8}S_{-1}S_2^2 + \frac{1}{6}S_2 S_{-3} \\ &\quad + \frac{1}{5}S_{-5} \qquad (58) \\ \underbrace{S_{-1,\ldots,-1}}_{6} &= \frac{1}{720}S_{-1}^6 + \frac{1}{48}S_2 S_{-1}^4 + \frac{1}{18}S_{-3}S_{-1}^3 + \frac{1}{8}S_4 S_{-1}^2 + \frac{1}{5}S_{-5}S_{-1} + \frac{1}{16}S_{-1}^2 S_2^2 \\ &\quad + \frac{1}{6}S_{-1}S_2 S_{-3} + \frac{1}{48}S_2^3 + \frac{1}{8}S_2 S_4 + \frac{1}{18}S_{-3}^2 + \frac{1}{6}S_6 \qquad (59) \\ \underbrace{S_{-1,\ldots,-1}}_{7} &= \frac{1}{5040}S_{-1}^7 + \frac{1}{240}S_{-1}^5 S_2 + \frac{1}{72}S_{-1}^4 S_{-3} + \frac{1}{24}S_{-1}^3 S_4 + \frac{1}{10}S_{-1}^2 S_{-5} + \frac{1}{6}S_{-1}S_6 \\ &\quad + \frac{1}{10}S_2 S_{-5} + \frac{1}{24}S_2^2 S_{-3} + \frac{1}{48}S_2^2 S_{-1}^3 + \frac{1}{48}S_2^3 S_{-1} + \frac{1}{12}S_{-3}S_4 + \frac{1}{18}S_{-3}^2 S_{-1} \\ &\quad + \frac{1}{8}S_{-1}S_2 S_4 + \frac{1}{12}S_{-1}^2 S_2 S_{-3} + \frac{1}{7}S_{-7} \qquad (60) \\ \underbrace{S_{-1,\ldots,-1}}_{8} &= \frac{1}{40320}S_{-1}^8 + \frac{1}{1440}S_{-1}^6 S_2 + \frac{1}{360}S_{-1}^5 S_{-3} + \frac{1}{96}S_{-1}^4 S_4 + \frac{1}{30}S_{-1}^3 S_{-5} \\ &\quad + \frac{1}{12}S_{-1}^2 S_6 + \frac{1}{7}S_{-1}S_{-7} + \frac{1}{36}S_{-1}^3 S_2 S_{-3} + \frac{1}{16}S_{-1}^2 S_2 S_4 + \frac{1}{24}S_{-1}S_2^2 S_{-3} \\ &\quad + \frac{1}{12}S_{-1}S_{-3}S_4 + \frac{1}{96}S_{-1}^2 S_2^3 + \frac{1}{36}S_{-1}^2 S_{-3}^2 + \frac{1}{384}S_2^4 + \frac{1}{192}S_2^2 S_{-1}^4 + \frac{1}{32}S_2^2 S_4 \\ &\quad + \frac{1}{36}S_2 S_{-3}^2 + \frac{1}{12}S_2 S_6 + \frac{1}{10}S_2 S_{-1}S_{-5} + \frac{1}{15}S_{-3}S_{-5} + \frac{1}{32}S_4^2 + \frac{1}{8}S_8 \qquad (61) \end{aligned}$$

The corresponding relations for $\underbrace{S_{1,\ldots,1}}_{\mathrm{n}}$ are obtained by substituting the indicees $-k \rightarrow k$ in eqs. (58–61). One may evaluate these sums also using the recursion relations

$$\underbrace{S_{-1,\ldots,-1}}_{\mathrm{k}} = \frac{1}{k}\sum_{l=1}^{k} S_{(-1)^l l}\, \underbrace{S_{-1,\ldots,-1}}_{\mathrm{k-1}} \qquad (62)$$

SIMILAR RELATIONS: RAMANUJAN
and others.

S. Ramanujan, 1919
(From G. H. Hardy, *Ramanujan, Twelve Lectures on Subjects Suggested by His Life and Work.* Cambridge University Press, 1940.)

EINLEITUNG

IN DIE

THEORIE DER BINÄREN FORMEN

VON

F. FAÀ DI BRUNO.

MIT UNTERSTÜTZUNG VON PROFESSOR M. NOETHER

DEUTSCH BEARBEITET

VON

DR. THEODOR WALTER.

LEIPZIG,
VERLAG VON B. G. TEUBNER.
1881.

BRUCE C. BERNDT, ED.

RAMANUJAN'S NOTEBOOKS

→ POWERSUMS

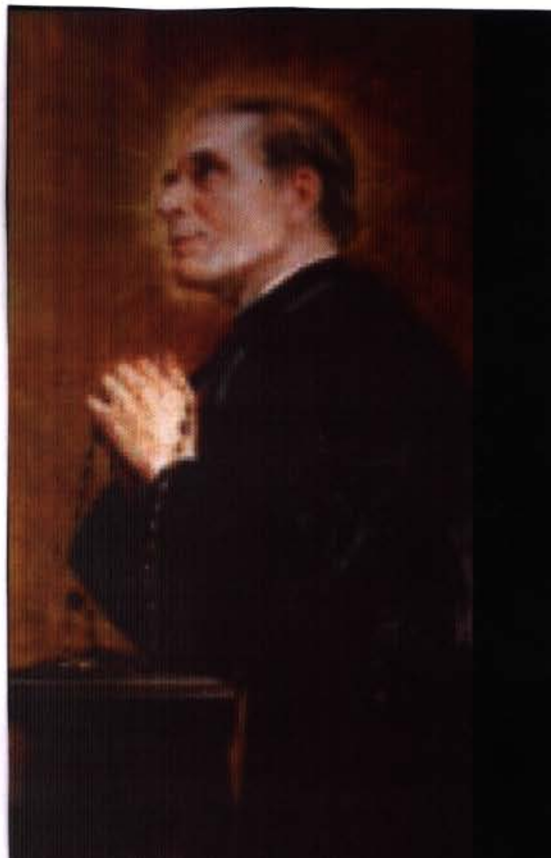
FAA DI BRUNO:

CAPTAIN OF SARDINIAN ARMY,

MATHEMATICIAN, PRIEST

CANONIZED SEPT 25, 1988

(BEATUS)



ALGEBRAIC RELATIONS:

L. EULER (1775):

$$S_{m,n} + S_{n,m} = S_m \cdot S_n + S_{m+n}$$

$m, n > 0.$

FIRST ALGEBRAIC RELATION!

ONE MAY GENERALIZE THIS TO $m, n \lessgtr 0$

$$S_{m,n} + S_{n,m} = S_m \cdot S_n + S_{m \wedge n}$$

$$m \wedge n = [|m| + |n|] \operatorname{sign}(m) \operatorname{sign}(n).$$

TERNARY RELATION: SITA RAMACHANDRA RAO 1984

4-ARY —"— : JB, KURTH 1998.

THESE & OTHER RELATIONS HOLD WIDELY INDEPENDENT OF THE VALUE & TYPE OF THESE OBJECTS.

DETERMINED BY:
- INDEX STRUCTURE
- MULTIPLICATION RELATION

→ QUASI-SHUFFLE ALGEBRAS

FREE LIE ALGEBRAS etc.

LINEAR REPRESENTATIONS OF MELLIN TRANSFORMS THROUGH HARMONIC SUMS:

$$M[F_w(x)](N) = S^{w}_{k_1,\dots,k_m}(N)$$

ZETA VALUES

$$+ P\left(S^{\tau'}_{k_1\dots k_r},\ \sigma^{\tau''}_{k_1\dots k_p}\right)$$

$w = \sum_{i=1}^{m} |k_i|$ WEIGHT

HARMONIC SUMS

$\tau', \tau'' < w$

P is a polynomial.

NUMBER OF FUNCTIONS $F_w(x)$ & SUMS:

w	#	Σ			
1	2	2			EXPL. KNOWN IN ALL DETAILS
2	6	8			
3	18	26	2 LOOP	ANOM. DIM.	
4	54	80		COEFF. FCT.	
5	162	242	3 LOOP	ANOM. DIM.	ALGEBRA FULLY KNOWN.
6	486	728		COEFF FCT.	
	$2 \cdot 3^{w-1}$	$3^w - 1$			

SHUFFLE PRODUCTS

(MAC LANE 1950)
(LYNDON 1954)

Depth 2 :

$$S_{a_1}(N) ⧢ S_{a_2}(N) = S_{a_1,a_2}(N) + S_{a_2,a_1}(N)$$

Depth 3 :

$$S_{a_1}(N) ⧢ S_{a_2,a_3}(N) = S_{a_1,a_2,a_3}(N) + S_{a_2,a_1,a_3}(N) + S_{a_2,a_3,a_1}(N) \tag{2.8}$$

Depth 4 :

$$S_{a_1}(N) ⧢ S_{a_2,a_3,a_4}(N) = S_{a_1,a_2,a_3,a_4}(N) + S_{a_2,a_1,a_3,a_4}(N) + S_{a_2,a_3,a_1,a_4}(N) + S_{a_2,a_3,a_4,a_1}(N) \tag{2.9}$$

$$\begin{aligned} S_{a_1,a_2}(N) ⧢ S_{a_3,a_4}(N) &= S_{a_1,a_2,a_3,a_4}(N) + S_{a_1,a_3,a_2,a_4}(N) + S_{a_1,a_3,a_4,a_2}(N) \\ &+ S_{a_3,a_4,a_1,a_2}(N) + S_{a_3,a_1,a_4,a_2}(N) + S_{a_3,a_1,a_2,a_4}(N) \end{aligned} \tag{2.10}$$

