

Perturbation Theory for High-Precision Lattice QCD

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HPQCD Collaboration

KITP Jan. 15 2004

Major recent developments

(1) Unquenched simulations

- ▶ a new “staggered” discretization for light quarks
(*much* more efficient & accurate than other disc’ns)
- ▶ *much* smaller sea quark masses
(factor 3-5 smaller; also run *at* 2+1 flavours)
- ▶ provides reliable χ extrapolation
- ▶ reduces systematic errors on “Gold-Plated” quantities to few %
(cf. 10-20% errors in quenched approx.)

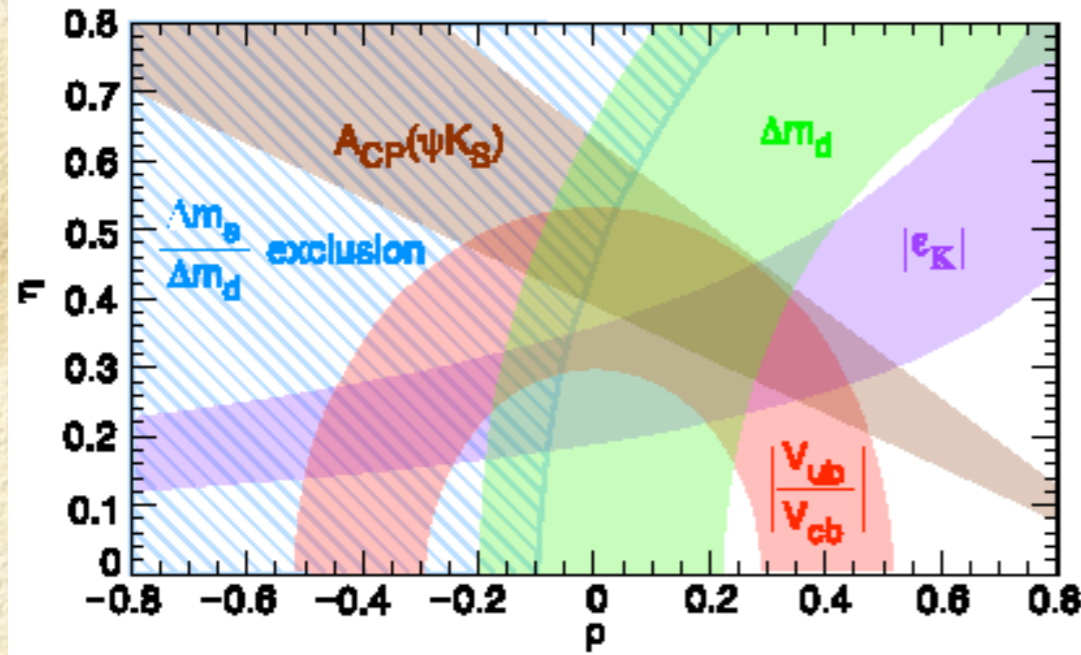
Major recent developments

(2) Perturbation theory

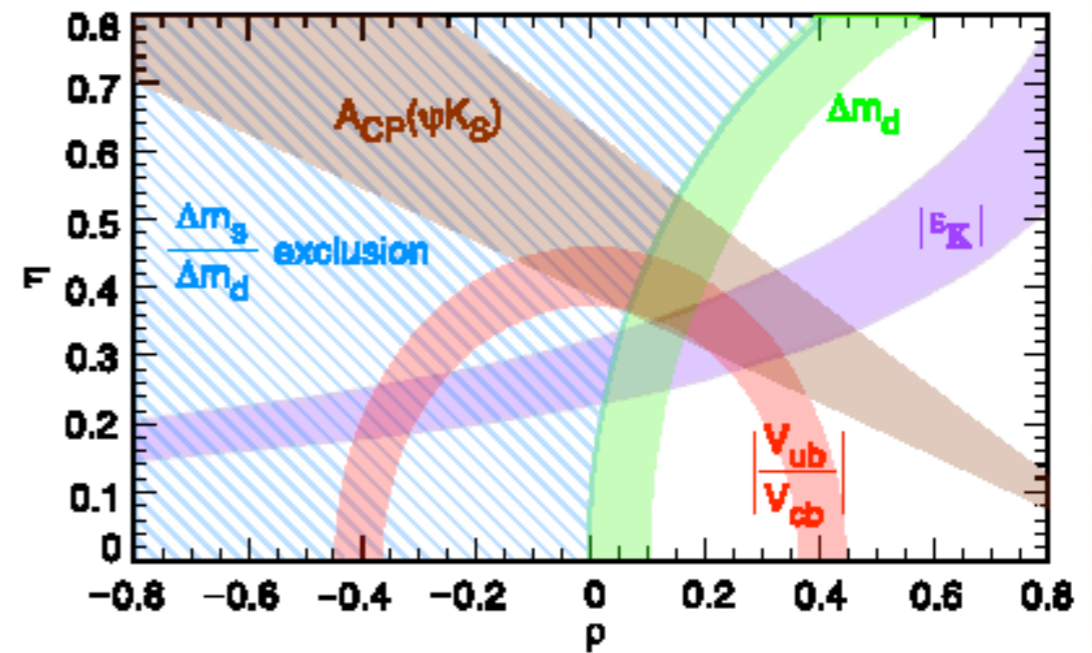
- ▶ matching lattice \Leftrightarrow continuum QCD
crucial for future progress (CKM parameters)
 - ▶ must routinely do **two-loop** matching
- ▶ “novel” challenges PT with lattice regulator
 - ▶ enormous algebraic bottleneck: lattice Feynman rules
 - ▶ how to deal with infrared divergences beyond one-loop
- ▶ bring continuum expertise to bear?

Implication of few-% LQCD on CKM

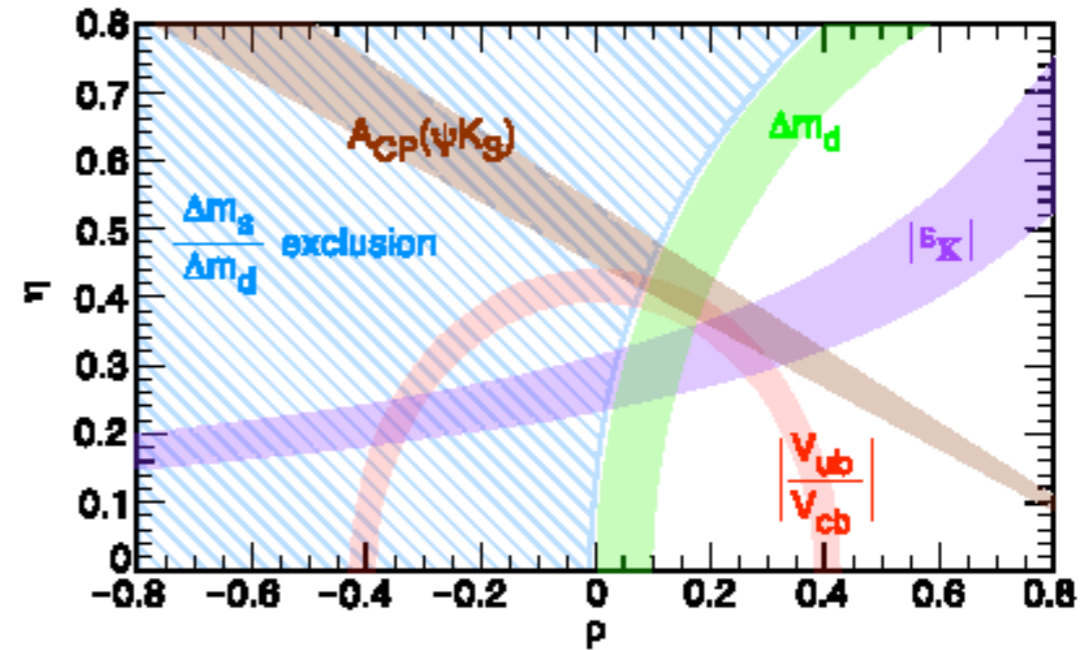
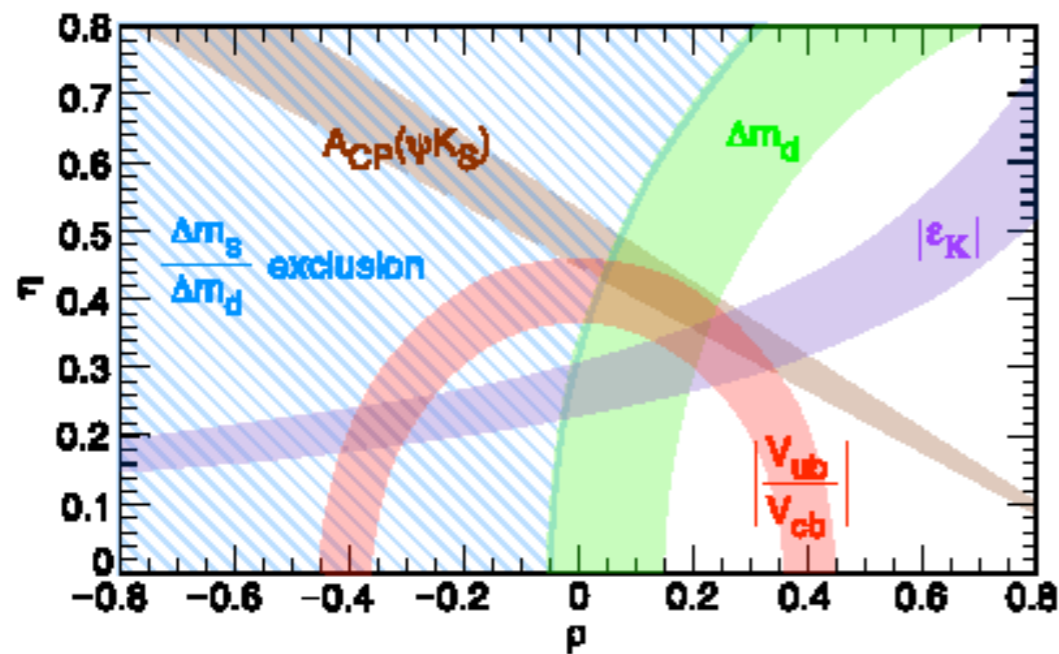
CKM today ...



... and with 2–3% theory errors



And with B Factories ...



95% confidence levels; CLEO-c (2001).

A collaboration of collaborations

PHYSICAL REVIEW LETTERS

High-Precision Lattice QCD Confronts Experiment

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The recently developed Symanzik-improved staggered-quark discretization allows unquenched lattice-QCD simulations with much smaller (and more realistic) quark masses than previously possible. To test this formalism, we compare experiment with a variety of nonperturbative calculations in QCD drawn from a restricted set of “gold-plated” quantities. We find agreement to within statistical and systematic errors of 3% or less. We discuss the implications for phenomenology and, in particular, for heavy-quark physics.

▶ generate unquenched backgrounds (MILC)

▶ generate & analyze correlators

▶ χ extrap'n

▶ perturbative LQCD \Leftrightarrow continuum

LQCD can't do everything (yet)

- ▶ Unstable hadrons e.g. $\rho \rightarrow \pi\pi \rightarrow \rho$
 - ▶ intermediate particles propagate to lattice boundaries, induces large finite volume errors
- ▶ Hadrons near decay thresholds e.g. $\psi' \rightarrow D\bar{D} \rightarrow \psi'$

“Gold-plated” LQCD quantities

- ▶ Narrow/Stable hadrons sufficiently below threshold
 - ▶ e.g. $\pi, K, p, D, D_s, B, B_s, J/\psi, \Upsilon, \Upsilon', \dots$ but not ρ, D^*, ψ', \dots
- ▶ At most **one hadron** in **initial** and **final** states
 - ▶ i.e. semileptonic ✓ while nonleptonic ✗

Must work if LQCD is to be trusted at all

“Gold-plated” LQCD meets CKM

V_{ud}

$\pi \rightarrow \ell\nu$

V_{us}

$K \rightarrow \ell\nu$

$K \rightarrow \pi\ell\nu$

V_{ub}

$B \rightarrow \pi\ell\nu$

V_{cd}

$D \rightarrow \ell\nu$

$D \rightarrow \pi\ell\nu$

V_{cs}

$D_s \rightarrow \ell\nu$

$D \rightarrow K\ell\nu$

V_{cb}

$B \rightarrow D\ell\nu$

V_{td}

$\langle B_d | \overline{B}_d \rangle$

V_{ts}

$\langle B_s | \overline{B}_s \rangle$

V_{tb}

(1) Recent Unquenched LQCD

$$\langle \mathcal{O} \rangle = \int [dU_\mu(x)] [d\bar{\psi}d\psi] \mathcal{O} e^{-\beta(S_{\text{gluon}} + S_{\text{quark}}^{\text{stagg}})}$$

► Only 5 input parameters (same as in continuum QCD)

► $m_u (= m_d), m_s, m_c, m_b, a$ ($\Leftrightarrow \alpha_s$)

