Hidden Symmetry and Quantum Phases in Spin 3/2 Cold Atomic Systems

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Ref: C. Wu, Mod. Phys. Lett. B 20, 1707, (2006);
C. Wu, Phys. Rev. Lett. 95, 266404 (2005);
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Rapid progress in cold atomic physics


• Magnetic traps: spin degrees of freedom are frozen.

• Optical traps and lattices: spin degrees of freedom are released; a controllable way to study high spin physics.

• In optical lattices, interaction effects are adjustable. New opportunity to study strongly correlated high spin systems.
High spin physics with cold atoms

• Most atoms have high hyperfine spin multiplets.

\[ F = L \text{ (nuclear spin)} + S \text{ (electron spin)} \]

• Different from high spin transition metal compounds.

• Spin-1 bosons: \(^{23}\text{Na}\) (antiferro), \(^{87}\text{Rb}\) (ferromagnetic).

• High spin fermions: zero sounds and Cooper pairing structures.

Hidden symmetry: spin-3/2 atomic systems are special!

• Spin 3/2 atoms: $^{132}$Cs, $^9$Be, $^{135}$Ba, $^{137}$Ba, $^{201}$Hg.

• Hidden $\text{SO}(5)$ symmetry without fine tuning!

Continuum model (s-wave scattering); the lattice-Hubbard model.

Exact symmetry regardless of the dimensionality, lattice geometry, impurity potentials.

$\text{SO}(5)$ in spin 3/2 systems $\leftrightarrow \text{SU}(2)$ in spin $1/2$ systems

• This $\text{SO}(5)$ symmetry is qualitatively different from the $\text{SO}(5)$ theory of high $T_c$ superconductivity.

What is SO(5)/Sp(4) group?

• SO(3) /SU(2) group.

3-vector: x, y, z.

3-generator: \( L_z = L_{12}, \quad L_x = L_{23}, \quad L_y = L_{31}. \)

2-spinor: \( |\uparrow\rangle, \quad |\downarrow\rangle \)

• SO(5) /Sp(4) group.

5-vector: \( n_1, n_2, n_3, n_4, n_5 \)

10-generator: \( L_{ab} \ (1 \leq a < b \leq 5) \)

4-spinor: \( \uparrow |\frac{3}{2}\rangle \quad \uparrow |\frac{1}{2}\rangle \quad \downarrow |\frac{-1}{2}\rangle \quad \downarrow |\frac{-3}{2}\rangle \)
Quintet superfluidity and half-quantum vortex

- High symmetries do not occur frequently in nature, since every new symmetry brings with itself a possibility of new physics, they deserve attention.
  
  ---D. Controzzi and A. M. Tsvelik, cond-mat/0510505

- Cooper pairing with $S_{\text{pair}}=2$.

- Half-quantum vortex (non-Abelian Alice string) and quantum entanglement.

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Multi-particle clustering order

- Quartetting order in spin 3/2 systems.

- 4-fermion counter-part of Cooper pairing.

- Feshbach resonances: Cooper pairing superfluidity.

- Driven by logic, it is natural to expect the quartetting order as a possible focus for the future research.

Strong quantum fluctuations in magnetic systems

- Intuitively, quantum fluctuations are weak in high spin systems.

\[
\left[ \frac{S_i}{S}, \frac{S_j}{S} \right] = i \varepsilon_{ijk} \frac{1}{S} \frac{S_k}{S}
\]

- However, due to the high SO(5) symmetry, quantum fluctuations here are even **stronger** than those in spin \( \frac{1}{2} \) systems.

large N (N=4) v.s. large S.

From Auerbach’s book:
• The proof of the exact SO(5) symmetry.

• Quintet superfluids and non-Abelian topological defects.

• Quartetting v.s pairing orders in 1D spin 3/2 systems.

• SO(5) (Sp(4)) Magnetism in Mott-insulating phases.
The SO(4) symmetry in Hydrogen atoms

• An obvious SO(3) symmetry: \( \vec{L} \) (angular momentum).

• The energy level degeneracy \( n^2 \) is mysterious.

• Not accidental! \( 1/r \) Coulomb potential gives rise to a hidden conserved quantity.

  Runge-Lentz vector \( \vec{A} \); SO(4) generators \( \vec{A}, \vec{L} \).