Depth 5 :

$$\begin{aligned} S_{a_1}(N) ⧢ S_{a_2,a_3,a_4,a_5}(N) &= S_{a_1,a_2,a_3,a_4,a_5}(N) + S_{a_2,a_1,a_3,a_4,a_5}(N) + S_{a_2,a_3,a_1,a_4,a_5}(N) \\ &+ S_{a_2,a_3,a_4,a_1,a_5}(N) + S_{a_2,a_3,a_4,a_5,a_1}(N) \end{aligned} \tag{2.11}$$

$$\begin{aligned} S_{a_1,a_2}(N) ⧢ S_{a_3,a_4,a_5}(N) &= S_{a_1,a_2,a_3,a_4,a_5}(N) + S_{a_1,a_3,a_2,a_4,a_5}(N) + S_{a_1,a_3,a_4,a_2,a_5}(N) \\ &+ S_{a_1,a_3,a_4,a_5,a_2}(N) + S_{a_3,a_1,a_2,a_4,a_5}(N) + S_{a_3,a_1,a_4,a_2,a_5}(N) \\ &+ S_{a_3,a_1,a_4,a_5,a_2}(N) + S_{a_3,a_4,a_5,a_1,a_2}(N) + S_{a_3,a_4,a_1,a_5,a_2}(N) \\ &+ S_{a_3,a_4,a_1,a_2,a_5}(N) \end{aligned} \tag{2.12}$$

Depth 6 :

$$\begin{aligned} S_{a_1}(N) ⧢ S_{a_2,a_3,a_4,a_5,a_6}(N) &= S_{a_1,a_2,a_3,a_4,a_5,a_6}(N) + S_{a_2,a_1,a_3,a_4,a_5,a_6}(N) + S_{a_2,a_3,a_1,a_4,a_5,a_6}(N) \\ &+ S_{a_2,a_3,a_4,a_1,a_5,a_6}(N) + S_{a_2,a_3,a_4,a_5,a_1,a_6}(N) + S_{a_2,a_3,a_4,a_5,a_6,a_1}(N) \end{aligned} \tag{2.13}$$

$$\begin{aligned} S_{a_1,a_2}(N) ⧢ S_{a_3,a_4,a_5,a_6}(N) &= S_{a_1,a_2,a_3,a_4,a_5,a_6}(N) + S_{a_1,a_3,a_2,a_4,a_5,a_6}(N) + S_{a_1,a_3,a_4,a_2,a_5,a_6}(N) \\ &+ S_{a_1,a_3,a_4,a_5,a_2,a_6}(N) + S_{a_1,a_3,a_4,a_5,a_6,a_2}(N) + S_{a_3,a_1,a_2,a_4,a_5,a_6}(N) \\ &+ S_{a_3,a_1,a_4,a_2,a_5,a_6}(N) + S_{a_3,a_1,a_4,a_5,a_2,a_6}(N) + S_{a_3,a_1,a_4,a_5,a_6,a_2}(N) \\ &+ S_{a_3,a_4,a_1,a_2,a_5,a_6}(N) + S_{a_3,a_4,a_1,a_5,a_2,a_6}(N) + S_{a_3,a_4,a_1,a_5,a_6,a_2}(N) \\ &+ S_{a_3,a_4,a_5,a_6,a_1,a_2}(N) + S_{a_3,a_4,a_5,a_1,a_6,a_2}(N) + S_{a_3,a_4,a_5,a_1,a_2,a_6}(N) \end{aligned} \tag{2.14}$$

$$\begin{aligned} S_{a_1,a_2,a_3}(N) ⧢ S_{a_4,a_5,a_6}(N) &= S_{a_1,a_2,a_3,a_4,a_5,a_6}(N) + S_{a_1,a_2,a_4,a_3,a_5,a_6}(N) + S_{a_1,a_2,a_4,a_5,a_3,a_6}(N) \\ &+ S_{a_1,a_2,a_4,a_5,a_6,a_3}(N) + S_{a_1,a_4,a_2,a_3,a_5,a_6}(N) + S_{a_1,a_4,a_2,a_5,a_3,a_6}(N) \\ &+ S_{a_1,a_4,a_2,a_5,a_6,a_3}(N) + S_{a_1,a_4,a_5,a_6,a_2,a_3}(N) + S_{a_1,a_4,a_5,a_2,a_6,a_3}(N) \\ &+ S_{a_1,a_4,a_5,a_2,a_3,a_6}(N) + S_{a_4,a_5,a_6,a_1,a_2,a_3}(N) + S_{a_4,a_5,a_1,a_6,a_2,a_3}(N) \\ &+ S_{a_4,a_5,a_1,a_2,a_6,a_3}(N) + S_{a_4,a_5,a_1,a_2,a_3,a_6}(N) + S_{a_4,a_1,a_5,a_6,a_2,a_3}(N) \\ &+ S_{a_4,a_1,a_5,a_2,a_6,a_3}(N) + S_{a_4,a_1,a_5,a_2,a_3,a_6}(N) + S_{a_4,a_1,a_2,a_3,a_5,a_6}(N) \\ &+ S_{a_4,a_1,a_2,a_5,a_3,a_6}(N) + S_{a_4,a_1,a_2,a_5,a_6,a_3}(N)\,. \end{aligned} \tag{2.15}$$

THE ALGEBRAIC EQUATIONS

Depth 2 :

$$S_{a_1}(N) \sqcup\!\sqcup S_{a_2}(N) - S_{a_1}(N)S_{a_2}(N) - S_{a_1\wedge a_2}(N) = 0 \quad [36] \tag{2.17}$$

Depth 3 :

$$S_{a_1}(N) \sqcup\!\sqcup S_{a_2,a_3}(N) - S_{a_1}(N)S_{a_2,a_3}(N) - S_{a_1\wedge a_2,a_3}(N) - S_{a_2,a_1\wedge a_3}(N) = 0 \tag{2.18}$$

Depth 4 :

$$\begin{aligned} S_{a_1}(N) \sqcup\!\sqcup S_{a_2,a_3,a_4}(N) &- S_{a_1}(N)S_{a_2,a_3,a_4}(N) - S_{a_1\wedge a_2,a_3,a_4}(N) - S_{a_2,a_1\wedge a_3,a_4}(N) \\ &- S_{a_2,a_3,a_1\wedge a_4}(N) = 0 \end{aligned} \tag{2.19}$$

$$\begin{aligned} S_{a_1,a_2}(N) \sqcup\!\sqcup S_{a_3,a_4}(N) &- S_{a_1,a_2}(N)S_{a_3,a_4}(N) - S_{a_1,a_2\wedge a_3,a_4}(N) - S_{a_1,a_3,a_2\wedge a_4}(N) \\ &- S_{a_3,a_1\wedge a_4,a_2}(N) - S_{a_3,a_1,a_2\wedge a_4}(N) - S_{a_1\wedge a_3,a_2,a_4}(N) \\ &- S_{a_1\wedge a_3,a_4,a_2}(N) + S_{a_1\wedge a_3,a_2\wedge a_4} = 0 \end{aligned} \tag{2.20}$$

Depth 5 :

$$\begin{aligned} S_{a_1}(N) \sqcup\!\sqcup S_{a_2,a_3,a_4,a_5}(N) &- S_{a_1}(N)S_{a_2,a_3,a_4,a_5}(N) - S_{a_1\wedge a_2,a_3,a_4,a_5}(N) - S_{a_2,a_1\wedge a_3,a_4,a_5}(N) \\ &- S_{a_2,a_3,a_1\wedge a_4,a_5}(N) - S_{a_2,a_3,a_4,a_1\wedge a_5}(N) = 0 \end{aligned} \tag{2.21}$$

$$\begin{aligned} S_{a_1,a_2}(N) \sqcup\!\sqcup S_{a_3,a_4,a_5}(N) &- S_{a_1,a_2\wedge a_3,a_4,a_5}(N) - S_{a_1,a_3,a_2\wedge a_4,a_5}(N) - S_{a_1,a_3,a_4,a_2\wedge a_5}(N) \\ &- S_{a_3,a_1,a_2\wedge a_4,a_5}(N) - S_{a_3,a_1,a_4,a_2\wedge a_5}(N) - S_{a_3,a_4,a_1\wedge a_5,a_2}(N) \\ &- S_{a_3,a_4,a_1,a_2\wedge a_5}(N) - S_{a_3,a_1\wedge a_4,a_2,a_5}(N) - S_{a_3,a_1\wedge a_4,a_5,a_2}(N) \\ &- S_{a_1,a_2}(N)S_{a_3,a_4,a_5}(N) - S_{a_1\wedge a_3,a_2,a_4,a_5}(N) - S_{a_1\wedge a_3,a_4,a_2,a_5}(N) \\ &- S_{a_1\wedge a_3,a_4,a_5,a_2}(N) + S_{a_1\wedge a_3,a_2\wedge a_4,a_5}(N) + S_{a_1\wedge a_3,a_4,a_2\wedge a_5}(N) = 0 \end{aligned} \tag{2.22}$$