► $m_{u/d} \leftarrow m_\pi^2$

► $m_s \leftarrow 2m_K^2 - m_\pi^2$

► $m_c \leftarrow m_D$

► $m_b \leftarrow m_\Upsilon$

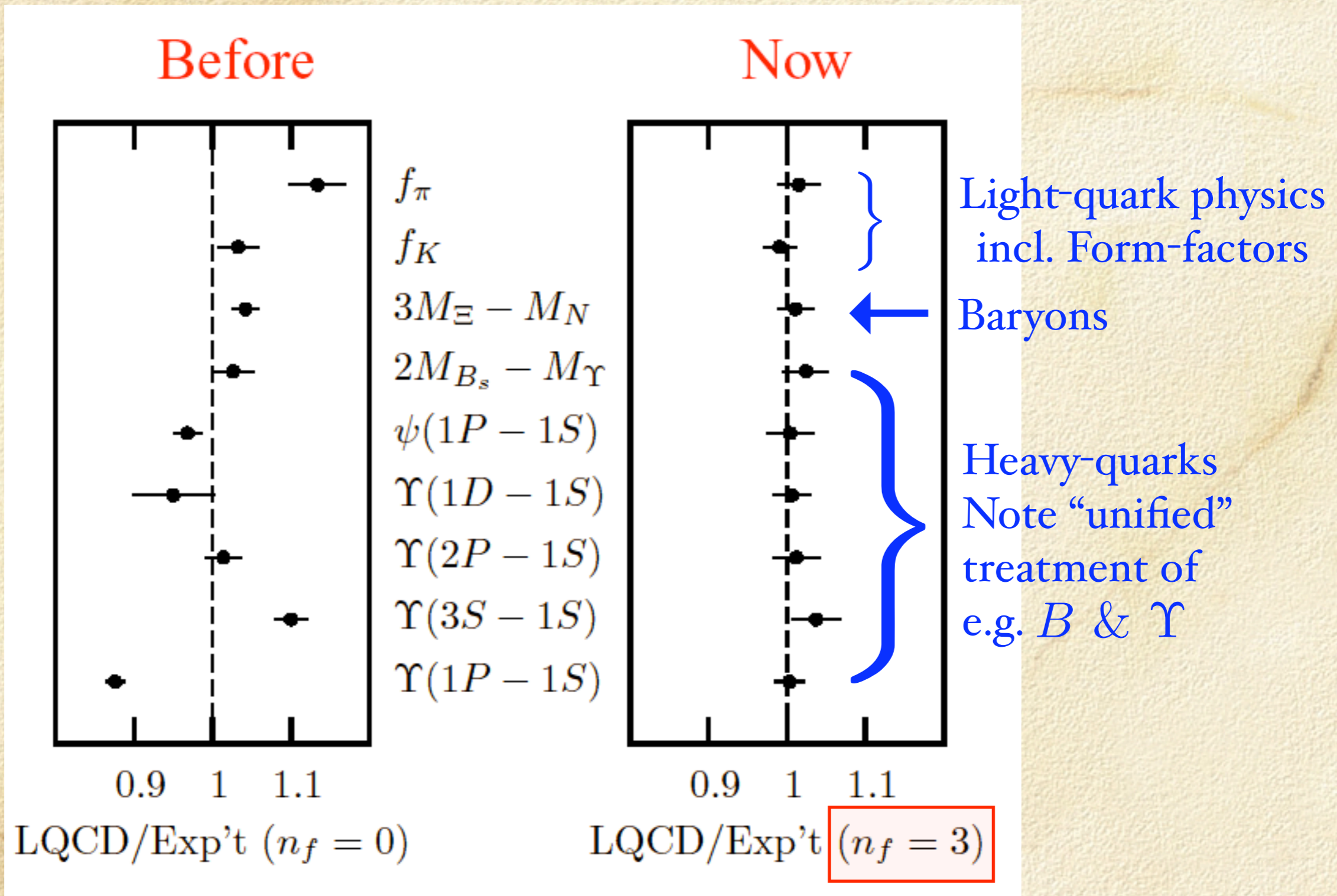
► $a \leftarrow m_{\Upsilon'} - m_\Upsilon$

Each experimental quantity roughly \propto the one m_{quark} and roughly independent of the other masses

This mass difference roughly independent of all m_{quark} 's

Next: Predict gold-plated quantities

LQCD / Experiment



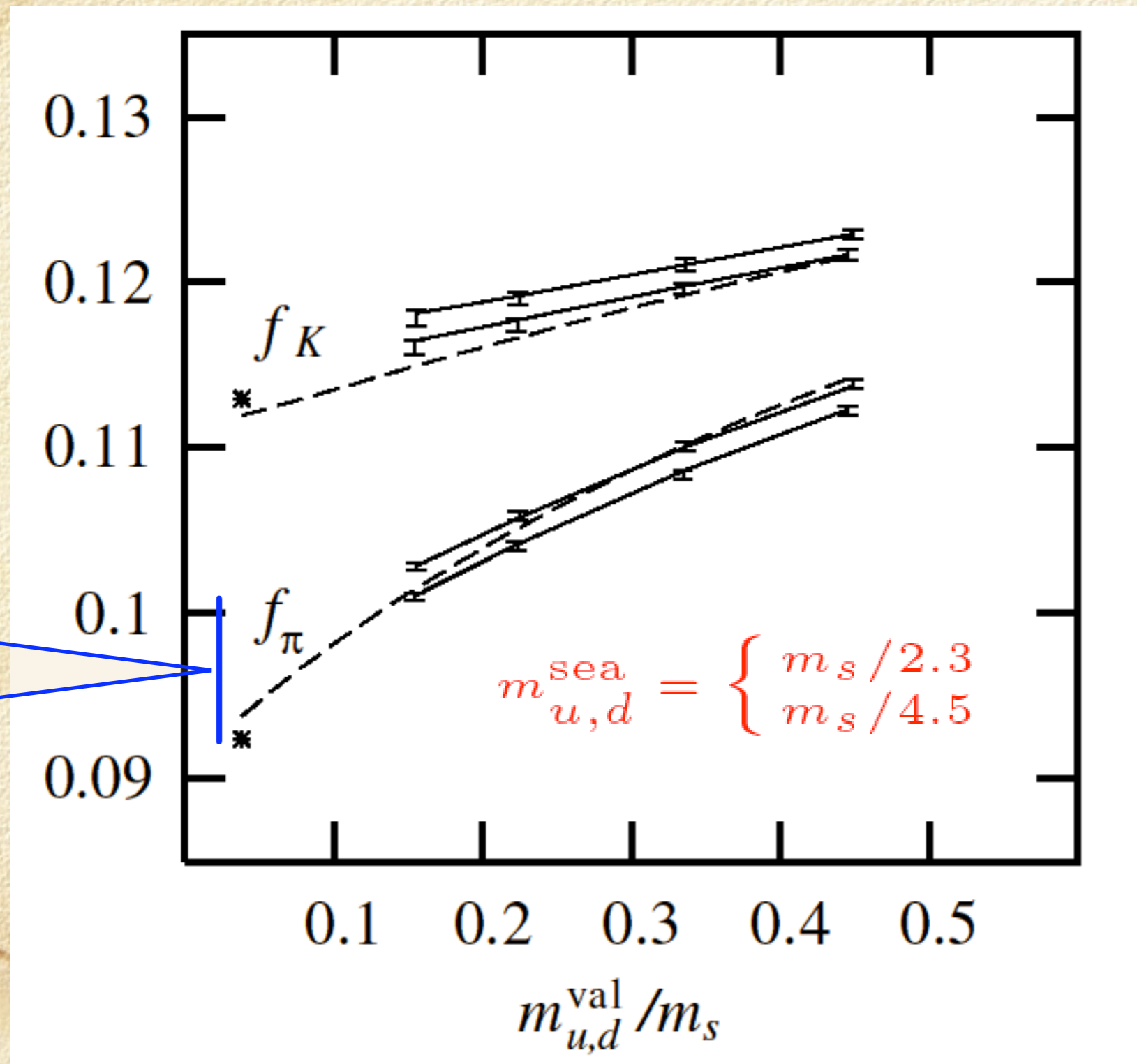
Errors $\sim 10-15\%$

Errors $< 3\%$

Partially Quenched χ PT

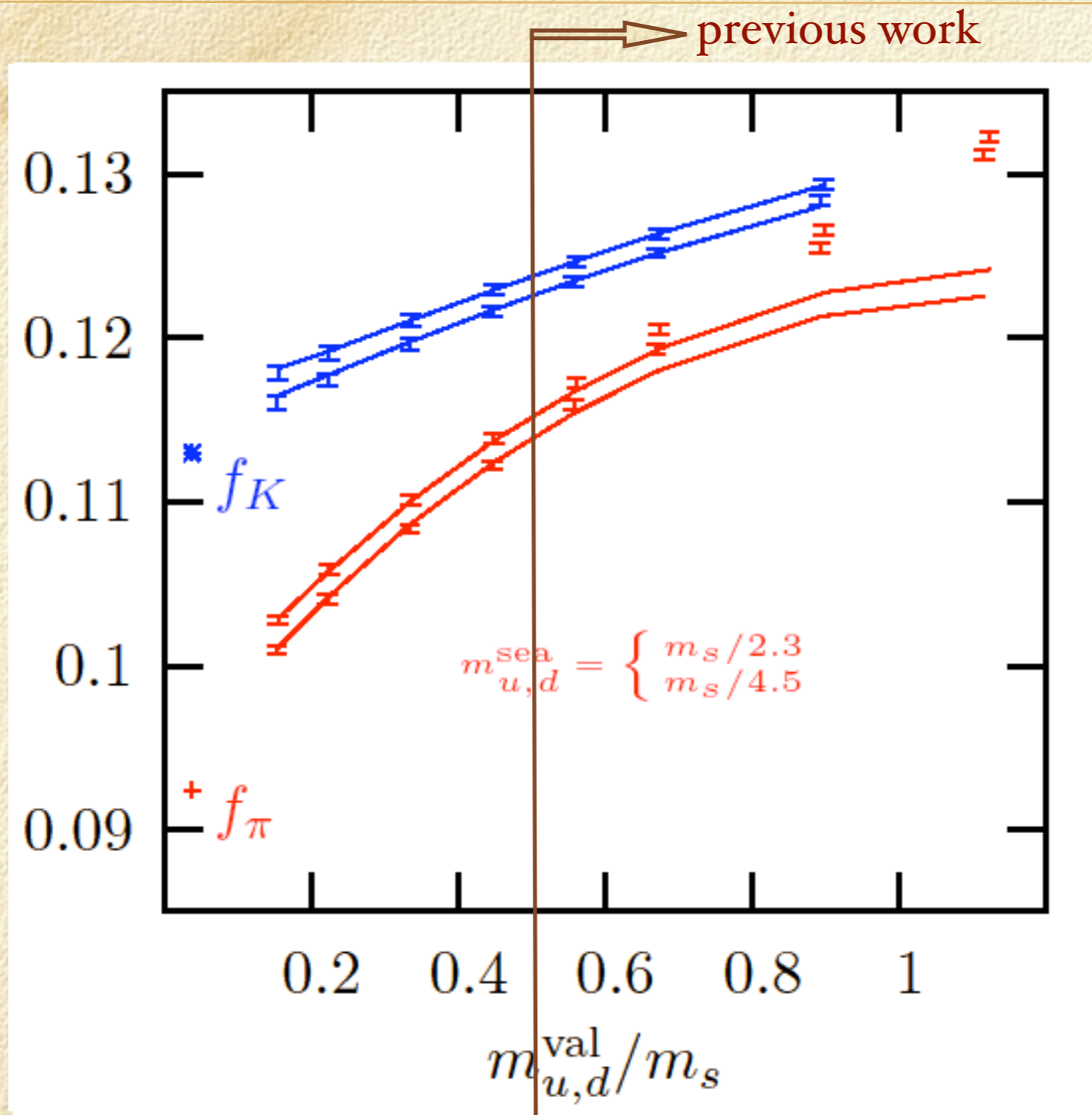
Lee & Sharpe
Bernard $N_f=2+1$

Run at several $m_{u/d,s}^{\text{valence}}$ not necessarily equal to m^{sea}



Sea quark masses this study $\sim 3-5$ times smaller than in previous unquenched

Impact of small sea-quark masses



- ▶ χ PT expansion poor for $m_{u,d} \gtrsim m_s/2$
- ▶ f_π - high mass extrapolation looks linear, but 10% high

Why speedup now?