• Q: What are the hidden conserved quantities in spin 3/2 systems?
Generic spin-3/2 Hamiltonian in the continuum

- The s-wave scattering interactions and spin SU(2) symmetry.

\[ H = \int d^d \vec{r} \sum_{\alpha=\pm3/2,\pm1/2} \psi^+_\alpha(\vec{r}) \left( -\frac{\hbar^2}{2m} \hat{\nabla}^2 - \mu \right) \psi_\alpha(\vec{r}) + \frac{g_0}{2} \eta^+ (\vec{r}) \eta (\vec{r}) + \frac{g_2}{2} \sum_{a=1}^{5} \chi^+_a (\vec{r}) \chi_a (\vec{r}) \]

- Pauli’s exclusion principle: only \( F_{\text{tot}} = 0, 2 \) are allowed; \( F_{\text{tot}} = 1, 3 \) are forbidden.

**Singlet:** \( \eta^+ (\vec{r}) = \sum_{\alpha\beta} \langle 00 \mid \frac{3}{2} \frac{3}{2} ; \alpha\beta \rangle \psi^+_\alpha (\vec{r}) \psi^+_\beta (\vec{r}) \)

**Quintet:** \( \chi^+_a (\vec{r}) = \sum_{\alpha\beta} \langle 2a \mid \frac{3}{2} \frac{3}{2} ; \alpha\beta \rangle \psi^+_\alpha (\vec{r}) \psi^+_\beta (\vec{r}) \)
Spin-3/2 Hubbard model in optical lattices

\[
H = \sum_{\langle ij \rangle, \alpha} -t\{c_{i,\alpha}^+ c_{j,\alpha} + h.c.\} - \mu \sum_i c_{i,\alpha}^+ c_{i,\alpha} + U_0 \sum_i \eta^+ (i) \eta (i) + U_2 \sum_{a=1-5} \chi_a^+ (i) \chi_a (i)
\]

• The single band Hubbard model is valid away from resonances.

\[U_{0.2} < 0.1,\]
\[l_0 \sim 400\text{nm}, a_{s,0.2} \sim 100a_B\]
\[\left(\frac{V_0}{E_r}\right)^{1/4} \approx 1 \sim 2\]

\(a_s: \text{scattering length}, E_r: \text{recoil energy}\)
SO(5) symmetry: the single site problem

\[ E_0 = 0 \]
\[ E_1 = -\mu \]
\[ E_2 = U_2 - 2\mu \]
\[ E_3 = U_0 - 2\mu \]
\[ E_4 = \frac{1}{2}U_0 + \frac{5}{2}U_2 - 3\mu \]
\[ E_5 = U_0 + 5U_2 - 4\mu \]

2\(^4\)=16 states.

<table>
<thead>
<tr>
<th></th>
<th>SU(2)</th>
<th>SO(5)</th>
<th>degeneracy</th>
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<tbody>
<tr>
<td>E(_{0,3,5})</td>
<td>singlet</td>
<td>scalar</td>
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</tr>
<tr>
<td>E(_{1,4})</td>
<td>quartet</td>
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<tr>
<td>E(_2)</td>
<td>quintet</td>
<td>vector</td>
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</tr>
</tbody>
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\( U_0 = U_2 = U, \) SU(4) symmetry.

\[ H_{\text{int}} = \frac{U}{2} n(n-1) \]
Quintet channel \((S=2)\) operators as \(SO(5)\) vectors

- Kinetic energy has an obvious \(SU(4)\) symmetry; interactions break it down to \(SO(5)\) (\(Sp(4)\)); \(SU(4)\) is restored at \(U_0=U_2\).

\[
\begin{align*}
\hat{d}_{xy} : \chi_1^+(r) &= \begin{pmatrix}
\uparrow & \Downarrow \\
\Downarrow & \uparrow
\end{pmatrix} \\
\hat{d}_{xz} : \chi_2^+(r) &= i \begin{pmatrix}
\uparrow & - \\
- & \uparrow
\end{pmatrix} \\
\hat{d}_{yz} : \chi_3^+(r) &= \begin{pmatrix}
\uparrow & + \\
+ & \uparrow
\end{pmatrix} \\
\hat{d}_{3z^2-r^2} : \chi_4^+(r) &= i \begin{pmatrix}
\uparrow & + \\
+ & \uparrow
\end{pmatrix} \\
\hat{d}_{x^2-y^2} : \chi_5^+(r) &= i \begin{pmatrix}
\uparrow & + \\
+ & \uparrow
\end{pmatrix}
\end{align*}
\]
Particle-hole bi-linears $\psi_\alpha^+ M_{\alpha\beta} \psi_\beta$ (1)

- Total degrees of freedom: $4^2 = 16 = 1 + 3 + 5 + 7$.