Depth 6 :

$$\begin{aligned} S_{a_1}(N) \sqcup\!\sqcup S_{a_2,a_3,a_4,a_5,a_6}(N) &- S_{a_1}(N)S_{a_2,a_3,a_4,a_5,a_9}(N) - S_{a_1\wedge a_2,a_3,a_4,a_5,a_6}(N) \\ &- S_{a_2,a_1\wedge a_3,a_4,a_5,a_6}(N) - S_{a_2,a_3,a_1\wedge a_4,a_5,a_6}(N) \\ &- S_{a_2,a_3,a_4,a_1\wedge a_5,a_6}(N) - S_{a_2,a_3,a_4,a_5,a_1\wedge a_6}(N) = 0 \end{aligned} \tag{2.23}$$

$$\begin{aligned} S_{a_1,a_2}(N) \sqcup\!\sqcup S_{a_3,a_4,a_5,a_6}(N) &- S_{a_1,a_2\wedge a_3,a_4,a_5,a_6}(N) - S_{a_1,a_3,a_2\wedge a_4,a_5,a_6}(N) - S_{a_1,a_3,a_4,a_2\wedge a_5,a_6}(N) \\ &- S_{a_1,a_3,a_4,a_5,a_2\wedge a_6}(N) - S_{a_3,a_1,a_2\wedge a_4,a_5,a_6}(N) - S_{a_3,a_1,a_4,a_2\wedge a_5,a_6}(N) \\ &- S_{a_3,a_1,a_4,a_5,a_2\wedge a_6}(N) - S_{a_3,a_4,a_1,a_2\wedge a_5,a_6}(N) - S_{a_3,a_4,a_1,a_5,a_2\wedge a_6}(N) \\ &- S_{a_3,a_4,a_5,a_1\wedge a_6,a_2}(N) - S_{a_3,a_4,a_5,a_1,a_2\wedge a_6}(N) - S_{a_3,a_4,a_1\wedge a_5,a_2,a_6}(N) \\ &- S_{a_3,a_4,a_1\wedge a_5,a_6,a_2}(N) - S_{a_3,a_1\wedge a_4,a_2,a_5,a_6}(N) - S_{a_3,a_1\wedge a_4,a_5,a_2,a_6}(N) \\ &- S_{a_3,a_1\wedge a_4,a_5,a_6,a_2}(N) - S_{a_1\wedge a_3,a_2,a_4,a_5,a_6}(N) - S_{a_1\wedge a_3,a_4,a_2,a_5,a_6}(N) \\ &- S_{a_1\wedge a_3,a_4,a_5,a_2,a_6}(N) - S_{a_1\wedge a_3,a_4,a_5,a_6,a_2}(N) + S_{a_3,a_4,a_1\wedge a_5,a_2\wedge a_6}(N) \\ &+ S_{a_3,a_1\wedge a_4,a_2\wedge a_5,a_6}(N) + S_{a_3,a_1\wedge a_4,a_5,a_2\wedge a_6}(N) + S_{a_3,a_4,a_1\wedge a_5,a_2\wedge a_6}(N) \\ &+ S_{a_1\wedge a_3,a_2\wedge a_4,a_5,a_6}(N) + S_{a_1\wedge a_3,a_4,a_2\wedge a_5,a_6}(N) + S_{a_1\wedge a_3,a_4,a_5,a_2\wedge a_6}(N) \\ &- S_{a_1,a_2}(N)S_{a_3,a_4,a_5,a_6}(N) = 0 \end{aligned} \tag{2.24}$$

ALLOW FOR ANY INDEX PERMUTATION.

HOW MANY OF THESE EQ. ARE INDEPENDENT ?

\# BASIC SUMS = # PERM. − # IND. EQS.

$$
\begin{aligned}
S_{a_1,a_2,a_3}(N) \sqcup S_{a_4,a_5,a_6}(N) \; - \; & S_{a_1,a_2,a_3\wedge a_4,a_5,a_6}(N) - S_{a_1,a_2,a_4,a_3\wedge a_5,a_6}(N) - S_{a_1,a_2,a_4,a_5,a_3\wedge a_6}(N) \\
- \; & S_{a_1,a_4,a_2,a_3\wedge a_5,a_6}(N) - S_{a_1,a_4,a_2,a_5,a_3\wedge a_6}(N) - S_{a_1,a_4,a_5,a_2\wedge a_6,a_3}(N) \\
- \; & S_{a_1,a_4,a_5,a_2,a_3\wedge a_6}(N) - S_{a_1,a_4,a_2\wedge a_5,a_3,a_6}(N) - S_{a_1,a_4,a_2\wedge a_5,a_6,a_3}(N) \\
- \; & S_{a_1,a_2\wedge a_4,a_3,a_5,a_6}(N) - S_{a_1,a_2\wedge a_4,a_5,a_3,a_6}(N) - S_{a_1,a_2\wedge a_4,a_5,a_6,a_3}(N) \\
- \; & S_{a_4,a_5,a_1\wedge a_6,a_2,a_3}(N) - S_{a_4,a_5,a_1,a_2\wedge a_6,a_3}(N) - S_{a_4,a_5,a_1,a_2,a_3\wedge a_6}(N) \\
- \; & S_{a_4,a_1,a_5,a_2\wedge a_6,a_3}(N) - S_{a_4,a_1,a_5,a_2,a_3\wedge a_6}(N) - S_{a_4,a_1,a_2,a_3\wedge a_5,a_6}(N) \\
- \; & S_{a_4,a_1,a_2,a_5,a_3\wedge a_6}(N) - S_{a_4,a_1,a_2\wedge a_5,a_6,a_3}(N) - S_{a_4,a_1,a_2\wedge a_5,a_3,a_6}(N) \\
- \; & S_{a_4,a_1\wedge a_5,a_6,a_2,a_3}(N) - S_{a_4,a_1\wedge a_5,a_2,a_6,a_3}(N) - S_{a_4,a_1\wedge a_5,a_2,a_3,a_6}(N) \\
- \; & S_{a_1\wedge a_4,a_2,a_3,a_5,a_6}(N) - S_{a_1\wedge a_4,a_2,a_5,a_3,a_6}(N) - S_{a_1\wedge a_4,a_2,a_5,a_6,a_3}(N) \\
- \; & S_{a_1\wedge a_4,a_5,a_6,a_2,a_3}(N) - S_{a_1\wedge a_4,a_5,a_2,a_6,a_3}(N) - S_{a_1\wedge a_4,a_5,a_2,a_3,a_6}(N) \\
+ \; & S_{a_1,a_4,a_2\wedge a_5,a_3\wedge a_6}(N) + S_{a_1,a_2\wedge a_4,a_3\wedge a_5,a_6}(N) + S_{a_1,a_2\wedge a_4,a_5,a_3\wedge a_6}(N) \\
+ \; & S_{a_4,a_1,a_2\wedge a_5,a_3\wedge a_6}(N) + S_{a_4,a_1\wedge a_5,a_2\wedge a_6,a_3}(N) + S_{a_4,a_1\wedge a_5,a_2,a_3\wedge a_6}(N) \\
+ \; & S_{a_1\wedge a_4,a_2,a_3\wedge a_5,a_6}(N) + S_{a_1\wedge a_4,a_2,a_5,a_3\wedge a_6}(N) + S_{a_1\wedge a_4,a_5,a_2\wedge a_6,a_3}(N) \\
+ \; & S_{a_1\wedge a_4,a_5,a_2,a_3\wedge a_6}(N) + S_{a_1\wedge a_4,a_2\wedge a_5,a_3,a_6}(N) + S_{a_1\wedge a_4,a_2\wedge a_5,a_6,a_3}(N) \\
- \; & S_{a_1\wedge a_4,a_2\wedge a_5,a_3\wedge a_6}(N) - S_{a_1,a_2,a_3}(N) S_{a_4,a_5,a_6}(N) = 0 \, . \qquad (2.25)
\end{aligned}
$$

9 Appendix A: Coefficient Matrices for the Algebraic Relations of Harmonic Sums

$$M_{4a} = \begin{Vmatrix}
1&0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&1\\
1&0&1&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&1&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&1&1\\
1&1&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&0&1\\
1&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&1&1&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&1&1&0&1&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&0&0&0&0&0&1\\
1&0&1&1&0&0&0&0&0&0&0&0&1&1&0&0&1&0&0&0&0&0&0&0\\
0&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&0\\
1&1&1&0&0&0&1&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0\\
1&1&0&0&1&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&0&1&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&1&1&0&0&0&0&1&1&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&1&1&0&0&0&0&0&0&0&0&1&1&0&1\\
1&1&0&0&1&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&1&0&1&1&0&0\\
1&1&1&0&0&0&1&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&0&1&1&0&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&0&1&1&0&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&0&0&0&1&1&1\\
0&0&1&1&0&1&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&1&0&0&1&1\\
1&0&1&1&0&0&0&0&0&0&0&0&1&1&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&1&1&0&0&0&0&1&1&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&1&0&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&1&0&0&1&1\\
0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&0&0&0&1&1&1\\
0&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&0\\
0&0&0&0&0&0&0&1&0&0&1&1&0&0&0&0&0&0&0&0&1&1&0&1
\end{Vmatrix} . \tag{9.1}$$

DEPTH 4, MIN. WEIGHT 6.

W	MATRICES	
3	$6 \times 6 \leq$	
4	$24 \times 48 \leq$	
5	$120 \times 240 \leq$	
6	$720 \times 2160 \leq$	$\leftarrow$ 1 CPU day (2GHz, 2GBYTE)

$\longrightarrow$ NUMBER OF BASIS SUMS

& BASIS SUMS (EXPL. FORM.)

$\longrightarrow$ ALL RELATIONS.

4 The Fourfold Harmonic Sums

Five types of fourfold sums emerge. Their characteristics is summarized in the subsequent table.

Index Set	Number	Dep. Sums of Depth 4	min. Weight	Fraction of fund. Sums
$\{a,a,a,a\}$	1	1	4	0
$\{a,a,a,b\}$	4	3	4	1/4
$\{a,a,b,b\}$	6	5	4	1/6
$\{a,a,b,c\}$	12	9	5	1/4
$\{a,b,c,d\}$	24	18	6	1/4