Fast “staggered” quarks all along

► Problem: needed perturbative improvement to obtain sufficient accuracy (1999)

► flavour doubling: at the root of many ills

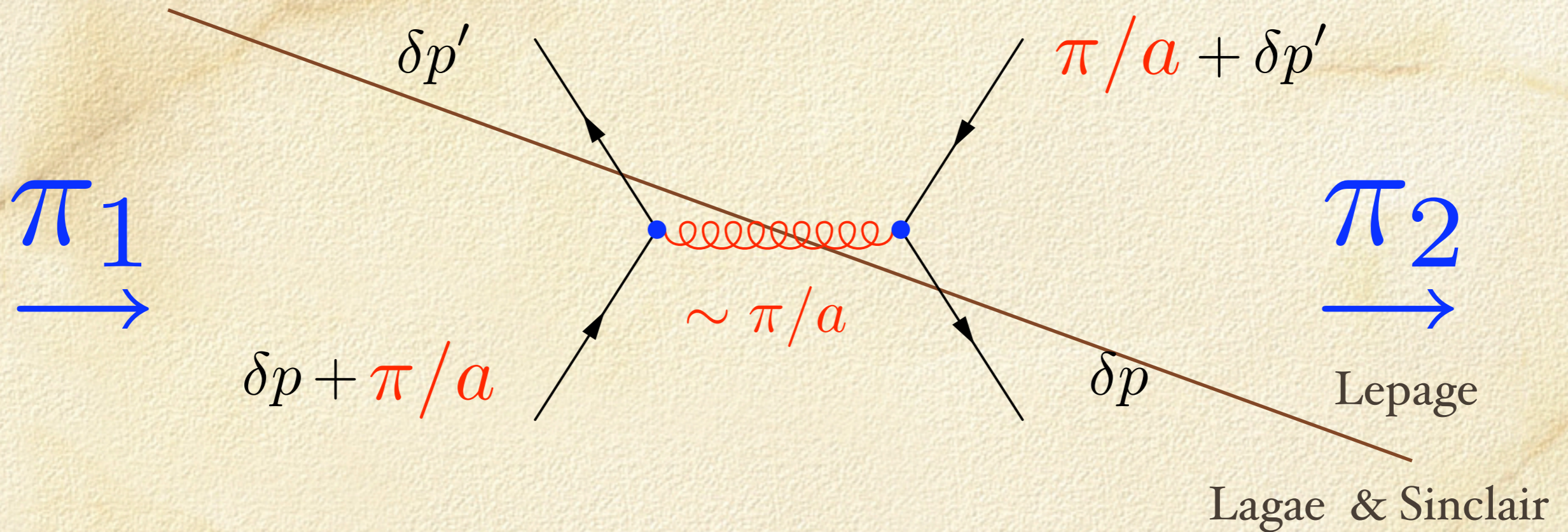
► $\mathcal{L}_{\text{naive}}^{\text{free}} = \bar{\psi}(x) \gamma_{\mu} \frac{1}{2a} [\psi(x + a\hat{\mu}) - \psi(x - a\hat{\mu})]$

$\Rightarrow \sin(p_{\mu}a) \times \bar{\psi}(p) \gamma_{\mu} \psi(p)$

\Rightarrow low-energy modes at $p_{\mu} = 0, \pi/a$

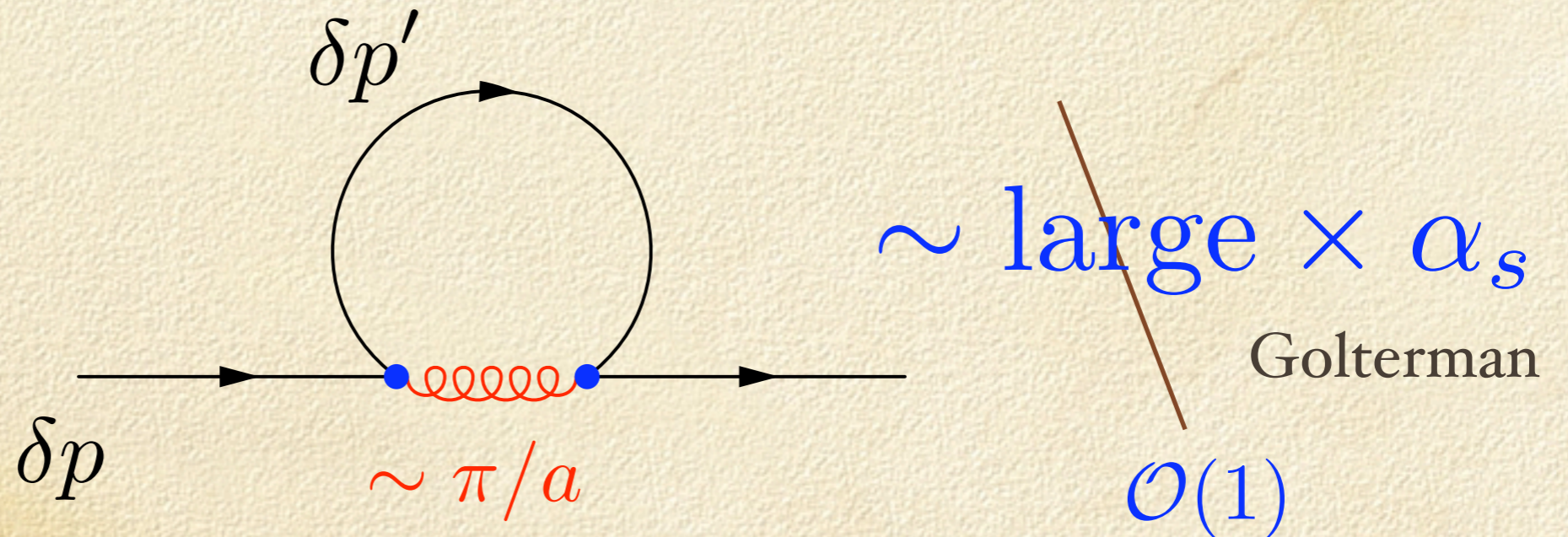
$\therefore 2^4$ degenerate copies (“tastes”) (reduce to 4 tastes by “staggering”)

Taste-changing interactions



Quark
Tadpole
Diagram

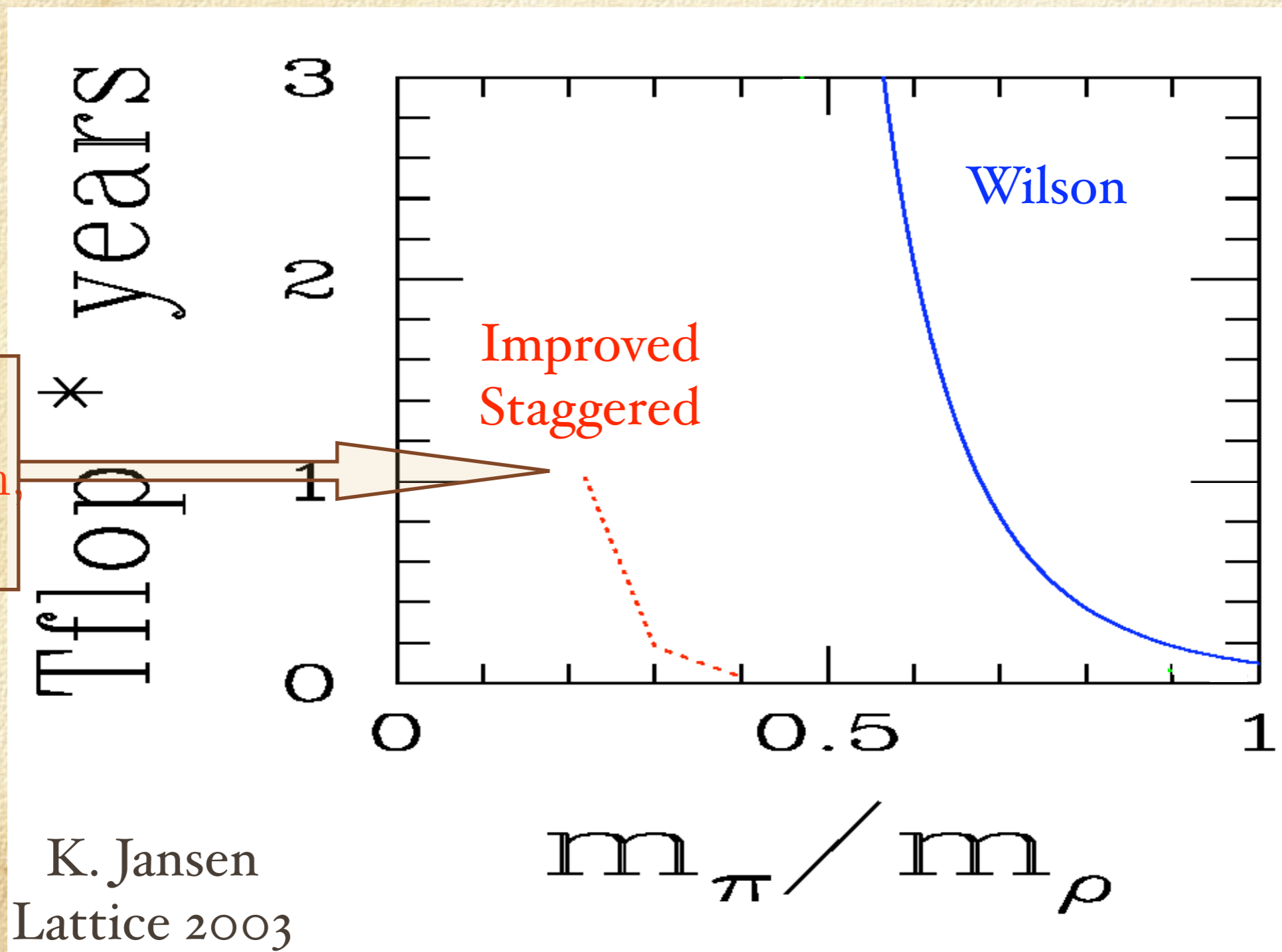
cf. Gluonic Tadpoles
Lepage & Mackenzie



Virtue of staggered quarks

- ▶ Preserves a χ symmetry at finite lattice space
 - ▶ eigenvalues of $\gamma \cdot D + m \rightarrow i\lambda + m$
- ▶ Contrast with Wilson quarks
 - ▶ $\mathcal{L}_{\text{Wilson}} = \mathcal{L}_{\text{naive}} - \frac{a}{2} \bar{\psi} \square \psi$
 - ▶ eliminates “tastes” but ~~χ symmetry~~,
leads to additive mass renormalization
 - ▶ zero modes slow down matrix inversion

This is why we use staggered!



A potential pit-fall

To get desired $(2+1)$ -flavours instead of 4
 $\det(\gamma \cdot D + m) \rightarrow \det(\gamma \cdot D + m)^{1/4}$

☂ potentially worrisome **non-localities**

☺ **No problems** at any order in PT (Batrouni et al 1985)

☺ **Chiral anomalies** correctly handled (Sharatchandra et al 1981;
Smit & Vink 1988)

☺ **Non-localities tied** to **taste-changing** interactions

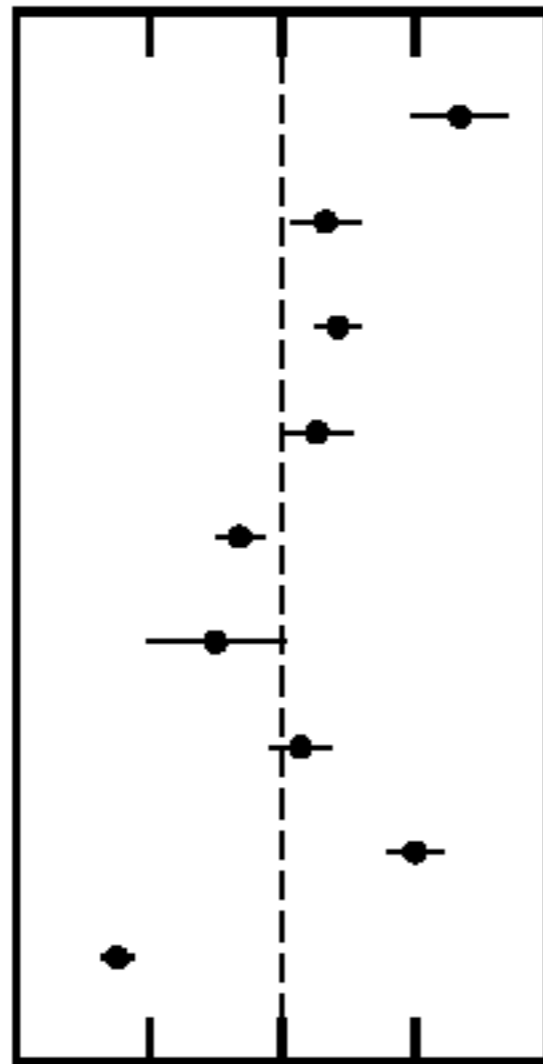
▶ **perturbative** in origin: well controlled

☂ missing strong ~~CP~~ phase at $m_u + m_d < 0$ (Dashen)

☺ **real world** is **not** in this phase

The price we pay for speed

Before

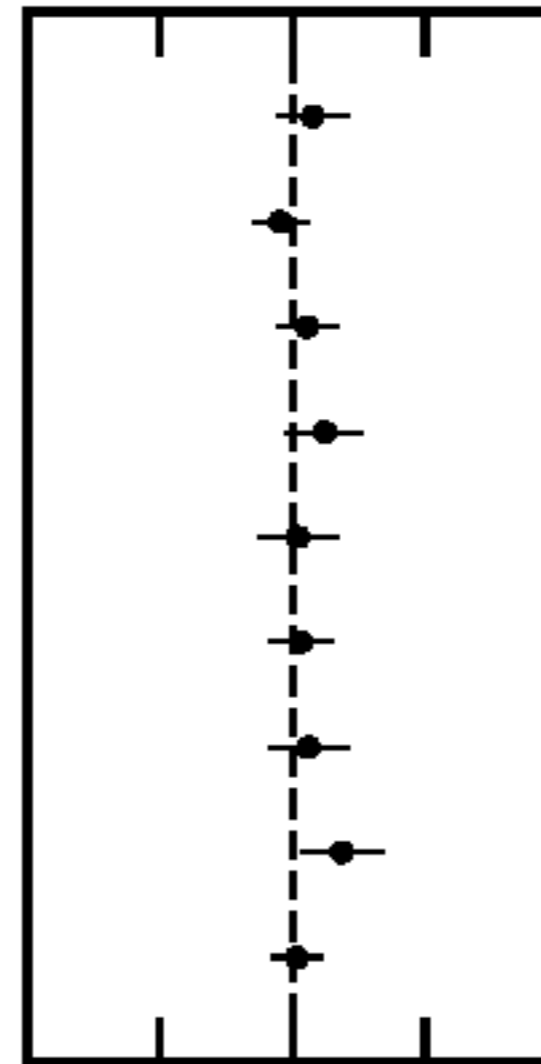


f_π
 f_K
 $3M_\Xi - M_N$
 $2M_{B_s} - M_\Upsilon$
 $\psi(1P - 1S)$
 $\Upsilon(1D - 1S)$
 $\Upsilon(2P - 1S)$
 $\Upsilon(3S - 1S)$
 $\Upsilon(1P - 1S)$