1 density operator and 3 spin operators are not complete.

\[
\begin{align*}
\text{rank: 0} & \quad 1, \\
1 & \quad F_x, F_y, F_z \\
2 & \quad \xi_{ij}^a F_i F_j \ (a = 1 \sim 5): \\
\text{3} & \quad \xi_{ijk}^a F_i F_j F_k \ (a = 1 \sim 7)
\end{align*}
\]

- **Spin-nematic matrices** (rank-2 tensors) form five-$\Gamma$ matrices (SO(5) vector).

\[
\Gamma^a = \xi_{ij}^a F_i F_j, \quad \{\Gamma^a, \Gamma^b\} = 2\delta_{ab}, \quad (1 \leq a, b \leq 5)
\]
Particle-hole channel algebra (II)

- Both $F_{x,y,z}$ and $\xi_{ijk} F_i F_j F_k$ commute with Hamiltonian. 10 generators of SO(5): 10=3+7.

**7 spin cubic tensors** are the hidden conserved quantities.

$$\Gamma^{ab} = \frac{i}{2} [\Gamma^a, \Gamma^b ] \quad (1 \leq a < b \leq 5)$$

- SO(5): 1 scalar + 5 vectors + 10 generators = 16

Time Reversal

1 density: \[ n = \psi^+ \psi; \quad \text{even} \]

5 spin nematic: \[ n_a = \frac{1}{2} \psi^+ \Gamma^a \psi; \quad \text{even} \]

3 spins + 7 spin cubic tensors: \[ L_{ab} = \frac{1}{2} \psi^+ \Gamma^{ab} \psi; \quad \text{odd} \]
Outline

• The proof of the exact SO(5) symmetry.

• Quintet superfluids and Half-quantum vortices.


• Quartetting v.s pairing orders in 1D spin 3/2 systems.

• SO(5) (Sp(4)) Magnetism in Mott-insulating phases.
g_2<0: s-wave quintet (S_{\text{pair}}=2) pairing

- BCS theory: polar condensation; order parameter forms an SO(5) vector.

\[
\begin{align*}
\hat{d}_x : \chi_1^+ (r) &= \begin{pmatrix} \uparrow \downarrow \end{pmatrix} \\
\hat{d}_z : \chi_2^+ (r) &= i \begin{pmatrix} \uparrow \downarrow \end{pmatrix} \\
\hat{d}_y : \chi_3^+ (r) &= \begin{pmatrix} \uparrow \downarrow \end{pmatrix} \\
\hat{d}_{3z^2-r^2} : \chi_4^+ (r) &= i \begin{pmatrix} \uparrow \downarrow \end{pmatrix} \\
\hat{d}_x^2-y^2 : \chi_5^+ (r) &= i \begin{pmatrix} \uparrow \downarrow \end{pmatrix}
\end{align*}
\]

\[\chi_a^+ = \sqrt{\rho} e^{i\theta} \hat{d}_a\]

5d unit vector

Ho and Yip, PRL 82, 247 (1999);
Wu, Hu and Zhang, cond-mat/0512602.
Superfluid with spin: half-quantum vortex (HQV)

\[ \chi^a = \sqrt{p} e^{i\theta} \hat{d}_a \]

remains single-valued.

\[ \pi \text{-disclination of } \hat{d} \text{ as a HQV.} \]

- \( \mathbb{Z}_2 \) gauge symmetry
- \( \hat{d} \rightarrow -\hat{d}, \ \theta \rightarrow \theta + \pi \)

\[ \pi_1(\text{RP}^4) = \mathbb{Z}_2 \]

Fundamental group of the manifold.

\[ \hat{d} : \text{RP}^4 = S^4 / \mathbb{Z}_2 \]

\( \hat{d} \) is not a rigorous vector, but a directionless director.