4.1 Harmonic Sums with 4 Different Indices

For this set of indices 24 different harmonic sums exist. As for the threefold harmonic sums one may write down the associated system of linear equations. The coefficient matrix has a size of 24×48. It is obtained considering all index permutations for Eqs. (2.19,2.20). The rank of the coefficient matrix is 18, i.e. 6 harmonic sums are chosen to express the remaining sums. Since none of the first 18 diagonal elements after bringing matrix into diagonal form vanishes we may use the last 6 sums as basic sums. Here and in the following we will not present the respective coefficient matrices being too large in size. One obtains the following relations :

$$\begin{aligned}
S_{a,b,c,d} &= -S_cS_{a,d,b} - S_cS_{a,b,d} - S_{c\wedge d,a,b} - S_cS_{d,a,b} + S_aS_{d,c,b} - S_{d,c,b,a} + S_{a\wedge d,b,c} \\
&\quad +S_dS_{a,b,c} + S_{a,b}S_{c,d} + S_{a\wedge d,c,b} - S_{a,c\wedge d,b} - S_{a,d,b\wedge c} - S_{d,a,b\wedge c} - S_{a\wedge c,b\wedge d} \\
&\quad +S_{a,c,b\wedge d} + S_{c,a\wedge d,b} + S_{c,a,b\wedge d} + S_{a,b\wedge d,c} + S_{d,c,a\wedge b} &(4.\text{-}1) \\
S_{a,b,d,c} &= S_cS_{a,d,b} + S_cS_{a,b,d} + S_{c\wedge d,a,b} + S_cS_{d,a,b} - S_aS_{d,c,b} + S_{d,c,b,a} - S_{a,b}S_{c,d} \\
&\quad +S_{d,a,b\wedge c} - S_{a\wedge d,b,c} - S_{a\wedge d,c,b} + S_{a,c\wedge d,b} - S_{d,a\wedge b,c} + S_{a,d,b\wedge c} + S_{a\wedge c,b\wedge d} \\
&\quad +S_{a,b,c\wedge d} - S_{a,c,b\wedge d} - S_{c,a\wedge d,b} - S_{c,a,b\wedge d} - S_{d,b,a\wedge c} - S_aS_{d,b,c} - S_{d,c,a\wedge b} \\
&\quad +S_{d,b,a,c} + S_{d,b,c,a} &(4.\text{-}3) \\
S_{a,c,b,d} &= S_dS_{a,c,b} - S_{a,d,b\wedge c} + S_{a\wedge d,b\wedge c} - S_{d,a,b\wedge c} - S_{a,c}S_{d,b} + S_{d,b,a\wedge c} + S_aS_{d,b,c} \\
&\quad +S_{a,c,b\wedge d} - S_{d,b,c,a} &(4.\text{-}3) \\
S_{a,c,d,b} &= -S_aS_{d,c,b} + S_{d,c,b,a} + S_{a,c\wedge d,b} + S_{a,d,b\wedge c} - S_{a\wedge d,b\wedge c} + S_{d,a,b\wedge c} + S_{a,c}S_{d,b} \\
&\quad -S_{d,a\wedge c,b} - S_{d,b,a\wedge c} - S_aS_{d,b,c} - S_{d,c,a\wedge b} + S_{d,c,a,b} + S_{d,b,c,a} &(4.\text{-}3) \\
S_{a,d,b,c} &= -S_{d,b,c,a} - S_{d,b,a,c} - S_{d,a,b,c} + S_aS_{d,b,c} + S_{a\wedge d,b,c} + S_{d,a\wedge b,c} + S_{d,b,a\wedge c} &(4.\text{-}2) \\
S_{a,d,c,b} &= -S_{d,c,b,a} - S_{d,c,a,b} - S_{d,a,c,b} + S_aS_{d,c,b} + S_{a\wedge d,c,b} + S_{d,a\wedge c,b} + S_{d,c,a\wedge b} &(4.\text{-}1) \\
S_{b,a,c,d} &= S_cS_{a,d,b} + S_cS_{a,b,d} + S_{c\wedge d,a,b} + S_cS_{d,a,b} - S_dS_{a,c,b} - S_bS_{c,a,d} - S_bS_{c,d,a} \\
&\quad -S_dS_{a,b,c} - S_{a,b}S_{c,d} + S_{d,a,b\wedge c} - S_{a\wedge d,b,c} - S_{a\wedge d,c,b} + S_aS_{b,c,d} + S_aS_{c,b,d} \\
&\quad +S_{a,d,b\wedge c} + S_{d,a\wedge c,b} + S_{a\wedge c,b\wedge d} - S_{a,b\wedge d,c} + S_aS_{c,d,b} + S_{c,b,a\wedge d} + S_{a\wedge c,d,b} \\
&\quad +S_{a\wedge b,c,d} + S_{b,a\wedge c,d} + S_{b,c,a\wedge d} - 2\,S_{a,c,b\wedge d} - 2\,S_{c,a,b\wedge d} + S_{a\wedge c,b,d} - S_{b\wedge c,a,d} \\
&\quad -S_{b\wedge c,d,a} - S_{c,b\wedge d,a} - S_{d,c,a,b} &(4.\text{-}4) \\
S_{b,a,d,c} &= -S_cS_{a,d,b} - S_cS_{a,b,d} - S_{c\wedge d,a,b} - S_cS_{d,a,b} + S_{a,b}S_{c,d} - S_{a,c\wedge d,b} + S_bS_{a,d,c} \\
&\quad -S_{d,a,b\wedge c} - S_{d,a\wedge c,b} - S_{a\wedge c,b\wedge d} + S_{a,b\wedge d,c} - S_{a,b,c\wedge d} + S_{a\wedge b,d,c} + S_{a,c,b\wedge d}
\end{aligned}$$

$$
\begin{aligned}
& +S_{c,a\wedge d,b}+S_{c,a,b\wedge d}+S_{d,a,b,c}+S_{d,c,a,b}+S_{d,a,c,b} && (4.\text{-}5)\\
S_{b,c,a,d} &= S_cS_{a,d,b}+S_cS_{a,b,d}+S_{c\wedge d,a,b}+S_cS_{d,a,b}+S_dS_{a,c,b}+S_bS_{c,a,d}+S_bS_{c,d,a}\\
& -S_{a,b}S_{c,d}-S_{a\wedge d,b,c}-S_aS_{b,c,d}-S_{a,b}S_{d,c}-S_aS_{c,b,d}+2\,S_{a,c\wedge d,b}-S_bS_{a,d,c} && (4.\text{-}5)\\
& -S_{a,d,b\wedge c}+S_{a\wedge c,b\wedge d}-S_{a,b\wedge d,c}+S_{a\wedge d,b\wedge c}+S_{b,a,c\wedge d}-S_{a\wedge b,c\wedge d}-S_aS_{c,d,b}\\
& +2\,S_{a,b,c\wedge d}-S_{c,b,a\wedge d}-S_{a\wedge c,d,b}-S_{b,c,a\wedge d}-2\,S_{c,a\wedge d,b}-S_{a\wedge c,b,d}+S_{b\wedge c,a,d}\\
& +S_{b\wedge c,d,a}+S_{c,b\wedge d,a}+S_{a,d}S_{b,c}-S_{d,a,c,b} && (4.\text{-}6)\\
S_{d,b,a,c} &= -S_cS_{a,d,b}-S_cS_{a,b,d}-S_{c\wedge d,a,b}-S_cS_{d,a,b}-S_aS_{d,c,b}+S_{d,c,b,a}+S_{a,b}S_{c,d}\\
& +S_{a\wedge d,b,c}+S_aS_{b,c,d}+S_{a,b}S_{d,c}-S_{a,c\wedge d,b}+S_bS_{a,d,c}+S_{a,d,b\wedge c}-S_{d,a\wedge c,b}\\
& -S_{a\wedge c,b\wedge d}+S_{a,b\wedge d,c}-S_{a\wedge d,b\wedge c}-S_{b,a,c\wedge d}+S_{a\wedge b,c\wedge d}-2\,S_{a,b,c\wedge d}+S_{b,c,a\wedge d}\\
& +S_{a,c,b\wedge d}+S_{c,a\wedge d,b}+S_{c,a,b\wedge d}-S_{a,d}S_{b,c}-S_{d,c,a\wedge b}+S_{d,c,a,b}+S_{d,a,c,b} && (4.\text{-}8)\\
S_{b,d,a,c} &= S_{b\wedge d,a,c}+S_bS_{d,a,c}+S_{d,a,b\wedge c}+S_{d,a\wedge b,c}-S_{d,a,b,c}-S_{d,a,c,b}-S_{d,b,a,c} && (4.\text{-}7)\\
S_{b,d,c,a} &= -S_{d,c,b,a}-S_{d,c,a,b}-S_{d,b,c,a}-S_bS_{d,a,c}-S_{b\wedge d,a,c}-S_{d,a,b\wedge c}-S_bS_{a,d,c}\\
& -S_{a,b\wedge d,c}-S_{a,d,b\wedge c}+S_aS_{b,d,c}+S_{b,a\wedge d,c}+S_{b,d,a\wedge c}+S_aS_{d,b,c}+S_{a\wedge d,b,c}\\
& +S_{d,b,a\wedge c}+S_aS_{d,c,b}+S_{a\wedge d,c,b}+S_{d,a\wedge c,b}+S_{d,c,a\wedge b} && (4.\text{-}8)\\
S_{c,a,b,d} &= S_cS_{a,b,d}-S_dS_{a,c,b}+S_{a\wedge c,b,d}-S_dS_{a,b,c}+S_{a,d,b\wedge c}-S_{a\wedge d,b\wedge c}+S_{d,a\wedge b,c}\\
& +S_{d,a,b\wedge c}+S_{a,c}S_{d,b}+S_{a,b\wedge c,d}-S_{a,b\wedge d,c}-S_{a,c,b\wedge d}-S_{d,b,a,c} && (4.\text{-}8)\\
S_{c,a,d,b} &= S_cS_{a,d,b}+S_{a\wedge c,d,b}-S_{a\wedge d,b,c}-S_{a\wedge d,c,b}+S_{a\wedge d,b\wedge c}-S_{d,a\wedge b,c}-S_{d,a,b\wedge c}\\
& -S_{a,c}S_{d,b}+S_{d,a,b,c}+S_{d,a,c,b}+S_{d,b,a,c} && (4.\text{-}8)\\
S_{c,b,a,d} &= -2\,S_cS_{a,d,b}-2\,S_cS_{a,b,d}-S_{c\wedge d,a,b}-S_cS_{d,a,b}-S_bS_{c,d,a}+S_dS_{a,b,c}+S_{a,b}S_{c,d}\\
& +2\,S_{a\wedge d,b,c}+S_{a\wedge d,c,b}+S_aS_{b,c,d}+S_{a,b}S_{d,c}+S_aS_{c,b,d}-2\,S_{a,c\wedge d,b}+S_bS_{a,d,c}\\
& -S_{a\wedge c,b\wedge d}+2\,S_{a,b\wedge d,c}-S_{a\wedge d,b\wedge c}-S_{b,a,c\wedge d}+S_{a\wedge b,c\wedge d}+S_aS_{c,d,b}-2\,S_{a,b,c\wedge d}\\
& +S_{c,a\wedge b,d}+S_{c,b,a\wedge d}+S_{b,c,a\wedge d}-S_{a,b\wedge c,d}+S_{a,c,b\wedge d}+2\,S_{c,a\wedge d,b}+S_{c,a,b\wedge d}\\
& -S_{b\wedge c,d,a}-S_{c,b\wedge d,a}-S_{a,d}S_{b,c}-S_{d,a,b,c} && (4.\text{-}11)\\
S_{c,b,d,a} &= 2\,S_cS_{a,d,b}+S_cS_{a,b,d}+S_{c\wedge d,a,b}+S_cS_{d,a,b}+S_bS_{c,d,a}-S_{a,b}S_{c,d}-2\,S_{a\wedge d,b,c}\\
& -S_{a\wedge d,c,b}-S_aS_{b,c,d}-S_{a,b}S_{d,c}+2\,S_{a,c\wedge d,b}-S_bS_{a,d,c}-S_{d,a\wedge b,c}+S_{a\wedge c,b\wedge d}\\
& -S_{a,b\wedge d,c}+S_{a\wedge d,b\wedge c}+S_{b,a,c\wedge d}-S_{a\wedge b,c\wedge d}-S_aS_{c,d,b}+2\,S_{a,b,c\wedge d}-S_{b,c,a\wedge d}\\
& -S_{a,c,b\wedge d}-2\,S_{c,a\wedge d,b}-S_{c,a,b\wedge d}+S_{b\wedge c,d,a}+S_{c,b\wedge d,a}-S_{d,b,a\wedge c}-S_aS_{d,b,c}\\
& +S_{a,d}S_{b,c}+S_{d,a,b,c}+S_{d,b,a,c}+S_{d,b,c,a} && (4.\text{-}14)\\
S_{c,d,a,b} &= S_{c\wedge d,a,b}+S_cS_{d,a,b}+S_{d,a\wedge c,b}+S_{d,a,b\wedge c}-S_{d,a,b,c}-S_{d,c,a,b}-S_{d,a,c,b} && (4.\text{-}13)\\
S_{c,d,b,a} &= -S_{d,c,b,a}-S_{d,b,c,a}-S_{d,b,a,c}-S_cS_{d,a,b}-S_{c\wedge d,a,b}-S_{d,a,b\wedge c}-S_cS_{a,d,b}\\
& -S_{a,c\wedge d,b}-S_{a,d,b\wedge c}+S_aS_{c,d,b}+S_{a\wedge d,b,c}+S_{c,a\wedge d,b}+S_{c,d,a\wedge b}+S_aS_{d,b,c}+S_{a\wedge d,c,b}\\
& +S_{d,a\wedge b,c}+S_{d,b,a\wedge c}+S_aS_{d,c,b}+S_{d,c,a\wedge b}\,. && (4.\text{-}14)
\end{aligned}
$$