0.9 1 1.1

LQCD/Exp't ($n_f = 0$)

Now

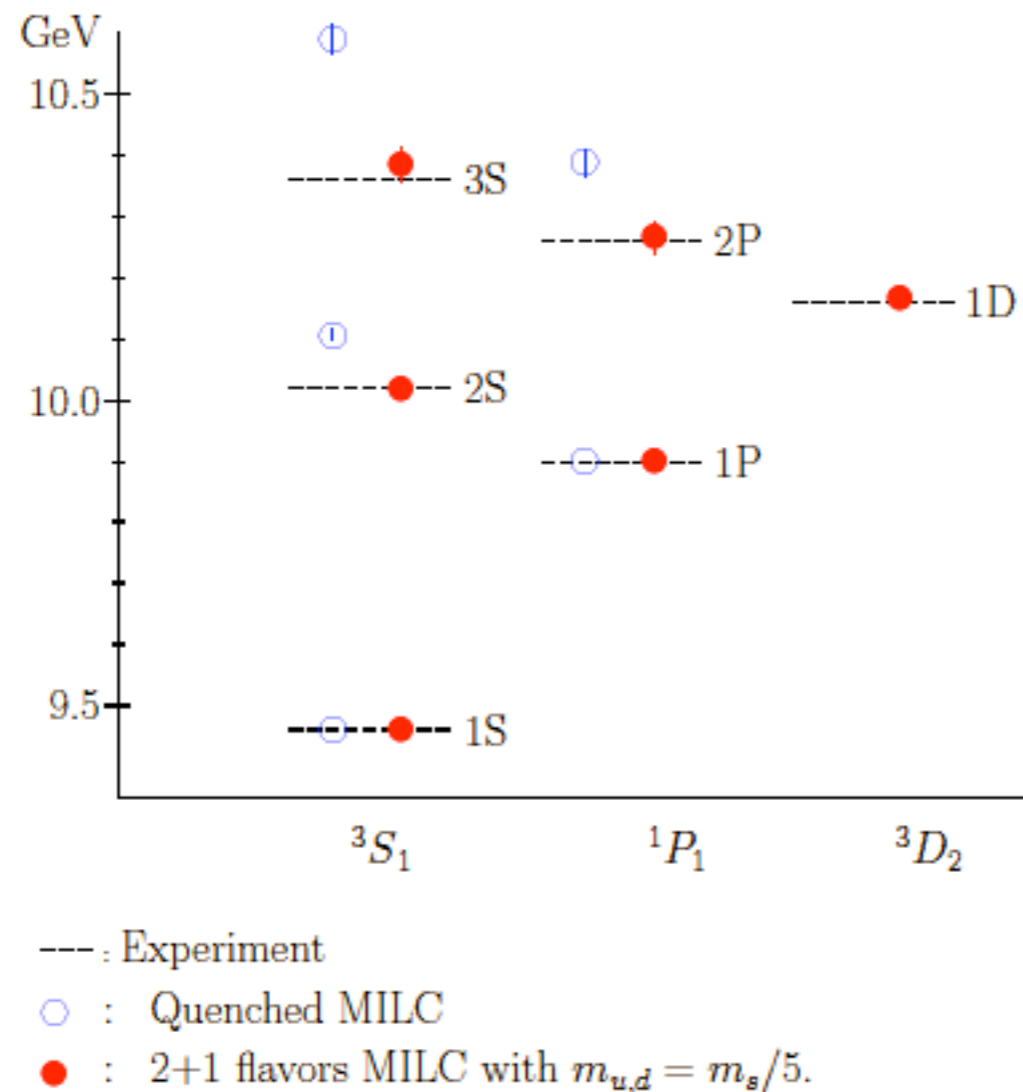


0.9 1 1.1

LQCD/Exp't ($n_f = 3$)

Sampler of Recent Calculations

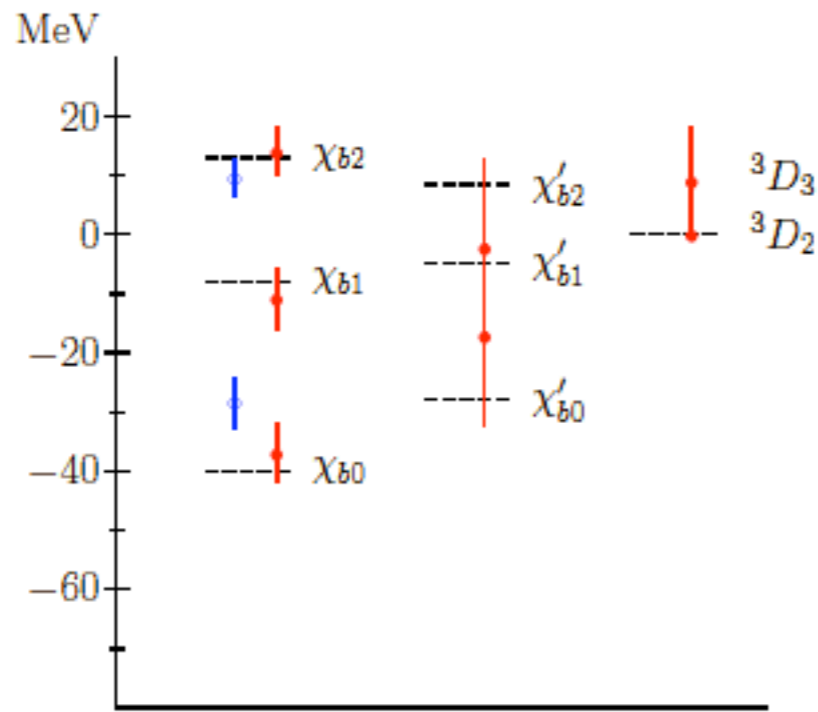
Υ Spectrum



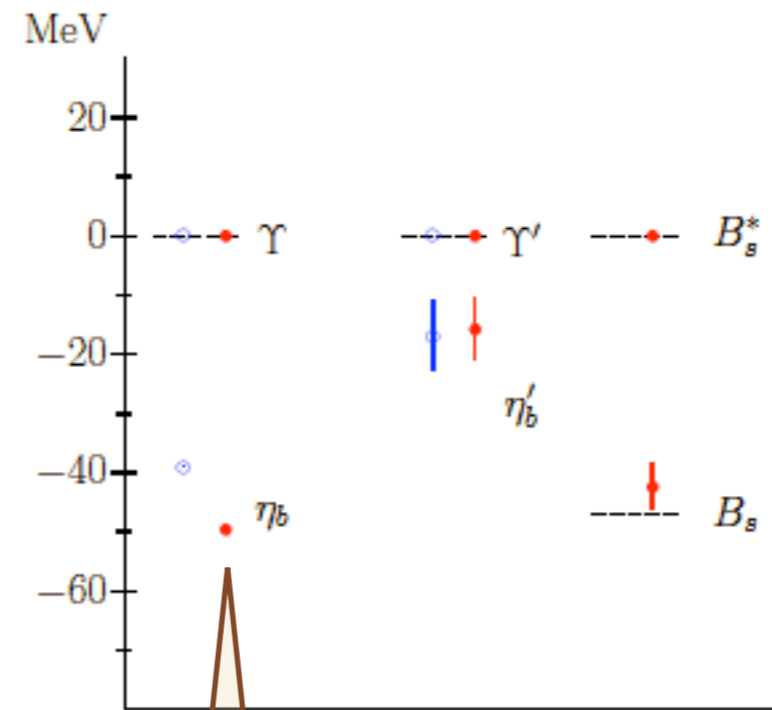
Note:

- Direct from QCD path integral; no potential model....
- Tests/tunes b quark action for use in B physics \Rightarrow overconstrained.
- Other tests: leptonic widths, photon transitions, fine structure.
- Statistical and systematic errors of 2–3%; 1S and 1P used in tuning.

Υ Fine Structure



- : Experiment
- ◊ : Quenched
- : 2+1 flavours MILC with $m_{u,d} = m_s/5$.



- : Experiment
- ◊ : Quenched
- : 2+1 flavours MILC with $m_{u,d} = m_s/5$.

Note: 20–30% systematic error due to use of tree-level pert'n theory.

Davies, Gray et al. (HPQCD, 2002).

Hyperfine splitting & e.g. f_B related

Depend on perturbation theory: $\mathcal{L}_{\text{HeavyQ}} = -c_B \frac{g}{2M_Q^0} \psi^\dagger \vec{\sigma} \cdot \vec{B} \psi + \dots$

(2) Lattice PT key to further progress

$$= \left(\begin{array}{cccc} Z \times & \text{triangle} & + & \text{triangle} & + & \text{triangle} & + & \dots \\ & \text{blue lines} & & \text{blue lines} & & \text{blue lines} & & \text{vertical dots} \end{array} \right)_{\text{lat}}$$

$$= \left(\begin{array}{cccc} & \text{triangle} & + & \text{triangle} & + & \text{triangle} & + & \dots \\ & \text{blue lines} & & \text{blue lines} & & \text{blue lines} & & \text{vertical dots} \end{array} \right)_{\text{cont}}$$

$$Z = 1 + c_1 \alpha_s (\pi/a) + c_2 \alpha_s^2 (\pi/a) + \dots$$

Demands an ambitious perturbative program

$$\alpha_V(1/a) \approx 0.2-0.3 \quad (a \approx 0.1 \text{ fm})$$

- ▶ few % precision => match through second-order
- ▶ *very few* previously in lattice PT

HPQCD Perturbation Theory “Subgroup”

▶ Quentin Mason ▶ Matthew Nobes

▶ G.P. Lepage

▶ C. Davies, A. Gray

▶ K. Wong, R. Woloshyn

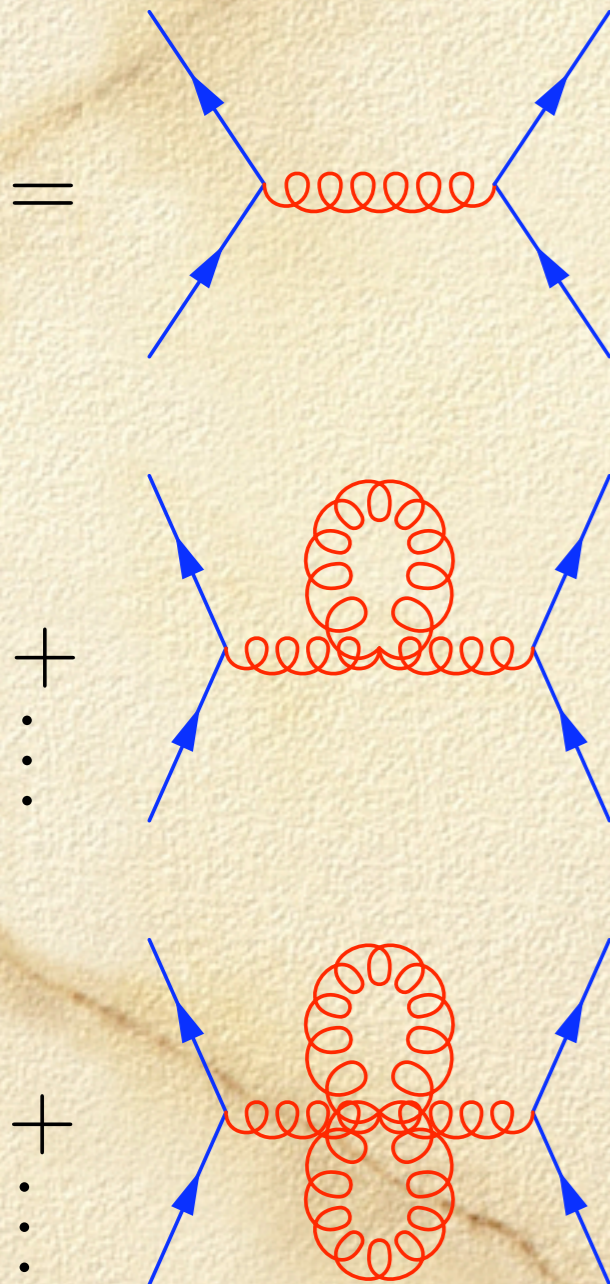
▶ A. El-Khadra, A. Kronfeld, P. Mackenzie, B. Oktay

▶ J. Shigemitsu, E. Gulez, M. Wingate

▶ I.T. Drummond, A. Hart, R.R. Horgan, L.C. Storoni

Perturbation Theory: We need the Feynman rules

$$-4\pi C_F \frac{\alpha_V(q^2)}{q^2}$$

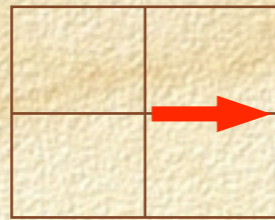


Needless to say, the calculation of the four gluon vertex (see fig. on page 212) from the fourth order contribution in θ_i^A to the effective action is quite tedious and we shall not present it here. The expression is very lengthy and has been given in the appendix of the paper by Kawai et al. (1981):*