\[ \hat{d} \]

is not a rigorous vector, but a directionless director.
Stability of half-quantum vortices

\[ E = \int dr \frac{\hbar^2}{4M} \left\{ \rho_{sf} (\nabla \theta)^2 + \rho_{sp} (\nabla \hat{d})^2 \right\} \]

• Single quantum vortex:

\[ E = \frac{\hbar}{4M^2} \rho_{sf} \log \frac{L}{\xi} \]

• A pair of HQV:

\[ E = \frac{\hbar}{4M^2} \left\{ \frac{\rho_{sf} + \rho_{sp}}{2} \log \frac{L}{\xi} + \frac{\rho_{sf} - \rho_{sp}}{2} \log \frac{L}{R} \right\} \]

• Stability condition: \[ \rho_{sp} < \rho_{sf} \]
Example: HQV as Alice string ($^3$He-A phase)

- A particle flips the sign of its spin after it encircles HQV.
- Example: $^3$He-A, triplet Cooper pairing.

Configuration space $U(1)$

$\hat{d} : S^2 / \mathbb{Z}_2$

$\left| \uparrow \right\rangle \rightarrow e^{i\alpha} \left| \downarrow \right\rangle$, $\left| \downarrow \right\rangle \rightarrow e^{-i\alpha} \left| \uparrow \right\rangle$

$U(\hat{n}) = \begin{pmatrix} 0, & e^{i\alpha} \\ e^{-i\alpha}, & 0 \end{pmatrix}$

A. S. Schwarz et al., Nucl. Phys. B 208, 141(1982);
F. A. Bais et al., Nucl. Phys. B 666, 243 (2003);
The HQV pair in 2D or HQV loop in 3D

SO(3) → SO(2)

- Phase state.

\[ |\alpha\rangle_{\text{vort}} = \exp(iS_z \alpha) |\alpha = 0\rangle_{\text{vort}} \]

SO(2) Cheshire charge ($^3$He-A)

- HQV pair or loop can carry spin quantum number.

- For each phase state, SO(2) symmetry is only broken in a finite region, so it should be dynamically restored.

- Cheshire charge state ($S_z$) vs phase state.

\[
|m\rangle_{\text{vort}} = \int d\alpha \, \exp(i m \alpha) |\alpha\rangle_{\text{vort}}
\]

\[
S_z |m\rangle_{\text{vort}} = m |m\rangle_{\text{vort}}
\]

\[
|m = 0\rangle_{\text{vort}} = \int d\alpha |\alpha\rangle_{\text{vort}}
\]

Spin conservation by exciting Cheshire charge

- Initial state: particle spin up and HQV pair (loop) zero charge.

- Final state: particle spin down and HQV pair (loop) charge 1.

\[
|\text{init}\rangle = |\uparrow\rangle_p \otimes |m = 0\rangle_{\text{vort}} = |\uparrow\rangle_p \otimes \int d\alpha |\alpha\rangle_{\text{vort}}
\]

\[
|\text{final}\rangle = |\downarrow\rangle_p \otimes \int d\alpha e^{i\alpha} |\alpha\rangle_{\text{vort}}
\]

= \[
|\downarrow\rangle_p \otimes |m = 1\rangle_{\text{vort}}
\]

Quintet pairing as a non-Abelian generalization

- HQV configuration space $|\hat{n}_{\text{vort}}\rangle$, SU(2) instead of U(1).

**equator:** $\hat{n} \in S^3 = SU(2)$

$$\hat{d}(\phi, \hat{n}) = \cos \frac{\phi}{2} \hat{e}_4 - \sin \frac{\phi}{2} \hat{n}$$

\[\begin{align*}
S^4 & \quad \hat{e}_4 \\
\hat{n} & \quad \hat{e}_{1,2,3,5}
\end{align*}\]

\[\begin{align*}
\hat{d}(\phi = 0) \parallel \hat{e}_4 \\
\pi & \quad \text{-disclination}
\end{align*}\]
SU(2) Berry phase