Harmonic sums of this type occur for the first time at the level of weight 6.

4.2 Harmonic Sums with 3 Different Indices

This class contains 12 different sums. The coefficient matrix is M_{4b} is of rank 9 and again we may choose the last 3 harmonic sums to express the remaining 9. The relations for the sums are

$$
\begin{aligned}
S_{a,a,b,c} &= \frac{1}{2}\left[S_{a,a\wedge c,b}-S_aS_{a,c,b}-S_aS_{c,a,b}+S_aS_{c,b,a}-S_{c,a,a\wedge b}\right.\\
& \left.+S_{a\wedge c,a,b}+S_{a\wedge c,b,a}-S_{a\wedge a,c,b}-S_{a,c,a\wedge b}+S_{c,b,a\wedge a}-S_{c,a\wedge a,b}\right]+S_cS_{a,a,b}
\end{aligned}
$$

$$-\frac{1}{6}S_{b,b,a\wedge a,b,a}+\frac{1}{3}S_{b,a,b,a\wedge a,b}+\frac{1}{3}S_{a,a\wedge a,b,b,b}+\frac{1}{3}S_{b,b,a\wedge a,a,b}+\frac{1}{3}S_{b,a\wedge a,a,b,b}$$
$$-\frac{1}{2}S_{a,a,b,a,b\wedge b}-\frac{1}{6}S_{b,a,b,b,a\wedge a}-S_{b,b,b,a,a,a}+\frac{1}{3}S_{a\wedge b,a,a,b,b}+\frac{1}{3}S_aS_{b,b,a,a,b}$$
$$-\frac{1}{6}S_aS_{a,b,b,b,a}-\frac{1}{6}S_aS_{b,b,a,b,a}+\frac{1}{3}S_aS_{b,a,a,b,b}+\frac{1}{3}S_aS_{a,b,a,b,b}-\frac{1}{6}S_aS_{b,a,b,b,a}$$
$$-\frac{1}{2}S_bS_{a,a,b,a,b}-\frac{1}{2}S_bS_{a,b,a,a,b}+\frac{1}{3}S_aS_{b,b,b,a,a}-\frac{1}{2}S_bS_{b,a,a,a,b}+\frac{1}{3}S_aS_{a,b,b,a,b} \qquad (6.-12)$$

6.1.2 Index-Set $\{a,a,a,a,b,b\}$

This class contains 15 different sums which are represented by two basic sums. For $S_{a,a,a,a,b,b}$ one obtains

$$\begin{aligned}
S_{a,a,a,a,b,b} = & -\frac{1}{4}S_aS_{b,a,a,a,b}+\frac{3}{4}S_{a\wedge b,a,a,a,b}-\frac{1}{4}S_{b,a,a,a,a\wedge b}+\frac{1}{12}S_aS_{a,a,b,b,a}+S_{a,a,a,a,b\wedge b}\\
& -\frac{1}{12}S_{a\wedge a,b,b,a,a}-\frac{1}{12}S_{a,b,b,a\wedge a,a}-\frac{1}{12}S_{a,b,b,a,a\wedge a}-\frac{1}{4}S_{b,a\wedge a,a,a,b}-\frac{1}{4}S_{b,a,a\wedge a,a,b}\\
& -\frac{1}{4}S_{b,a,a,a\wedge a,b}-\frac{1}{4}S_{a,a\wedge a,a,b,b}-\frac{1}{4}S_{a,a,a\wedge a,b,b}-\frac{1}{4}S_{a,b,a\wedge a,a,b}-\frac{1}{4}S_{a,b,a,a\wedge a,b}\\
& +\frac{1}{12}S_{a\wedge a,b,a,b,a}+\frac{1}{12}S_{a,b,a\wedge a,b,a}-\frac{1}{4}S_{a,a,a,b,a\wedge b}-\frac{1}{4}S_{a,a,b,a,a\wedge b}+\frac{1}{12}S_{a,a,a\wedge b,b,a}\\
& +\frac{3}{4}S_{a,a,a\wedge b,a,b}-S_{b,b,a,a,a,a}+\frac{1}{4}S_{b,b,a,a\wedge a,a}-\frac{1}{4}S_{a\wedge a,a,a,b,b}+\frac{1}{12}S_{a,b,a,b,a\wedge a}\\
& +\frac{1}{12}S_{b,a\wedge a,a,b,a}+\frac{1}{12}S_{b,a,a\wedge a,b,a}+\frac{1}{12}S_{b,a,a,b,a\wedge a}-\frac{1}{12}S_{b,a\wedge a,b,a,a}-\frac{1}{12}S_{b,a,b,a\wedge a,a}\\
& +\frac{1}{4}S_{b,b,a\wedge a,a,a}+\frac{1}{4}S_{b,b,a,a,a\wedge a}-\frac{1}{4}S_{a\wedge a,a,b,a,b}-\frac{1}{4}S_{a,a\wedge a,b,a,b}-\frac{1}{4}S_{a,a,b,a\wedge a,b}\\
& +\frac{1}{12}S_{a\wedge a,a,b,b,a}+\frac{1}{12}S_{a,a\wedge a,b,b,a}+\frac{1}{12}S_{a,a,b,b,a\wedge a}-\frac{1}{4}S_{a\wedge a,b,a,a,b}-\frac{1}{12}S_{b,a,b,a,a\wedge a}\\
& +\frac{1}{12}S_{a\wedge b,a,a,b,a}+\frac{1}{12}S_{b,a,a,a\wedge b,a}-\frac{1}{12}S_{a\wedge b,a,b,a,a}+\frac{1}{4}S_{a\wedge b,b,a,a,a}+\frac{1}{4}S_{b,a\wedge b,a,a,a}\\
& -\frac{1}{12}S_{b,a,a\wedge b,a,a}+\frac{3}{4}S_{a,a,a,a\wedge b,b}+\frac{1}{12}S_{a,a,b,a\wedge b,a}+\frac{3}{4}S_{a,a\wedge b,a,a,b}-\frac{1}{4}S_{a,b,a,a,a\wedge b}\\
& +\frac{1}{12}S_{a,a\wedge b,a,b,a}+\frac{1}{12}S_{a,b,a,a\wedge b,a}-\frac{1}{12}S_{a,a\wedge b,b,a,a}-\frac{1}{12}S_{a,b,a\wedge b,a,a}-\frac{1}{4}S_aS_{a,a,a,b,b}\\
& -\frac{1}{12}S_aS_{a,b,b,a,a}+\frac{1}{12}S_aS_{a,b,a,b,a}-\frac{1}{12}S_aS_{b,a,b,a,a}+\frac{1}{12}S_aS_{b,a,a,b,a}-\frac{1}{4}S_aS_{a,b,a,a,b}\\
& +S_bS_{a,a,a,a,b}+\frac{1}{4}S_aS_{b,b,a,a,a}-\frac{1}{4}S_aS_{a,a,b,a,b}
\end{aligned} \qquad (6.-24)$$

6.1.3 Index-Set $\{a,a,a,a,a,b\}$

This class contains 6 different sums. The coefficient matrix reads

$$M_{6c}=\left\|\begin{array}{cccccc} 5&1&0&0&0&0\\ 0&4&2&0&0&0\\ 0&0&3&3&0&0\\ 0&0&0&2&4&0\\ 0&0&0&0&1&5\\ 1&1&1&1&1&1 \end{array}\right\| \qquad (6.-24)$$