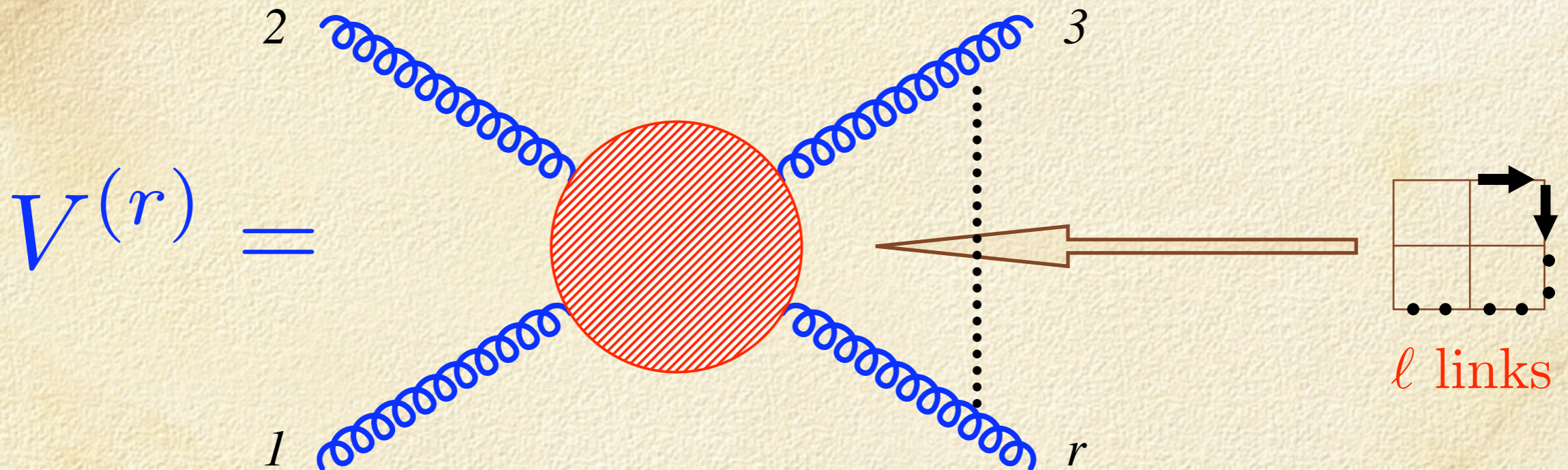
$$\begin{aligned} \Gamma_{\mu\nu\lambda\rho}^{ABCD}(p, q, r, s) = & \\ & -g^2 f_{ABEF} f_{CDE} \left\{ \delta_{\mu\lambda} \delta_{\nu\rho} \left[\cos \frac{a(q-s)_\mu}{2} \cos \frac{a(p-r)_\nu}{2} - \frac{a^4}{12} \hat{p}_\nu \hat{q}_\mu \hat{r}_\nu \hat{s}_\mu \right] \right. \\ & - \delta_{\mu\rho} \delta_{\nu\lambda} \left[\cos \frac{a(q-r)_\mu}{2} \cos \frac{a(p-s)_\nu}{2} - \frac{a^4}{12} \hat{p}_\nu \hat{q}_\mu \hat{r}_\mu \hat{s}_\nu \right] \\ & + \frac{a^2}{12} \delta_{\mu\nu} \delta_{\mu\lambda} \delta_{\mu\rho} \sum_{\sigma} (\hat{q}_\sigma e^{-i\frac{a}{2}p_\sigma} - \hat{p}_\sigma e^{-i\frac{a}{2}q_\sigma}) (\hat{s}_\sigma e^{-i\frac{a}{2}r_\sigma} - \hat{r}_\sigma e^{-i\frac{a}{2}s_\sigma}) \\ & - \frac{a^2}{6} \delta_{\mu\nu} \delta_{\mu\lambda} (\hat{q}_\rho e^{-i\frac{a}{2}p_\rho} - \hat{p}_\rho e^{-i\frac{a}{2}q_\rho}) \hat{s}_\mu \cos \frac{ar_\rho}{2} \\ & + \frac{a^2}{6} \delta_{\mu\nu} \delta_{\mu\rho} (\hat{q}_\lambda e^{-i\frac{a}{2}p_\lambda} - \hat{p}_\lambda e^{-i\frac{a}{2}q_\lambda}) \hat{r}_\mu \cos \frac{as_\lambda}{2} \\ & - \frac{a^2}{6} \delta_{\mu\lambda} \delta_{\mu\rho} (\hat{s}_\nu e^{-i\frac{a}{2}r_\nu} - \hat{r}_\nu e^{-i\frac{a}{2}s_\nu}) \hat{q}_\mu \cos \frac{ap_\nu}{2} \\ & \left. + \frac{a^2}{6} \delta_{\nu\lambda} \delta_{\nu\rho} (\hat{s}_\mu e^{-i\frac{a}{2}r_\mu} - \hat{r}_\mu e^{-i\frac{a}{2}s_\mu}) \hat{p}_\nu \cos \frac{aq_\mu}{2} \right\} \\ & + (B \leftrightarrow C, \nu \leftrightarrow \lambda, q \leftrightarrow r) + (B \leftrightarrow D, \nu \leftrightarrow \rho, q \leftrightarrow s) \\ & + g^2 \frac{a^4}{12} \left\{ \frac{2}{3} (\delta_{AB} \delta_{CD} + \delta_{AC} \delta_{BD} + \delta_{AD} \delta_{BC}) \right. \\ & \left. + (d_{ABED} d_{CDE} + d_{ACED} d_{BDE} + d_{ADE} d_{BCE}) \right\} \left\{ \delta_{\mu\nu} \delta_{\mu\lambda} \delta_{\mu\rho} \sum_{\sigma} \hat{p}_\sigma \hat{q}_\sigma \hat{r}_\sigma \hat{s}_\sigma \right. \\ & - \delta_{\mu\nu} \delta_{\mu\lambda} \hat{p}_\rho \hat{q}_\rho \hat{r}_\rho \hat{s}_\mu - \delta_{\mu\nu} \delta_{\mu\rho} \hat{p}_\lambda \hat{q}_\lambda \hat{s}_\lambda \hat{r}_\mu \\ & - \delta_{\mu\lambda} \delta_{\mu\rho} \hat{p}_\nu \hat{r}_\nu \hat{s}_\nu \hat{q}_\mu - \delta_{\nu\lambda} \delta_{\nu\rho} \hat{q}_\mu \hat{r}_\mu \hat{s}_\mu \hat{p}_\nu \\ & \left. + \delta_{\mu\nu} \delta_{\lambda\rho} \hat{p}_\lambda \hat{q}_\lambda \hat{r}_\mu \hat{s}_\mu + \delta_{\mu\lambda} \delta_{\nu\rho} \hat{p}_\nu \hat{r}_\nu \hat{q}_\mu \hat{s}_\mu + \delta_{\mu\rho} \delta_{\nu\lambda} \hat{p}_\nu \hat{s}_\nu \hat{q}_\mu \hat{r}_\mu \right\} \end{aligned} \tag{14.44}$$

* The expression given in the above reference is however not completely correct. We give here the corrected form which was provided to us by W. Wetzel.

Complication is due to link field



$$U_\mu(x) = e^{igaA_\mu(x)}$$



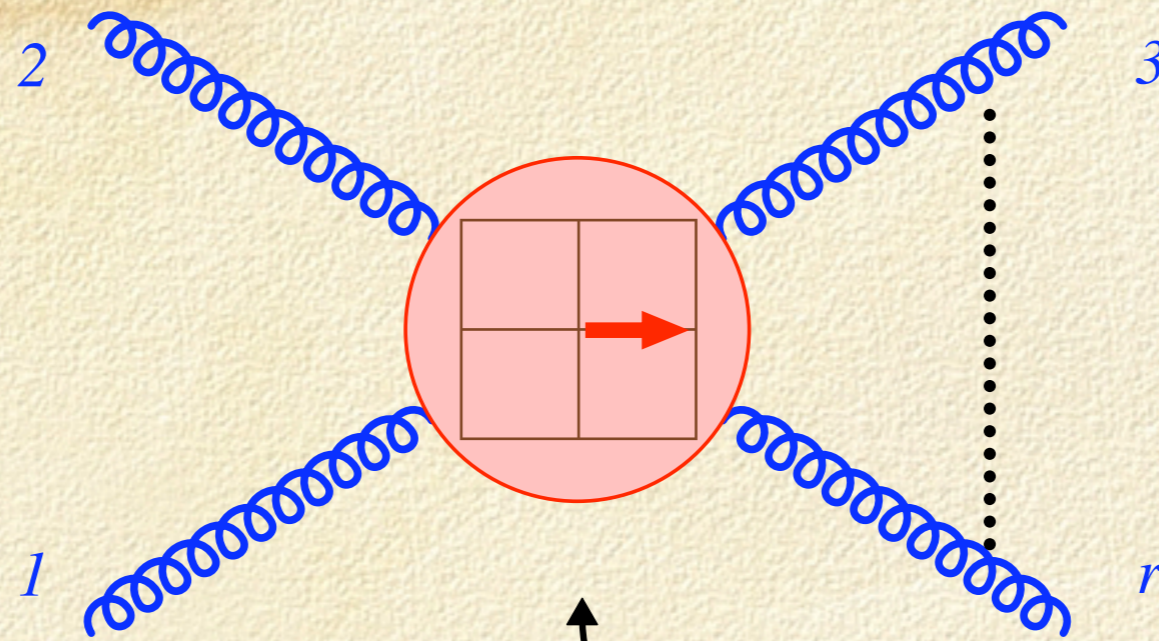
$$\begin{aligned} \# \text{ terms in vertex} &= \frac{2}{(r-1)!} \ell(\ell+1) \dots (\ell+r-1) \\ &= 5544 \text{ for } \ell = r = 6 \end{aligned}$$

- ▶ Many effective theories (light- q , heavy- Q , glue)
- ▶ actions continue to evolve

There exists a class of remarkably simple automated algorithms

- ▶ entirely symbolic/numeric manipulation
- ▶ generate Feynman rules for essentially arbitrarily complicated lattice actions
- ▶ **M. Lüscher and P. Weisz**, Nucl. Phys. B266, 309 (1986)
- ▶ C. Morningstar
- ▶ B. Allés, M. Campostrini, A. Feo, H. Panagopoulos
- ▶ S. Capitani, G. Rossi

Simplest Case: Gluon “action” with links in a *single* direction



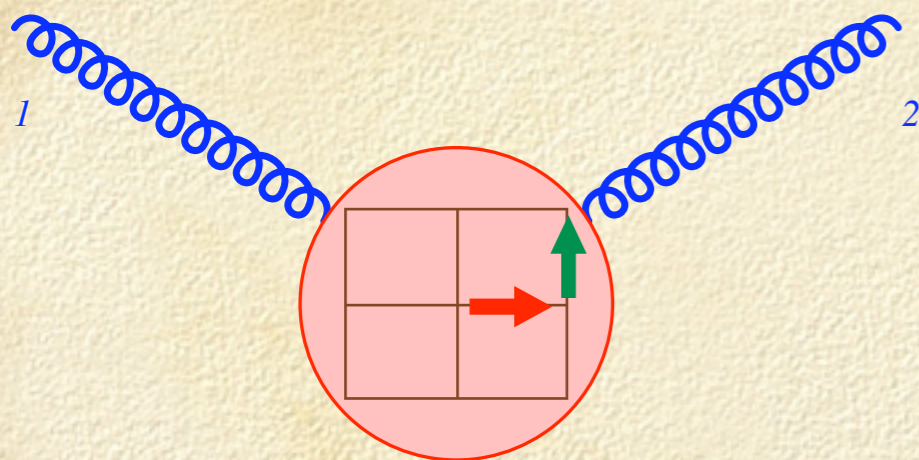
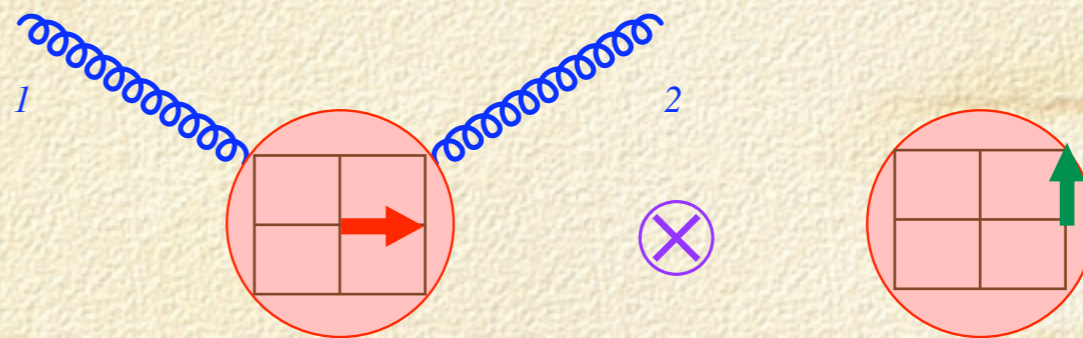
$$\mathcal{S} = \sum_x U_\ell(x) = \sum_x e^{A_{\hat{\ell}}(x + \frac{1}{2} a_\ell)}$$

$$V_{\text{link unsym}}^{\text{link}} \left(\left\{ \begin{matrix} k_1 \\ \mu_1 \\ a_1 \end{matrix} \right\}, \left\{ \begin{matrix} k_2 \\ \mu_2 \\ a_2 \end{matrix} \right\}, \dots, \left\{ \begin{matrix} k_r \\ \mu_r \\ a_r \end{matrix} \right\} \right) = (2\pi)^4 \delta(k_1 + k_2 + \dots + k_r = k_{\text{tot}}) \times$$

$$\frac{1}{r!} \delta_{\hat{\mu}_1 = \hat{\mu}_2 = \dots = \hat{\mu}_r = \hat{\ell}} \times e^{i(k_1 \cdot \frac{a_\ell}{2} + k_2 \cdot \frac{a_\ell}{2} + \dots + k_r \cdot \frac{a_\ell}{2})} \times (T^{a_1} T^{a_2} \dots T^{a_r})$$