• After a particle moves around HQV, or passes a HQV pair:

\[
\psi = \begin{pmatrix} \frac{3}{2} \\ -\frac{3}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \quad \psi \rightarrow \begin{pmatrix} 0 & W \\ W^* & 0 \end{pmatrix} \psi
\]

\[
W(\hat{n}) = \begin{pmatrix} n_3 + in_2 & -n_1 + in_5 \\ n_1 - in_5 & n_3 - in_2 \end{pmatrix}
\]

\[
\frac{3}{2}, -\frac{3}{2} \leftrightarrow \frac{1}{2}, -\frac{1}{2}
\]

\[
\hat{n} = n_1 \hat{e}_1 + n_2 \hat{e}_2 + n_3 \hat{e}_3 + n_5 \hat{e}_5
\]

\[S^3 \approx SU(2)\]
Non-Abelian Cheshire charge

• Construct SO(4) Cheshire charge state for a HQV pair (loop) through $S^3$ harmonic functions.

\[ \left| TT_3; T'T'_3 \right\rangle_{vort} = \int_{n \in S^3} d\hat{n} \ Y_{TT_3;T'T'_3} (\hat{n}) \left| \hat{n} \right\rangle_{vort} \]

\[ \vec{T} (\vec{T}') = L_{12}, \pm L_{35}, L_{13}, \pm L_{25}, L_{23} \pm L_{15}. \]

• Zero charge state.

\[ \left| 00;00 \right\rangle_{vort} = \int_{n \in S^3} d\hat{n} \left| \hat{n} \right\rangle_{vort} \]
Entanglement through non-Abelian Cheshire charge!

• SO(4) spin conservation. For simplicity, only $S_z = T_3 + \frac{3}{2} T_3'$ is shown.

\[
|\text{init}\rangle = \left|\frac{3}{2}\right>_p \otimes |\text{zero charge}\rangle_{\text{vort}} \quad \rightarrow \\
|\text{final}\rangle = \left|\frac{1}{2}\right>_p \otimes |S_z = 1\rangle_{\text{vort}} - \left|\frac{1}{2}\right>_p \otimes |S_z = 2\rangle_{\text{vort}}
\]

• Generation of entanglement between the particle and HQV loop!
• The proof of the exact SO(5) symmetry.

• Alice string in quintet superfluids and non-Abelian Cheshire charge.

• Quartetting and pairing orders in 1D systems.


• SO(5) (Sp(4)) Magnetism in Mott-insulating phases.
Multiple-particle clustering (MPC) instability

• Pauli’s exclusion principle allows MPC. More than two particles form bound states.

  baryon (3-quark); alpha particle (2p+2n); bi-exciton (2e+2h)

• Spin 3/2 systems: quartetting order.

SU(4) singlet:  
4-body maximally entangled states

\[ O_{qt} = \psi_{3/2}^+(r)\psi_{1/2}^+(r)\psi_{-1/2}^+(r)\psi_{-3/2}^+(r) \]

• Difficulty: lack of a BCS type well-controlled mean field theory.

1D systems: strongly correlated but understandable

• Bethe ansatz results for 1D SU(2N) model:

2N particles form an SU(2N) singlet; Cooper pairing is not possible because 2 particles can not form an SU(2N) singlet.


• Competing orders in 1D spin 3/2 systems with SO(5) symmetry.

  Both quartetting and singlet Cooper pairing are allowed.
  Transition between quartetting and Cooper pairing.

Phase diagram at incommensurate fillings

- Gapless charge sector.
- Spin gap phases B and C: pairing v.s. quartetting.
- Singlet pairing in purely repulsive regime.
- Ising transition between B and C.
- Quintet pairing is not allowed.
Phase B: the quartetting phase

• Quartetting superfluidity v.s. CDW of quartets ($2k_f$-CDW).

\[ O_{qt} = \psi_{3/2}^{+} \psi_{1/2}^{+} \psi_{-1/2}^{+} \psi_{-3/2}^{+} \text{ wins at } K_c > 2; \]
\[ N_{2k_f} = \psi_{R\alpha}^{+} \psi_{L\alpha} \text{ wins at } K_c < 2. \]

\( K_c \): the Luttinger parameter in the charge channel.

\[ d = \frac{2\pi}{(2k_f)} \]
Phase C: the singlet pairing phase

• Singlet pairing superfluidity v.s CDW of pairs (4k_f-CDW).