DEPTH = 3

Index Set	Number	Dep. Sums of Depth 3	min. Weight	Fraction of fund. Sums
$\{a,a,a\}$	1	1	3	0
$\{a,a,b\}$	3	2	3	1/3
$\{a,b,c\}$	6	4	4	1/3

DEPTH = 4

Index Set	Number	Dep. Sums of Depth 4	min. Weight	Fraction of fund. Sums
$\{a,a,a,a\}$	1	1	4	0
$\{a,a,a,b\}$	4	3	4	1/4
$\{a,a,b,b\}$	6	5	4	1/6
$\{a,a,b,c\}$	12	9	5	1/4
$\{a,b,c,d\}$	24	18	6	1/4

DEPTH = 5

Index Set	Number	Dep. Sums of Depth 5	min. Weight	Fraction of fund. Sums
$\{a,a,a,a,a\}$	1	1	5	0
$\{a,a,a,a,b\}$	5	4	5	1/5
$\{a,a,a,b,b\}$	10	8	5	1/5
$\{a,a,a,b,c\}$	20	16	6	1/5
$\{a,a,b,b,c\}$	30	24	6	1/5
$\{a,a,b,c,d\}$	60	48	7	1/5
$\{a,b,c,d,e\}$	120	96	9	1/5

DEPTH = 6

Index Set	Number	Rel.1	Rel.2	Rel.3	Rel.1,2	Rel.1,2,3	min. Weight	Frac. of fund. Sums
$\{a,a,a,a,a,a\}$	1	1	1	1	1	1	6	0
$\{a,a,a,a,a,b\}$	6	5	5	5	5	5	6	1/6
$\{a,a,a,a,b,b\}$	15	11	9	7	12	13	6	2/15
$\{a,a,a,b,b,b\}$	20	14	12	8	16	17	6	3/20
$\{a,a,a,a,b,c\}$	30	22	18	12	24	25	7	1/6
$\{a,a,a,b,b,c\}$	60	41	35	23	47	50	7	1/6
$\{a,a,b,b,c,c\}$	90	60	52	36	70	76	8	7/45
$\{a,a,a,b,c,d\}$	120	81	70	45	94	100	8	1/6
$\{a,a,b,b,c,d\}$	180	118	104	67	140	150	8	1/6
$\{a,a,b,c,d,e\}$	360	232	208	132	280	300	10	1/6
$\{a,b,c,d,e,f\}$	720	455	416	261	560	600	12	1/6

6. Theory of Words

(LOTHAIRE
REUTENAUER)

CAN WE COUNT THE BASIS SIMPLER ?

YES.

INTRODUCE FREE LIE ALGEBRAS & THE THEORY OF CODES INTO PARTICLE PHYSICS.

EVERYTHING GOES THROUGH THE INDEX SET.

$\mathfrak{A} = \{a, b, c, d, \dots\}$ ALPHABET

$a < b < c < d < \dots$ ORDERED

$\mathfrak{A}^*(\mathfrak{A})$ SET OF WORDS OVER $\mathfrak{A}$

$W = a_1 \cdot a_2 \cdot a_{27} \, a_3 \cdot a_3 \dots a_{532}$ WORD

$\uparrow$ NON-COMMUTATIVE PRODUCT.

$W = p \cdot x \cdot s$

$\uparrow$ PREFIX $\uparrow$ SUFFIX

DEFINITION:

A LYNDON WORD IS SMALLER THAN ALL ITS SUFFIXES.

THEOREM [RADFORD, 1979]

THE SHUFFLE ALGEBRA $K\langle\mathfrak{A}\rangle$ IS FREELY GENERATED BY THE LYNDON WORDS.

$\longrightarrow$ I.E. THE NUMBER OF LYNDON WORDS IS THE NUMBER OF BASIC ELEMENTS.

EXERCISES:

$\{\underbrace{a, a, \dots, a, b}\}$ aaa..ab 1 LYNDON WORD

n PERMUTATIONS

$$\frac{n_{basic}}{n_{all}} = \frac{1}{n}$$

$n \equiv$ DEPTH (OF THE SUMS).

$\{a, a, a, b, b, b\}$

aaabbb
aababb
aabbab

3 LYNDON WORDS

$$\frac{n_{basic}}{n_{all}} = \frac{3}{20} < \frac{1}{6}$$

CAN ONE DERIVE A FORMULA ON THESE RELATIONS ?

(... DIG THE MATHEMATICAL LITERATURE.)

!

E. WITT (HH) (1937): JOURN. REINE & ANGEW. MATHEMATIK.

"TREUE DARSTELLUNG LIESCHER RINGE"

$$l_n(n_1, \dots, n_q) = \frac{1}{n} \sum_{d|n} \mu(d) \frac{\left(\frac{n}{d}\right)!}{\left(\frac{n_1}{d}\right)! \dots \left(\frac{n_q}{d}\right)!}$$

with $\sum_i n_i = n$.

$$l_6(\{a,a,a,b,b,b\}) = \frac{1}{6}\left[\mu(1)\frac{6!}{3!3!} + \mu(3)\frac{2!}{1!1!}\right] = 3$$

$$n_6(\{a,a,a,b,b,b\}) = \frac{6!}{3!\,3!} = \frac{4\cdot 5\cdot 6}{2\cdot 3} = 20$$

$$\frac{l_6(\{\dots\})}{n_6(\{\dots\})} = \frac{3}{20} < \frac{1}{6} \qquad ;\ \mu(3) = -1.$$

SUM OVER THE 2^{nd} WITT FORMULA.

↓

Weight	# Sums	Cum. # Sums	# Basic Sums	Cum. # Basic Sums	Cum. Fraction
1	2	2	0	0	0.0
2	6	8	1	1	0.1250
3	18	26	6	7	0.2692
4	54	80	16	23	0.2875
5	162	242	46	69	0.2851
6	486	728	114	183	0.2513

THE ALGEBRAIC RELATIONS REDUCE THE NUMBER OF MELLIN TRANSFORMS TO ~~24~~ 23 (OUT OF 80):

$$
\begin{array}{cccc}
\dfrac{\log(1+x)}{x+1} & \dfrac{\log^2(1+x)-\log^2(2)}{x-1} & \dfrac{\log^2(1+x)}{x+1} & \dfrac{\mathrm{Li}_2(x)}{x+1} \\[2ex]
\dfrac{\mathrm{Li}_2(x)-\zeta(2)}{x-1} & \dfrac{\mathrm{Li}_2(-x)}{x+1} & \dfrac{\mathrm{Li}_2(-x)+\zeta(2)/2}{x-1} \rightarrow & \dfrac{\log(x)\mathrm{Li}_2(x)}{x+1} \\[2ex]
\rightarrow \dfrac{\log(x)\mathrm{Li}_2(x)}{x-1} & \dfrac{\mathrm{Li}_3(x)}{x+1} & \cancel{\dfrac{\mathrm{Li}_3(x)-\zeta(3)}{x-1}} & \dfrac{\mathrm{Li}_3(-x)}{x+1} \\[2ex]
\dfrac{\mathrm{Li}_3(-x)-3\zeta(3)/4}{x-1} & \dfrac{\mathrm{S}_{1,2}(x)}{x+1} & \dfrac{\mathrm{S}_{1,2}(x)-\zeta(3)}{x-1} & \dfrac{\mathrm{S}_{1,2}(-x)-\zeta(3)/8}{x-1} \\[2ex]
\dfrac{\mathrm{S}_{1,2}(-x)}{x+1} & \dfrac{\mathrm{S}_{1,2}(x^2)}{x+1} & \dfrac{\mathrm{S}_{1,2}(x^2)-\zeta(3)}{x-1} & \log(1-x)\dfrac{\mathrm{Li}_2(-x)}{x+1} \\[2ex]
\dfrac{\log(1+x)-\log(2)}{x-1}\mathrm{Li}_2(x) & \cancel{\dfrac{\log(1+x)-\log(2)}{x-1}}\mathrm{Li}_2(-x) & \cancel{\dfrac{\log(1-x)\mathrm{Li}_2(x)}{1+x}} & \dfrac{\log(1+x)}{1+x}\mathrm{Li}_2(x)
\end{array}
$$

$$\rightarrow \frac{\log(x)\log^2(1+x)}{1-x}, \quad \frac{1}{1+x}\left[2\mathrm{Li}_3\left(\frac{1-x}{2}\right) - \ln(1-x)\,\mathrm{Li}_2\left(\frac{1-x}{2}\right)\right] \tag{191}$$

$\longrightarrow$ IN 2-LOOP PHYSICAL PROCESSES EVEN A LOWER NUMBER OF BASIC TRANSFORMS IS GOING TO OCCUR!

$\longrightarrow$ ANY 2-LOOP QUANTITY ($m \to 0$) CAN BE REPRESENTED AS A MELLIN POLYNOMIAL OF THE ABOVE FUNCTIONS.

7. Deeper Relations

CAN ONE REDUCE THE BASIS FURTHER ?

PRESENT RESULTS ONLY FINISHED FOR $O(\alpha^2)$, I.E. DEPTH = 4 SUMS ($w=4$).

FIG.

- NO SYSTEMATIC MATHEMATICAL THEORY YET.

THE NUMBER OF LYNDON WORDS $\ell_n(\{a_1 \ldots a\}) = 0$ $\left(\sum_{d|n} \mu(d) = 0\right)$.

$\longrightarrow$ DO NOT COUNT SINGLE HARMONIC SUMS, POLYNOMIALS OR RAT. FCT'S IN N.

$$\boxed{\mathbf{M}[\ln^\ell(x) f(x)](N) = \frac{\partial^\ell}{(\partial N)^\ell} \mathbf{M}[f(x)](N)}$$

IF $\mathbf{M}[f(x)](N)$ IS KNOWN, ANY DERIVATIVE IS KNOWN (EASILY CALCULATED).