Handle arbitrarily complicated actions by convolution

$$V_{\text{two-link unsym}} \left(\left\{ \begin{matrix} k_1 \\ \mu_1 \\ a_1 \end{matrix} \right\}, \left\{ \begin{matrix} k_2 \\ \mu_2 \\ a_2 \end{matrix} \right\} \right) =$$

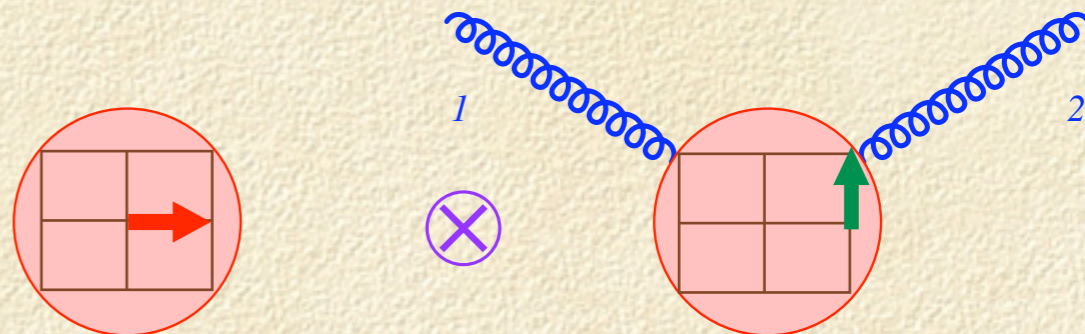


+



$\xrightarrow{k_1}$

+



Apply same algorithm to any gluon / quark action

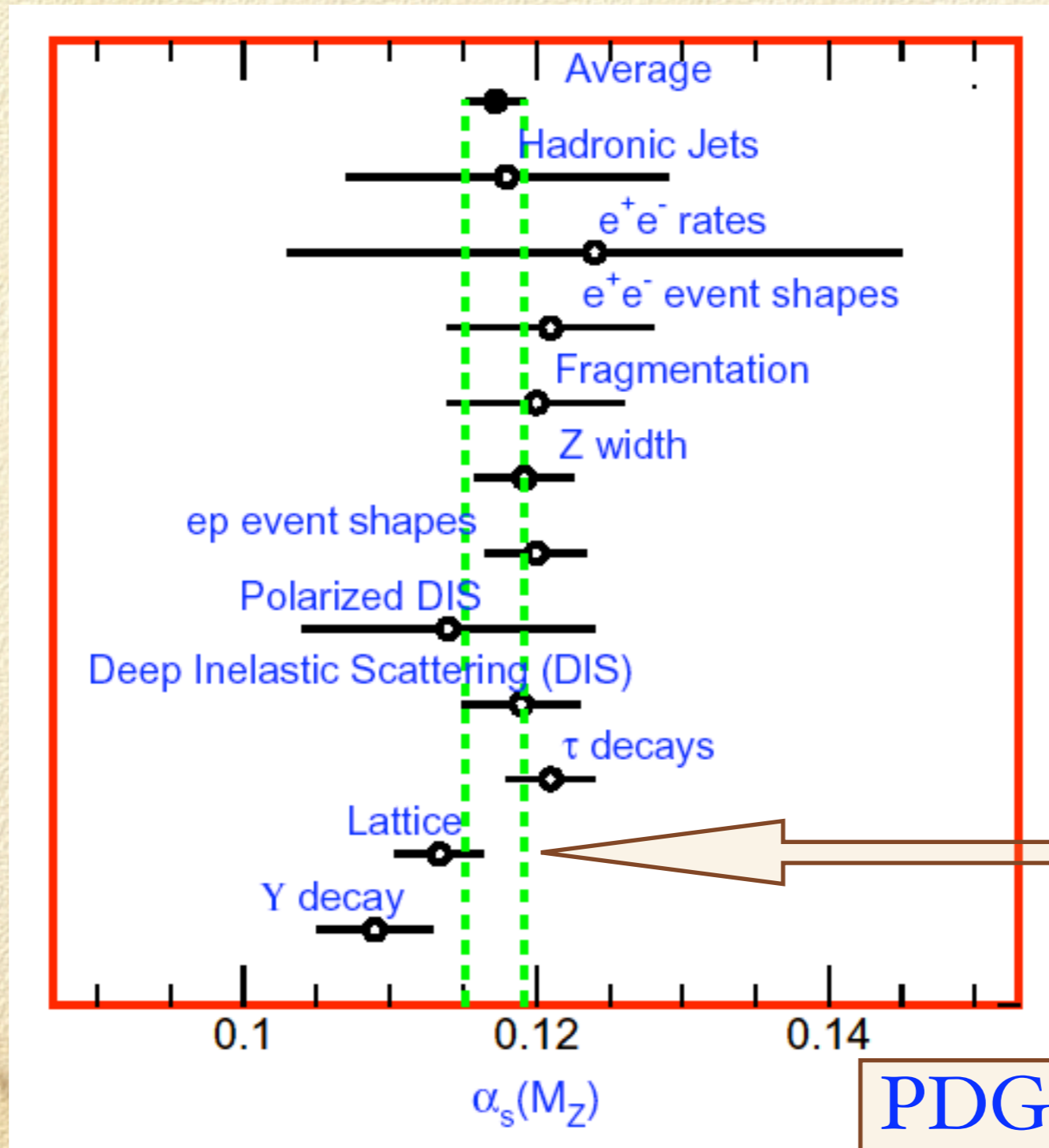
$$\mathcal{L}_{\text{Heavy } Q} = \bar{\psi} \left(1 + \frac{\Delta^{(2)}}{4nM_Q^0} \right)^n U_4^\dagger \left(1 + \frac{\Delta^{(2)}}{4nM_Q^0} \right)^n (1 - \delta H) \psi$$

$$\delta H = \dots - c_3 \frac{g}{8(M_Q^0)^2} \boldsymbol{\sigma} \cdot (\boldsymbol{\Delta} \times \mathbf{E} - \mathbf{E} \times \boldsymbol{\Delta}) - c_4 \frac{g}{2M_Q^0} \boldsymbol{\sigma} \cdot \mathbf{B} + \dots$$

- ▶ Convolute U_μ 's to get Feynman rules for $\boldsymbol{\Delta}, \mathbf{E}, \dots$
- ▶ Convolute $\boldsymbol{\Delta}, \mathbf{E}, \dots$ to get rules for $\mathcal{L}_{\text{Heavy } Q}$

Major effort over past year:

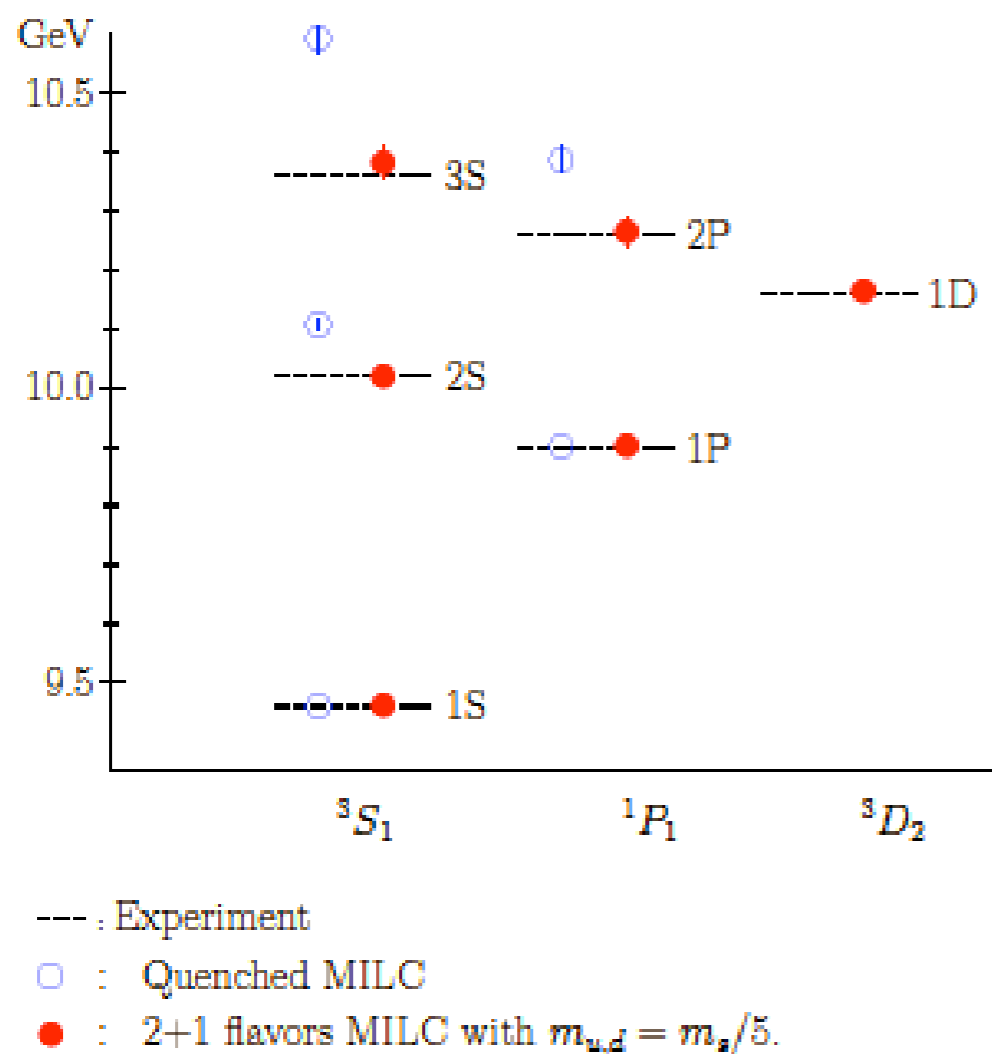
Third-order determination of $\alpha_{\overline{\text{MS}}}(M_Z)$



New
NLO
analysis
should
reduce
error by
factor ~ 3

Extracting $\alpha_{\overline{MS}}(M_Z)$ from LQCD

(NRQCD
Collab'n
1997)



(i) NPT input e.g. $\Upsilon' - \Upsilon \Rightarrow a$

(ii) Measure short-distance quantity

$$\langle \mathcal{O} \rangle = c_1 \alpha_V(q^*) + c_2 \alpha_V^2(q^*) + \dots$$

$$[q^* \propto 1/a]$$

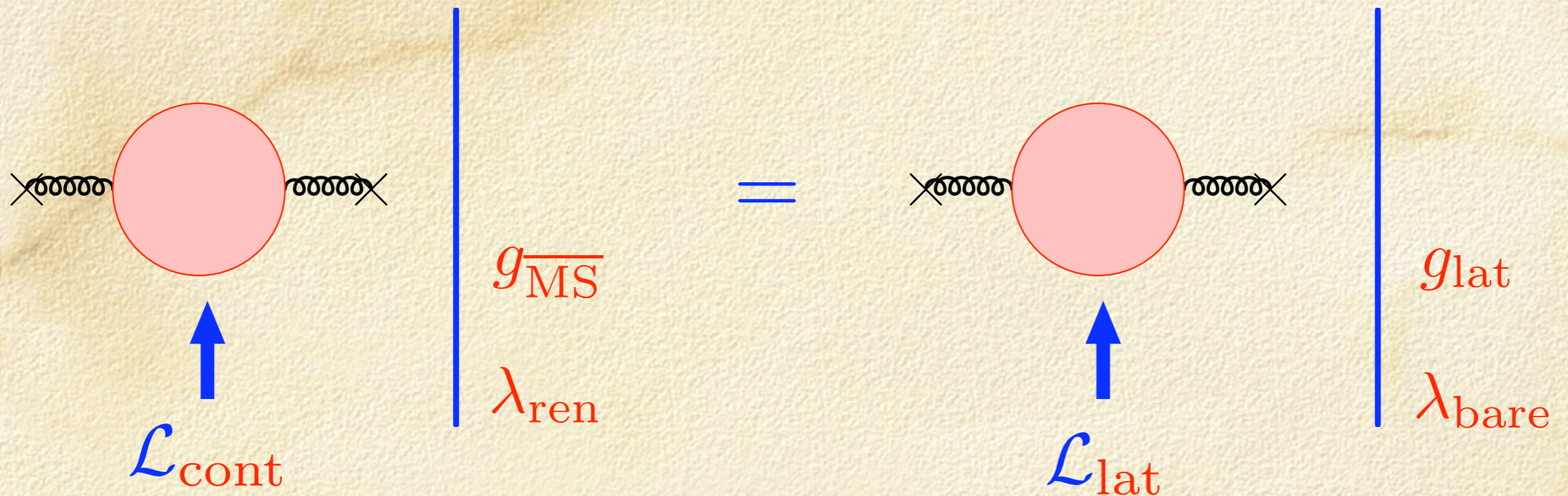
(iii) Evolve to M_Z , convert to $\alpha_{\overline{MS}}$

$\alpha_{\overline{MS}} \leftrightarrow \alpha_{\text{lat}}$ through **two-loops**