\[ \eta^+ = \psi_{3/2}^+ \psi_{-3/2}^+ - \psi_{1/2}^+ \psi_{-1/2}^+ \text{ wins at } K_c > \frac{1}{2}; \]

\[ O_{4k_f,\text{cdw}} = \psi_{R\alpha}^+ \psi_{R\beta}^+ \psi_{L\beta} \psi_{L\alpha} \text{ wins at } K_c < \frac{1}{2}. \]

\[ d = 2\pi / (4k_f) \]
Competition between quartetting and pairing phases


\begin{itemize}
  \item Phase locking problem in the two-band model.

  \[ \eta^+ = \Delta_1^+ - \Delta_2^+ \propto e^{i\sqrt{\pi} \theta_c} \cos \sqrt{\pi} \theta_r; \quad \Delta_1^+ = \psi_{3/2}^+ \psi_{-3/2}^+ \quad \Delta_2^+ = \psi_{1/2}^+ \psi_{-1/2}^+ \]

  \[ O_{\text{quar}} = \Delta_1^+ \Delta_2^+ \propto e^{i\sqrt{4} \pi \theta_c} \cos 2 \sqrt{\pi} \varphi_r. \]

  \[ \sqrt{\pi} \theta_c \text{ overall phase; } \sqrt{\pi} \theta_r \text{ relative phase.} \]

  \[ \sqrt{\pi} \varphi_r \text{ dual field of the relative phase} \]

  \[ \psi_{3/2, \pm 1/2}^+ \rightarrow \psi_{3/2, \pm 1/2}^+ e^{i\alpha} \text{ i.e. } \sqrt{\pi} \theta_c \rightarrow \sqrt{\pi} \theta_c + 2\alpha \]

\end{itemize}
Ising transition in the **relative phase** channel

\[
H_{\text{eff}} = \frac{1}{2} \left\{ (\partial_x \theta_r)^2 + (\partial_x \varphi_r)^2 \right\} + \frac{1}{2\pi a} (\lambda_1 \cos 2\sqrt{\pi} \theta_r + \lambda_2 \cos 2\sqrt{\pi} \varphi_r)
\]

- **\( \lambda_1 > \lambda_2 \)** the relative phase is pinned: pairing order;
- **\( \lambda_1 < \lambda_2 \)** the dual field is pinned: quartetting order.

Ising transition: two Majorana fermions with masses:  \( \lambda_1 \pm \lambda_2 \)

- **Ising symmetry:** \( \psi^+_{\frac{3}{2}} \rightarrow i\psi^+_{\frac{3}{2}}, \quad \psi^+_{\frac{1}{2}} \rightarrow -i\psi^+_{\frac{1}{2}} \).

  relative phase: \( \sqrt{\pi} \theta_r \rightarrow \sqrt{\pi} \theta_r \pm \pi \quad \sqrt{\pi} \varphi_r \rightarrow \sqrt{\pi} \varphi_r \)

- **Ising ordered phase:** \( \eta \rightarrow -\eta \),
- **Ising disordered phase:** \( O_{\text{quart}} \rightarrow O_{\text{quart}} \)
Experiment setup and detection

• Array of 1D optical tubes.

• RF spectroscopy to measure the excitation gap.

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Spin exchange (one particle per site)

- Spin exchange: bond singlet \((J_0)\), quintet \((J_2)\). No exchange in the triplet and septet channels.

\[
H_{ex} = \sum_{\langle ij \rangle} - J_0 Q_0(ij) - J_2 Q_2(ij)
\]

\[
J_0 = 4t^2 / U_0, J_2 = 4t^2 / U_2, J_1 = J_3 = 0 \quad \frac{3}{2} \times \frac{3}{2} = 0+2+1+3
\]

- Heisenberg model with bi-linear, bi-quadratic, bi-cubic terms.