$\psi(N)$ KNOWN $\longrightarrow$ $\psi^{(k)}(N)$ $\forall k$, KNOWN.

etc.

23 FCTS $\longrightarrow$ 20 FCTS $\qquad w \leq 4$.

RELATIONS BETWEEN MELLIN TRANSFORMS LEAD TO A FURTHER REDUCTION.

$\longrightarrow$ EXPLICIT CALCULATIONS.

WORK IN PROGRESS.

$\boxed{\text{THE LORD IS MERCY.}}$

FEYNMAN DIAGRAM CALCULATIONS SEEM NOT TO PRODUCE ALL POSSIBLE SUMS.

$\underline{O(\alpha^2)}$

$$\frac{1}{\varepsilon} \rightarrow \frac{\mathrm{Li}_2(x)}{1+x} \qquad \cancel{\ln(x)\frac{\mathrm{Li}_2(x)}{1+x}} \qquad \frac{\mathrm{Li}_3(x)}{1+x} \qquad \frac{S_{1,2}(x)}{1+x}$$

$$\left(\frac{\mathrm{Li}_2(x)}{1-x}\right)_+ \qquad \cancel{\ln(x)\frac{\mathrm{Li}_2(x)}{1-x}} \qquad \cancel{\left(\frac{\mathrm{Li}_3(x)}{1-x}\right)_+} \qquad \left(\frac{S_{1,2}(x)}{1-x}\right)_+$$

JB, S. MOCH

MASSLESS QCD @ 2 LOOPS DEPENDS ON ESSENTIALLY **5** FUNCTIONS FOR ANOM. DIMS. & WILSON COEFFICIENTS

REDUCTION : $77 \longrightarrow 5$

THANKS TO MELLIN SYMMETRY.

w		ALGEBRA		FURTHER MATH.	
5	162 →	46	⟶	? PROBABLY 14	'11/03
6	486 →	114	⟶	?	≤ 10

Appendix B: Overview on all Harmonic Sums up to Depth and Weight 6

Index Set	Number	Depth	Numb. of Relations
{−1}	1	1	1
{1}	1	1	1
{−2}	1	2	1
{2}	1	2	1
{−1, −1}	1	2	1
{−1, 1}	2	2	1
{1, 1}	1	2	1
{−3}	1	3	1
{3}	1	3	1
{−2, −1}	2	3	1
{−2, 1}	2	3	1
{2, −1}	2	3	1
{2, 1}	2	3	1
{−1, −1, −1}	1	3	1
{−1, −1, 1}	3	3	2
{−1, 1, 1}	3	3	2
{1, 1, 1}	1	3	1
{−4}	1	4	1
{4}	1	4	1
{−2, −2}	1	4	1
{−2, 2}	2	4	1
{2, 2}	1	4	1
{−3, −1}	2	4	1
{−3, 1}	2	4	1
{3, −1}	2	4	1
{3, 1}	2	4	1
{−2, −1, −1}	3	4	2
{−2, −1, 1}	6	4	4
{−2, 1, 1}	3	4	2
{2, −1, −1}	3	4	2
{2, −1, 1}	6	4	4
{2, 1, 1}	3	4	2

Index Set	Number	Depth	Numb. of Relations
{−1, −1, −1, −1}	1	4	1
{−1, −1, −1, 1}	4	4	3
{−1, −1, 1, 1}	6	4	5
{−1, 1, 1, 1}	4	4	3
{1, 1, 1, 1}	1	4	1
{−5}	1	5	1
{5}	1	5	1
{−4, −1}	2	5	1
{−4, 1}	2	5	1
{4, −1}	2	5	1
{4, 1}	2	5	1
{−3, −2}	2	5	1
{−3, 2}	2	5	1
{3, −2}	2	5	1
{3, 2}	2	5	1
{−3, −1, −1}	3	5	2
{−3, −1, 1}	6	5	4
{−3, 1, 1}	3	5	2
{3, −1, −1}	3	5	2
{3, −1, 1}	6	5	4
{3, 1, 1}	3	5	2
{−2, −2, −1}	3	5	2
{−2, −2, 1}	3	5	2
{−2, 2, −1}	6	5	4
{−2, 2, 1}	6	5	4
{2, 2, −1}	3	5	2
{2, 2, 1}	3	5	2
{−2, −1, −1, −1}	4	5	3
{−2, −1, −1, 1}	12	5	9
{−2, −1, 1, 1}	12	5	9
{−2, 1, 1, 1}	4	5	3
{2, −1, −1, −1}	4	5	3
{2, −1, −1, 1}	12	5	9
{2, −1, 1, 1}	12	5	9
{2, 1, 1, 1}	4	5	3

Index Set	Number	Depth	Numb. of Relations
{−1, −1, −1, −1}	1	4	1
{−1, −1, −1, 1}	4	4	3
{−1, −1, 1, 1}	6	4	5
{−1, 1, 1, 1}	4	4	3
{1, 1, 1, 1}	1	4	1
{−5}	1	5	1
{5}	1	5	1
{−4, −1}	2	5	1
{−4, 1}	2	5	1
{4, −1}	2	5	1
{4, 1}	2	5	1
{−3, −2}	2	5	1
{−3, 2}	2	5	1
{3, −2}	2	5	1
{3, 2}	2	5	1
{−3, −1, −1}	3	5	2
{−3, −1, 1}	6	5	4
{−3, 1, 1}	3	5	2
{3, −1, −1}	3	5	2
{3, −1, 1}	6	5	4
{3, 1, 1}	3	5	2
{−2, −2, −1}	3	5	2
{−2, −2, 1}	3	5	2
{−2, 2, −1}	6	5	4
{−2, 2, 1}	6	5	4
{2, 2, −1}	3	5	2
{2, 2, 1}	3	5	2
{−2, −1, −1, −1}	4	5	3
{−2, −1, −1, 1}	12	5	9
{−2, −1, 1, 1}	12	5	9
{−2, 1, 1, 1}	4	5	3
{2, −1, −1, −1}	4	5	3
{2, −1, −1, 1}	12	5	9
{2, −1, 1, 1}	12	5	9
{2, 1, 1, 1}	4	5	3

55

Index Set	Number	Depth	Numb. of Relations
{3, −2, −1}	6	6	4
{3, −2, 1}	6	6	4
{3, 2, −1}	6	6	4
{3, 2, 1}	6	6	4
{−3, −1, −1, −1}	4	6	3
{−3, −1, −1, 1}	12	6	9
{−3, −1, 1, 1}	12	6	9
{−3, 1, 1, 1}	4	6	3
{3, −1, −1, −1}	4	6	3
{3, −1, −1, 1}	12	6	9
{3, −1, 1, 1}	12	6	9
{3, 1, 1, 1}	4	6	3
{−2, −2, −1, −1}	6	6	5
{−2, −2, −1, 1}	12	6	9
{−2, −2, 1, 1}	6	6	5
{−2, 2, −1, −1}	12	6	9
{−2, 2, −1, 1}	24	6	18
{−2, 2, 1, 1}	12	6	9
{2, 2, −1, −1}	6	6	5
{2, 2, −1, 1}	12	6	9
{2, 2, 1, 1}	6	6	5
{−2, −1, −1, −1, −1}	5	6	4
{−2, −1, −1, −1, 1}	20	6	16
{−2, −1, −1, 1, 1}	30	6	24
{−2, −1, 1, 1, 1}	20	6	16
{−2, 1, 1, 1, 1}	5	6	4
{2, −1, −1, −1, −1}	5	6	4
{2, −1, −1, −1, 1}	20	6	16
{2, −1, −1, 1, 1}	30	6	24
{2, −1, 1, 1, 1}	20	6	16
{2, 1, 1, 1, 1}	5	6	4

Index Set	Number	Depth	Numb. of Relations
{−1, −1, −1, −1, −1, −1}	1	6	1
{−1, −1, −1, −1, −1, 1}	6	6	5
{−1, −1, −1, −1, 1, 1}	15	6	13
{−1, −1, −1, 1, 1, 1}	20	6	17
{−1, −1, 1, 1, 1, 1}	15	6	13
{−1, 1, 1, 1, 1, 1}	6	6	5
{1, 1, 1, 1, 1, 1}	1	6	1

SFB-CPP/03-???
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Mellin Representation for the Heavy Flavor Contributions to Deep Inelastic Structure Functions

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Abstract

We derive semi–analytic expressions for the analytic continuation of the Mellin transforms of the heavy flavor QCD coefficient functions for neutral current deep inelastic scattering in leading and next-to-leading order to complex values of the Mellin variable N. These representations are used in Mellin–space QCD evolution programs to provide fast evaluations of the heavy flavor contributions to the structure functions $F_2(x, Q^2), F_L(x, Q^2)$ and $g_1(x, Q^2)$.

[1]On leave of absence from IHEP, RU–142281 Protvino, Russia

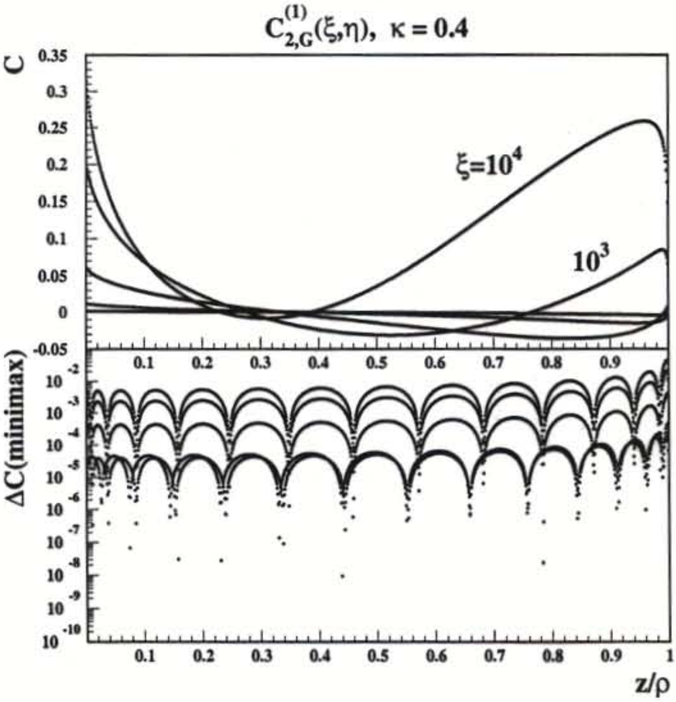


Figure 1d: Contribution to the NLO Wilson coefficient $c^{(1)}_{F_2,g}$. The MINIMAX-polynomial was determined choosing $\kappa = 0.4$ in (16). All other conditions are as in Figure 1a.