Wilson loops through *three-loops*

► Gluonic loops ► Fermionic loops

Background-Field Matching



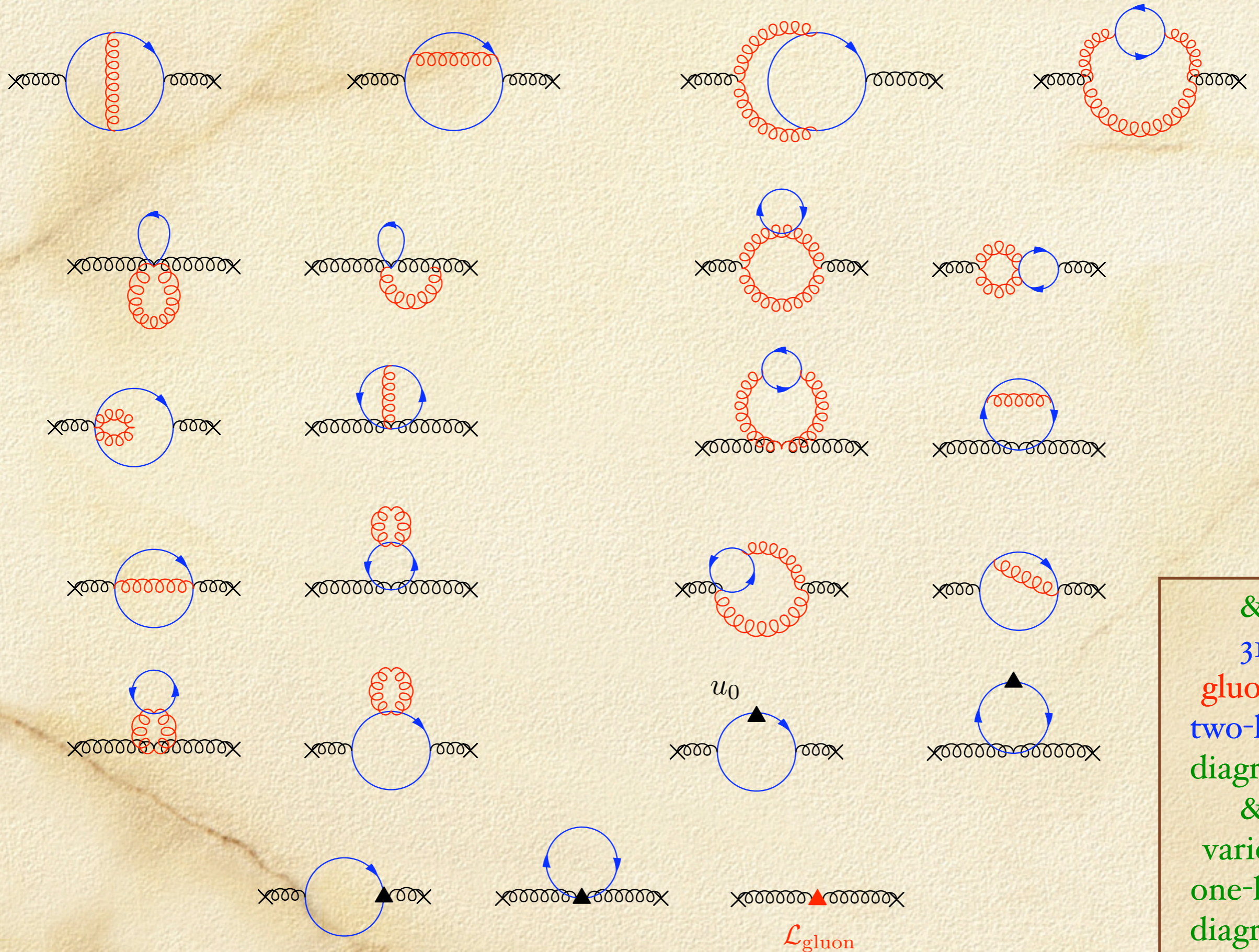
$$g_{\overline{\text{MS}}} = Z_g g_{\text{lat}}$$

(gauge parameter) $\lambda_{\text{ren}} = Z_3 \lambda_{\text{bare}}$

Continuum - Gluonic: K. Ellis (1984); L&W, van de Ven (1995)

- **Fermionic:** Panagopoulos et al.; HDT et al.

Lattice Two Loop N_f portion of $\alpha_V(q^*) = \alpha_{\text{lat}} + \dots$

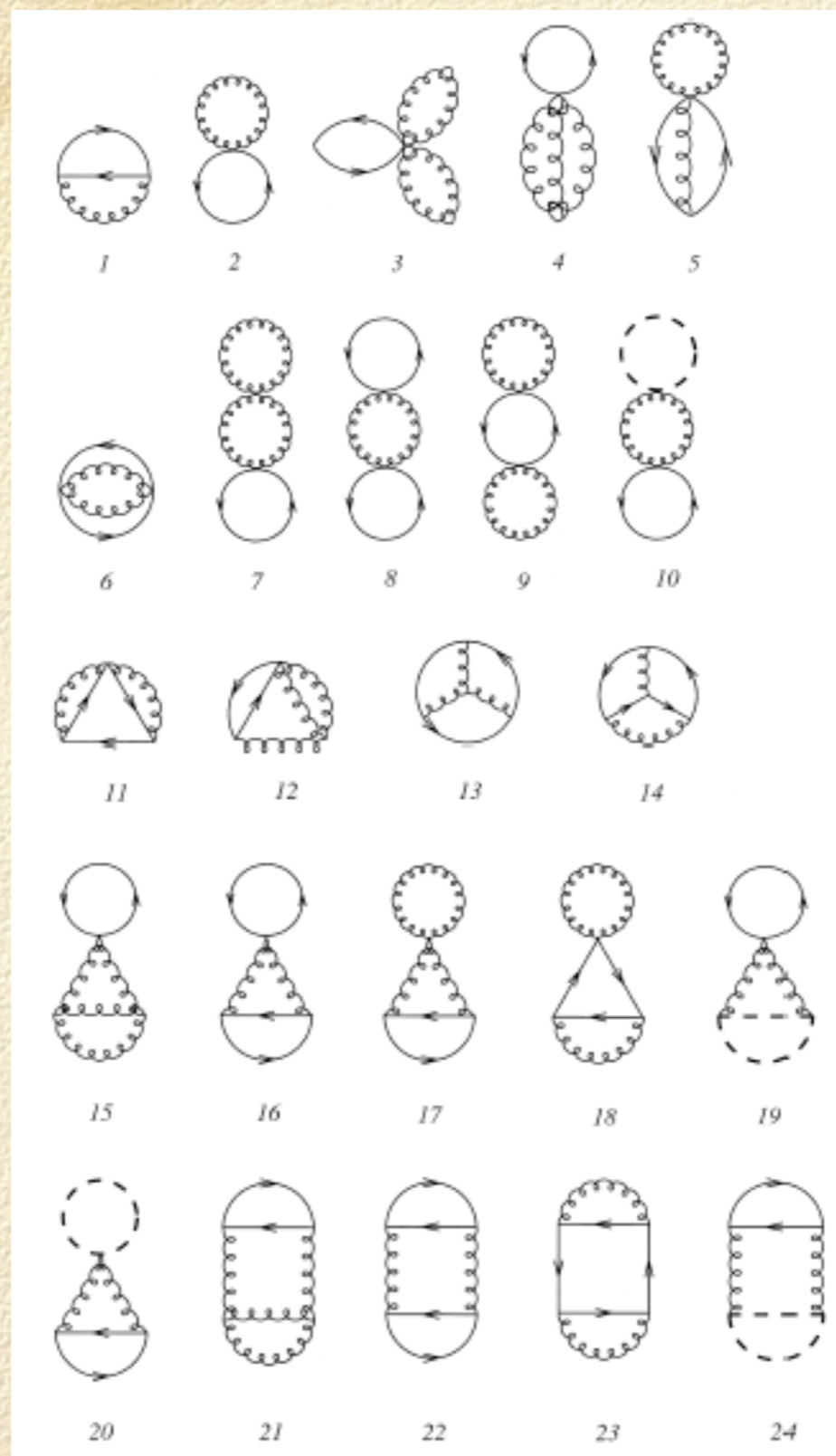


&
31
gluonic
two-loop
diagrams
&
various
one-loop
diagrams

$\mathcal{L}_{\text{gluon}}$

NNLO Wilson loops

Fermionic three-loop diagrams



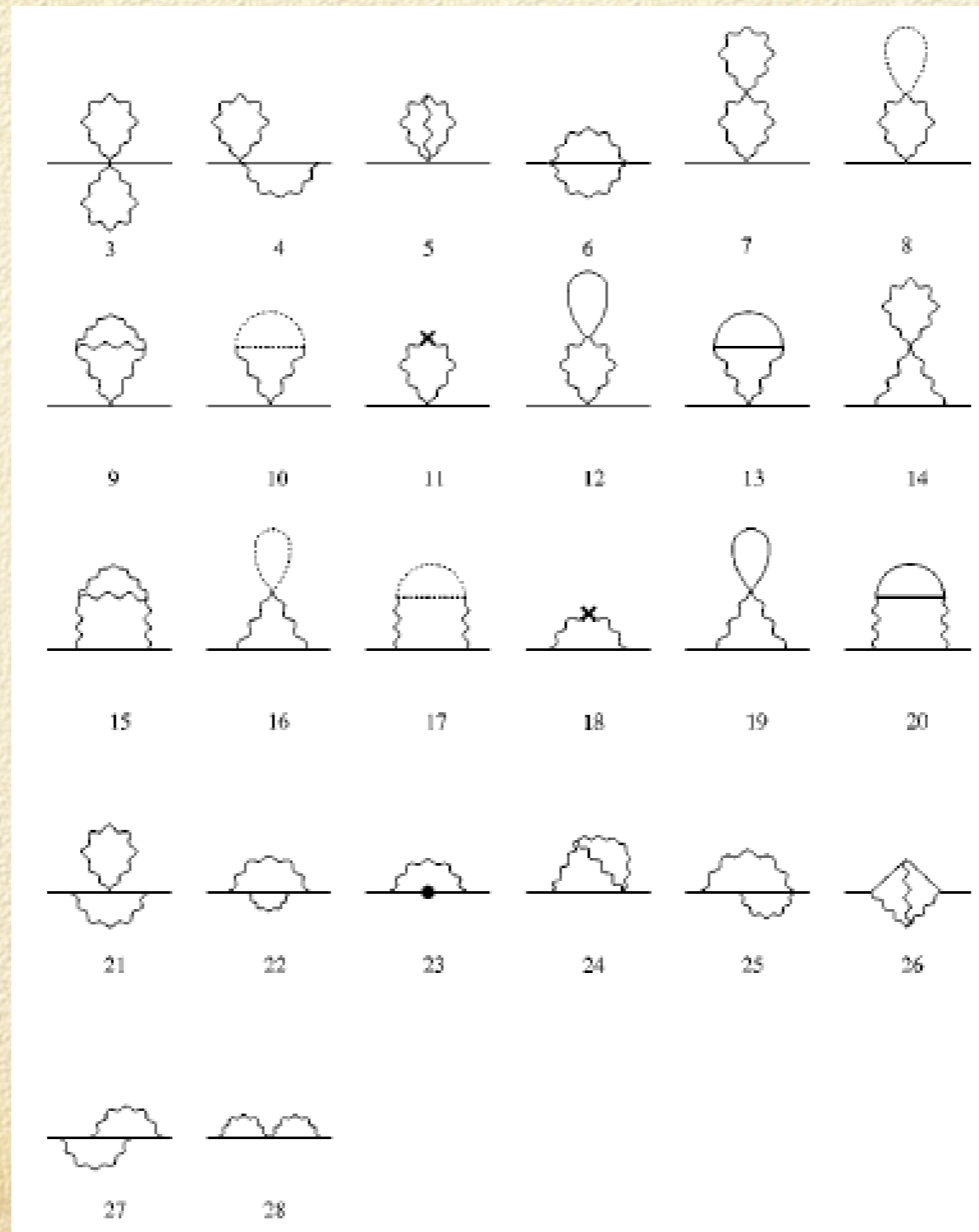
Plus comparable number of three-loop gluonic diagrams

... and the answer for $\alpha_{\overline{\text{MS}}}(M_Z)$ is ... coming soon!