- SO(5) or Sp(4) explicitly invariant form:

\[
H_{ex} = \sum_{ij} \frac{J_0 + J_2}{4} L_{ab} (i) L_{ab} (j) + \frac{-J_0 + 3J_2}{4} n_a (i) n_b (j) \quad a, b = 1 \sim 5
\]

\(L_{ab}\): 3 spins + 7 spin cubic tensors; \(n_a\): spin nematic operators; \(L_{ab}\) and \(n_a\) together form the 15 SU(4) generators.
Two different SU(4) symmetries

• A: $J_0 = J_2 = J$, SU(4) point.

\[
H_{ex} = J \sum_{ij} \{L_{ab}(i)L_{ab}(j) + n_a(i)n_b(j)\}
\]

\[
\begin{array}{c}
\text{singlet} \\
\text{quintet} \\
\text{triplet} \\
\text{septet}
\end{array}
\]

• B: $J_2 = 0$, the staggered SU’(4) point.

\[
H_{ex} = \frac{J_0}{4} \sum_{ij} \{L_{ab}(i)L'_{ab}(j) + n_a(i)n'_a(j)\}
\]

\[
\begin{array}{c}
\text{singlet} \\
\text{quintet} \\
\text{triplet} \\
\text{septet}
\end{array}
\]

In a bipartite lattice, a particle-hole transformation on odd sites:

\[
L_{ab}(j) = L'_{ab}(j) \quad n_{ab}(j) = -n'_{ab}(j)
\]

\[
H_{ex} = \frac{J_0}{4} \sum_{ij} \{L_{ab}(i)L'_{ab}(j) + n_a(i)n'_a(j)\}
\]

\[
\begin{array}{c}
\text{singlet} \\
\end{array}
\]

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Construction of singlets

- The SU(2) singlet: 2 sites.

- The uniform SU(4) singlet: 4 sites.

\[ \text{baryon} \quad \frac{\varepsilon_{\alpha\beta\gamma\delta}}{4!} \psi^\alpha_1 (1) \psi^\beta_2 (2) \psi^\gamma_3 (3) \psi^\delta_4 (4) |0\rangle \]

- The staggered SU'(4) singlet: 2 sites.

\[ \text{meson} \quad \frac{1}{2} \psi^\alpha_1 (1) R_{\alpha\beta} \psi^\beta_2 (2) \]
Phase diagram in 1D lattice (one particle per site)

- On the SU’(4) line, dimerized spin gap phase.
- On the SU(4) line, gapless spin liquid phase.

SU’(4) and SU(4) point at 2D

• At SU’(4) point ($J_2=0$), QMC and large N give the Neel order, but the moment is tiny.


• $J_2>0$, no conclusive results!

SU(4) point ($J_0=J_2$), 2D
Plaquette order at the SU(4) point?

Exact diagonalization on a 4*4 lattice

Exact result: SU(4) Majumdar-Ghosh ladder

- Exact dimer ground state in spin 1/2 M-G model.

\[ H = \sum_i H_{i,i+1,i+2}, \quad H_{i,i+1,i+2} = \frac{J}{2}(\vec{S}_i + \vec{S}_{i+1} + \vec{S}_{i+2})^2 \]

- SU(4) M-G: plaquette state.

\[ H = \sum_{\text{every six-site cluster}} H_i \]

\[ H_i = (\sum_{\text{six sites}} L_{ab})^2 + (\sum_{\text{six sites}} n_a)^2 \]

SU(4) Casimir of the six-site cluster

- Excitations as fractionalized domain walls.

SU(4) plaquette state: a four-site problem

- Bond spin singlet:
- Plaquette SU(4) singlet:
  \[ \frac{\epsilon_{\alpha\beta\gamma\delta}}{4!} \psi^+_\alpha \psi^+_\beta \psi^+_\gamma \psi^+_\delta |0\rangle \]
  4-body EPR state; no bond orders
- Level crossing:
  d-wave to s-wave
- Hint to 2D?
Speculations: 2D phase diagram?

- $J_2=0$, Neel order at the SU'(4) point (QMC).
  

- $J_2>0$, no conclusive results!

  2D Plaquette order at the SU(4) point?
  Exact diagonalization on a 4*4 lattice

- Phase transitions as $J_0/J_2$?
  Dimer phases? Singlet or magnetic dimers?
The SU’(4) model: dimensional crossover

• SU’(4) model: 1D dimer order; 2D Neel order.

• SU’(4) model in a rectangular lattice; phase diagram as $J_y/J_x$.

• Competition between the dimer and Neel order.

• No frustration; transition accessible by QMC.
Conclusion

• Spin 3/2 cold atomic systems open up a new opportunity to study high symmetry and novel phases.

• Quintet Cooper pairing: the Alice string and topological generation of quantum entanglement.

• Quartetting order and its competition with the pairing order.

• Strong quantum fluctuations in spin 3/2 magnetic systems.