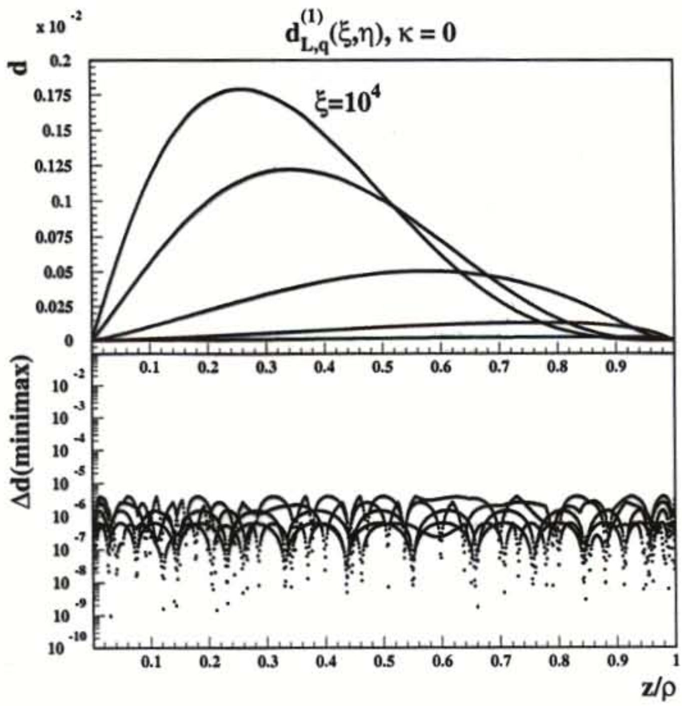


Figure 1e: Contribution to the NLO Wilson coefficient $d^{(1)}_{F_l,g}$. All other conditions are as in Figure 1a.

	max. absolute errors of the MINIMAX-polynomials					
Wilson Coeff.	κ	$\xi = 1$	$\xi = 10$	$\xi = 10^2$	$\xi = 10^3$	$\xi = 10^4$
$c^{(0)}_{F_L,g}$	0.	2.1e-5	4.5e-5	2.5e-5	5.7e-6	2.9e-7
$c^{(0)}_{F_2,g}$	0.5	1.4e-1	8.3e-3	3.0e-3	1.0e-3	3.8e-4
$c^{(0)}_{g_1,g}$	0.	1.1e-3	6.7e-4	2.4e-4	8.3e-5	3.0e-5
$c^{(1)}_{F_L,g}$	0.	4.1e-5	5.0e-5	1.4e-5	8.9e-6	6.9e-7
$\overline{c}^{(1)}_{F_L,g}$	0.	2.3e-5	5.0e-5	1.2e-6	1.8e-6	1.5e-7
$c^{(1)}_{F_L,q}$	0.	1.4e-5	2.2e-5	4.3e-6	3.3e-7	4.4e-7
$\overline{c}^{(1)}_{F_L,q}$	0.	6.0e-7	2.1e-6	3.7e-7	3.7e-8	3.1e-8
$d^{(1)}_{F_L,q}$	0.	4.0e-6	2.6e-6	6.1e-7	1.5e-6	6.3e-7
$c^{(1)}_{F_2,g}$	0.4	5.6e-2	2.6e-2	3.9e-3	1.0e-3	8.5e-4
$\overline{c}^{(1)}_{F_2,g}$	0.	8.9e-4	5.3e-3	1.9e-3	6.6e-4	2.3e-4
$c^{(1)}_{F_2,q}$	-0.5	2.6e-3	1.2e-3	2.2e-4	2.2e-5	6.3e-6
$\overline{c}^{(1)}_{F_L,q}$	0.	3.2e-4	1.3e-4	2.2e-5	1.8e-6	7.1e-7
$d^{(1)}_{F_2,q}$	0.	1.3e-4	5.1e-5	8.1e-6	1.0e-4	5.8e-4
$\overline{d}^{(1)}_{F_L,q}$	0.	1.4e-14	8.5e-55	–	–	–

Table 1: Maximal absolute errors of the MINIMAX–polynomialials for the LO and NLO Wilson coefficients as a function of ξ.

ANALYTIC CONTINUATION

$$\mathbb{M}[f(x)](N) = \int_0^1 dx\ x^{N-1} f(x) \qquad N \in \mathbb{N} \ \begin{smallmatrix}\text{even}\\ \text{or}\\ \text{odd}\end{smallmatrix}$$

$$N \longrightarrow \mathbb{C}$$

WHERE ARE SINGULARITIES ? SINGLE POLES

$\mathrm{Re}(N_s) \leqslant n_1$ fixed HARMONIC SUMS $\rightarrow$ N_s integer

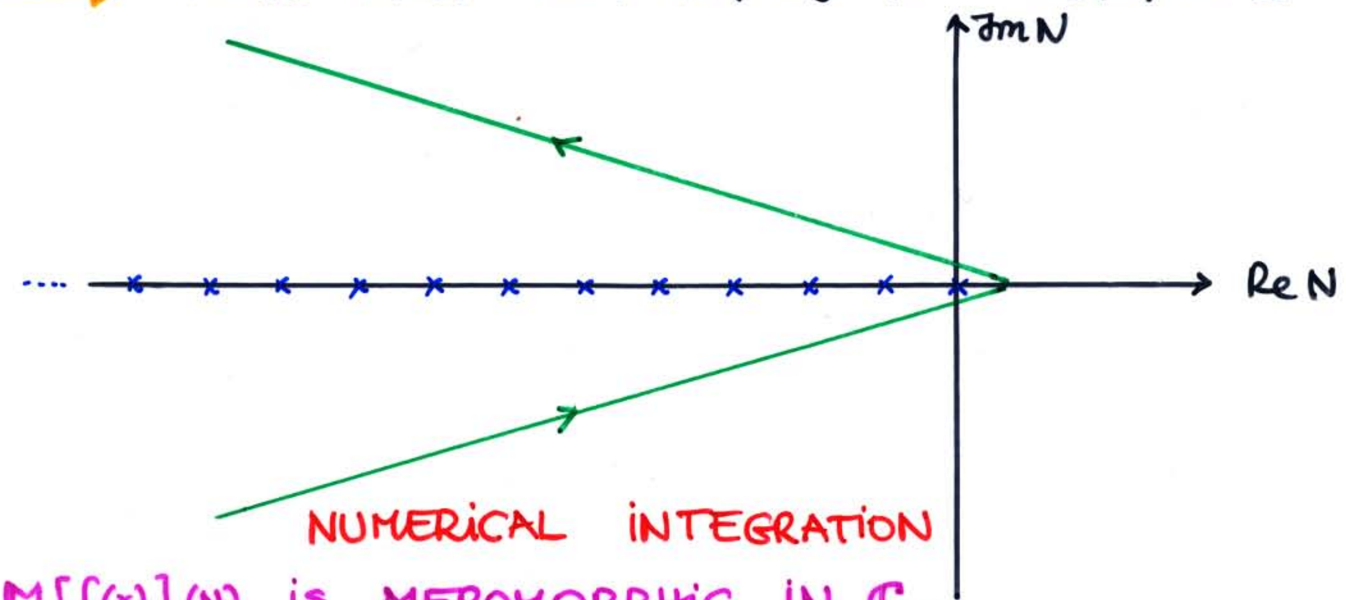
fixed order PT $\mathrm{Im}(N_s) = 0$

REPRESENTATION: (NIELSEN, MELLIN ~ 1905)

- STIRLING-LIKE ASYMPTOTIC REPRESENTATIONS.
- USE RECURSION RELATIONS:

$$S_{a b_1 \dots b_p}(N+1) - S_{a b_1 \dots b_p}(N) = \frac{[\mathrm{sign}(a)]^N}{N^{|a|}} S_{b_1 \dots b_p}(N)$$

$\Rightarrow$ MOVE FROM ANY $N \neq N_s$ TO $\mathrm{Re}(N) \gg 1$



$\mathbb{M}[f(x)](N)$ is MEROMORPHIC IN $\mathbb{C}$.

CONCLUSIONS

1) ALGEBRAIC RELATIONS BETWEEN MULTIPLE ζ-VALUES ARE KNOWN TO WEIGHT = 12 (MAY BE MORE: → LL 2004)

→ LIMIT: CPU & RUNNING TIME

→ M. WALDSCHMIDT: WOULD BE INTERESTING TO KNOW FOR $w > 20$ ... NUMBER THEORY.

2) BASIS COUNTING FOR FREE ALGEBRAS IS KNOWN IN GENERAL.

→ EXPLICIT RELATIONS: ALL WEIGHTS TO DEPTH = 6

→ HARMONIC SUMS, HARMONIC POLYLOGS. (MULTIPLE)

THIS PART HAS MANY MORE IMMEDIATE APPLICATIONS → MULTI-SCALE FUNCTIONS (JET PHYSICS etc.)

3) RELATIONS BY VALUE: HARM. SUMS

ALL 2-LOOP REDUCTIONS, WEIGHT = 5

3 LOOP REDUCTIONS ARE THERE

AS FAR AS NEEDED FOR THE PHYSICS

4) HARMONIC SUMS $S_{i_1 \dots i_k}(N)$ MAY BE ANALYTICALLY CONTINUED TO $N \in \mathbb{C}$ AND ARE MEROMORPHIC FUNCTIONS.