Some other work in progress

Two-loop m_s, m_c, m_b

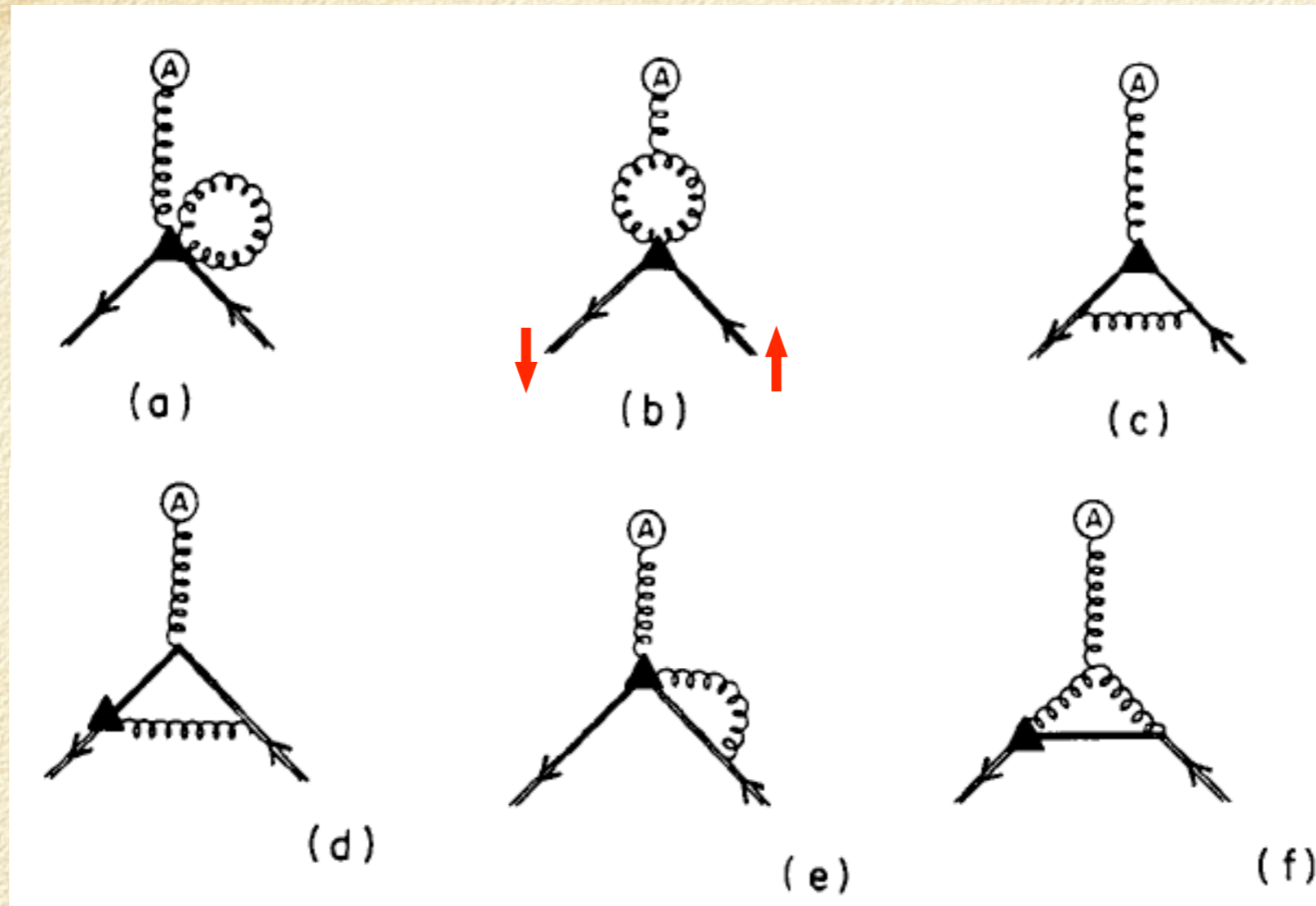
Q. Mason & HDT



Heretofore NLO
lattice PT only
done in the static
approximation

One-loop heavy quark action parameters

M. Nobes & HDT



► especially $\mathcal{L}_{\text{HeavyQ}} = -c_B \frac{g}{2M_Q^0} \psi^\dagger \vec{\sigma} \cdot \vec{B} \psi + \dots$

► impact on fine / hyperfine structure & f_B

Challenges / Open Problems

- ▶ **Continuum-side of two-loop matching**
 - ▶ lots of matching to be done
- ▶ **Continuum techniques for loop integration?**
 - ▶ loss of Lorentz invariance => very complex integrands
 - ▶ we do “brute-force” numerical integration: VEGAS
(may be unstable in some cases due to disparate scales)
 - ▶ bring continuum methods to bear (Becher & Melnikov)
- ▶ **“Optimal” infrared regulator?**
 - ▶ almost all lattice PT uses gluon mass
 - ▶ we have also applied “twisted” b.c. (provides gauge-invariant IR regulator)

Asymptotic Expansions

(Becher &
Melnikov)

Separation of soft- & hard-scales in lattice
loop integrals using analytic regularization

(alternative to IR subtractions, e.g. Luscher & Weisz)

$$G(m) \equiv \lim_{\delta \rightarrow 0} \int_{-\pi/a}^{\pi/a} \frac{d^4 k}{(2\pi)^4} \frac{1}{(\hat{k}^2 + m^2)^{1+\delta}} \quad \left[\hat{k}_\mu = 2a \sin\left(\frac{1}{2}ak_\mu\right) \right]$$
$$= G_{\text{soft}}(m) + G_{\text{hard}}(m) \quad \text{to take continuum limit}$$

- ▶ **Soft** ($k_i \sim m \ll \pi/a$): Reduce to continuum-like (IR-divergent) integrals, trivial to evaluate.
- ▶ **Hard** ($k_i \sim \pi/a \gg m$): IR-finite massless-tadpole integrals (lots!); Evaluate using standard techniques, e.g. recurrence relations
- ▶ **One-loop self-energy improved staggered quarks**

Twisted Boundary Conditions

(QCD on a torus: 't Hooft)

$$\blacktriangleright A_\mu(x + L\hat{\nu}) = \Omega_\nu A_\mu(x) \Omega_\nu^\dagger \quad (\Omega_1 \Omega_2 = z \Omega_2 \Omega_1, \\ z = e^{2\pi i/N})$$

$$\blacktriangleright \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^4}$$

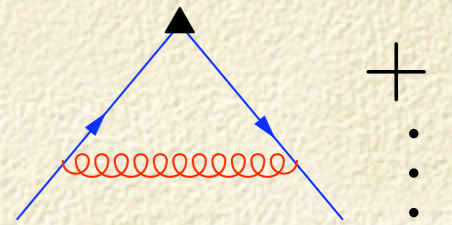
$$\longrightarrow \int \frac{dk_0 dk_3}{(2\pi)^2} \frac{1}{L^{D-2}} \sum_{n_1, \dots, n_{D-2}} \frac{1}{k^4}, \quad k_\perp = \frac{2\pi}{NL} n_\perp, \\ n_\perp \neq 0 \pmod{N}$$

$$= \frac{1}{2\pi^2} \left(\frac{2}{\epsilon} + \dots \right) + \frac{1}{\pi^2} \ln(\mu L) + c \quad (\text{Luscher})$$

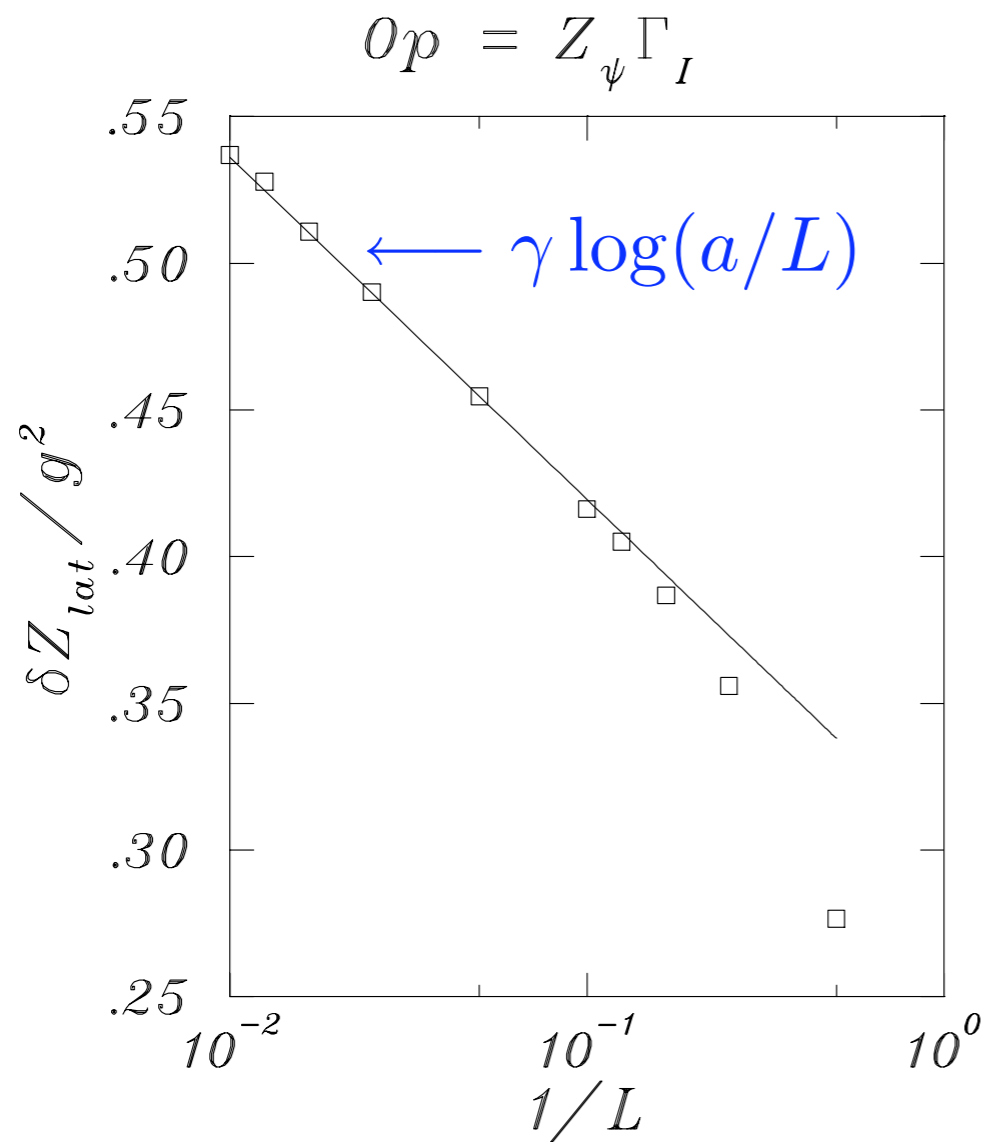
$$\blacktriangleright k_{\min} \sim \frac{1}{L} = \text{gauge invariant infrared cutoff}$$

\blacktriangleright cf. λ with gluon mass (at one-loop)

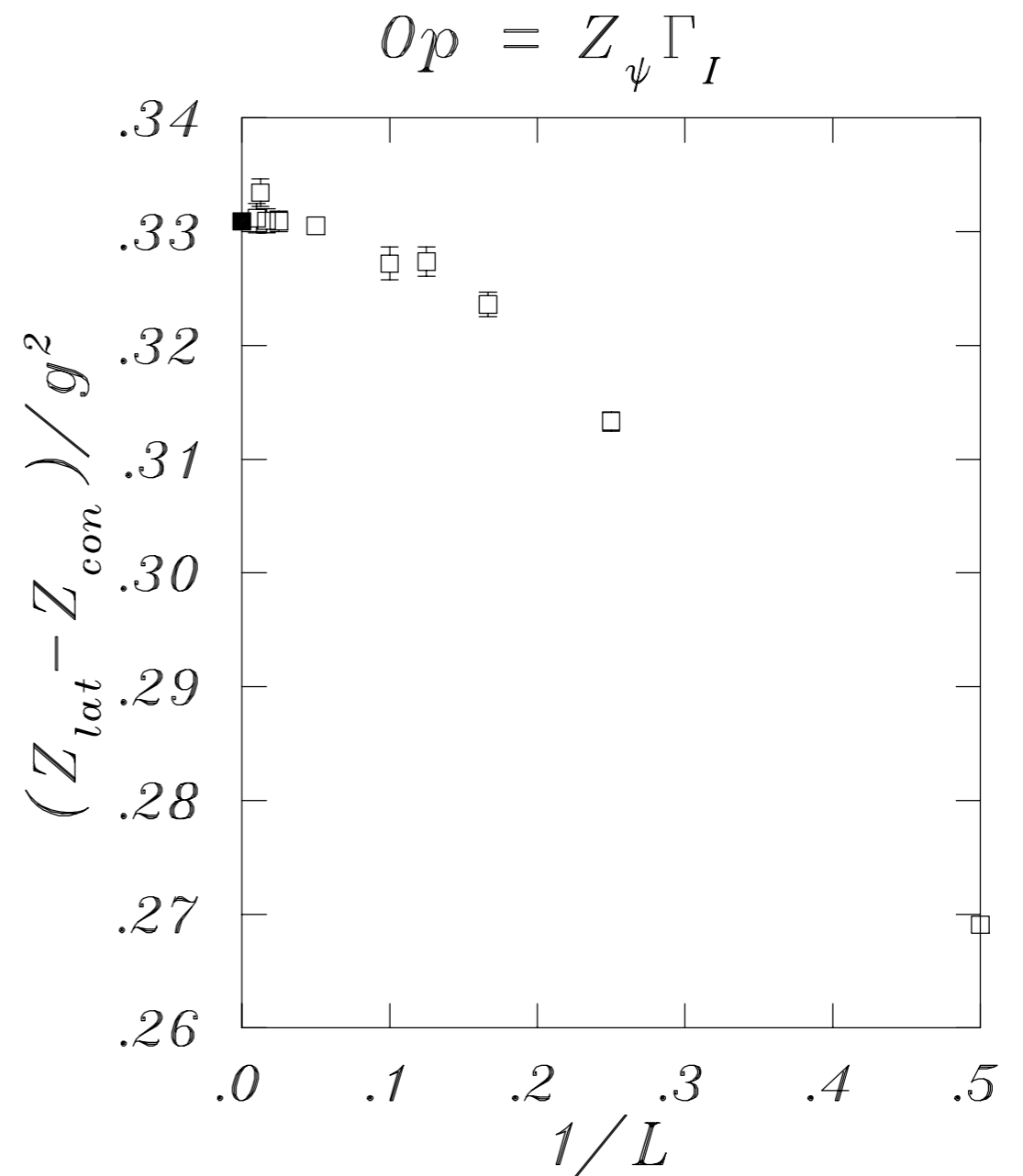
LQCD $\Leftrightarrow \overline{\text{MS}}$ One-Loop Current Matching with Twisted B.C.



Lattice



Lattice - Continuum



Summary

- ▶ Unquenched LQCD: few-% precision now possible for “gold-plated” quantities
- ▶ Perturbation theory key to further progress
 - ▶ automated methods allow one to routinely go to higher-orders for complex lattice actions
- ▶ We have already done some important two- and three-loop calculations for (2+1)-flavours
 - ▶ anticipate two-loop results soon for $\alpha_{\overline{\text{MS}}}(M_z)$, m_s , m_c , m_b
 - ▶ ultimately: two-loop hadronic matrix elements for CKM
- ▶ Lattice gauge theories: may be of broader import if physics @ LHC turns out to be strongly coupled