Phase space approach to quantum dynamics and hydrodynamics of interacting quantum systems

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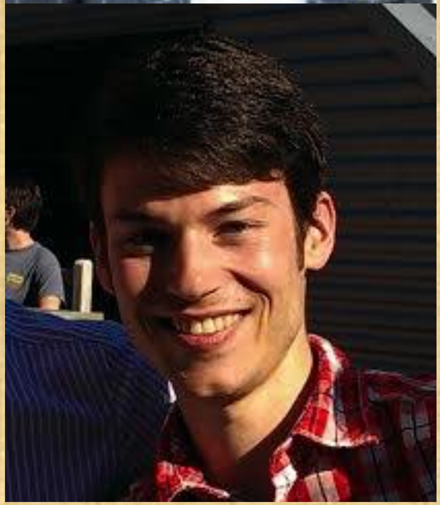


Plan

- Quantum Mechanics in phase space. Truncated Wigner approximations (TWA)
- Quantum correlations as extra phase space dimensions. Cluster TWA.
- Simulating diffusion in strongly coupled spin chains.
- Fermionic TWA. Applications to the SYK model. TWA as fluctuating mean-field approximation.

Major quest: reducing exponential complexity of quantum linear dynamics to polynomial complexity of classical nonlinear dynamics.

Why phase space?

Quantum mechanics – normally work in Schrödinger picture: evolve the wave function (density matrix) keep observables time independent. Equivalent classical picture – Liouville equation for the probability distribution:

$$\partial_t P(\vec{x},\vec{p},t) + \{P, H\} = 0$$

Extremely complicated (nearly impossible to solve) in many-particle systems

Quantum mechanics in Heisenberg picture – usually intractable with some exceptions. Equivalent classical picture – what we normally use

$$\overline{O(\vec{x},\vec{p},t)} = \int d\vec{x}_0 d\vec{p}_0 P_0(\vec{x}_0,\vec{p}_0)\, O(\vec{x}(t),\vec{p}(t)), \quad \dot{\vec{x}} = \{\vec{x}, H\},\ \dot{\vec{p}} = \{\vec{p}, H\}$$

Keep the initial probability distribution constant in time, evolve observables.

Quantum mechanics in phase space – a convenient tractable way to work near a classical (saddle point) limit

Phase space (Wigner-Weyl) representation of Quantum Mechanics

Observables – ordinary functions (Weyl symbols):

$$\Omega_W(x,p) = \int d\xi \left\langle x - \frac{\xi}{2} \right| \hat{\Omega}(\hat{x},\hat{p}) \left| x + \frac{\xi}{2} \right\rangle \mathrm{e}^{ip\xi/\hbar}$$

Wave function (density matrix) – Wigner quasiprobability distribution

$$W(x,p) = \int d\xi \langle x - \xi/2 | \hat{\rho} | x + \xi/2 \rangle \mathrm{e}^{ip\xi/\hbar} = \int d\xi \rho(x - \xi/2, x + \xi/2) \mathrm{e}^{ip\xi/\hbar}.$$

Expectation values: statistical averages

$$\langle \hat{\Omega}(\hat{x},\hat{p}) \rangle \equiv \mathrm{Tr}[\hat{\rho}\, \hat{\Omega}(\hat{x},\hat{p})] = \int \frac{dx dp}{2\pi\hbar} W(x,p) \Omega_W(x,p)$$

Rules of QM: noncommutativity, uncertainty,... – Moyal product

$$(\Omega_1 \Omega_2)_W(x,p) = \Omega_{1,W}(x,p) \exp\left[\frac{i\hbar}{2}\hat{\Lambda}\right] \Omega_{2,W}(x,p), \quad \hat{\Lambda} = \frac{\overleftarrow{\partial}}{\partial x}\frac{\partial}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\partial}{\partial x}$$

$$[\hat{\Omega}_1, \hat{\Omega}_2]_W = 2i\Omega_{1,W} \sin\left(\frac{\hbar}{2}\Lambda\right) \Omega_{2W} = i\hbar \left\{\Omega_{1,W}, \Omega_{2,W}\right\}_{MB},$$

Equations of motion

$$i\hbar \dot{\hat{\rho}} = [\hat{H}, \hat{\rho}] \quad \dot{W} = \frac{2}{\hbar} H_W \sin\left(\frac{\hbar}{2}\Lambda\right) W = \{H_W, W\} + O(\hbar^2)$$

Standard Truncated Wigner Approximation (TWA)

$$\langle \hat{\Omega}(\hat{x}, \hat{p}, t) \rangle \approx \int \int dx_0 dp_0 W(x_0, p_0) \Omega(x(t), p(t), t)$$

Coordinate-momentum

Phase Space

$$\langle \hat{\Omega}(\hat{a}, \hat{a}^\dagger, t) \rangle \approx \int \int da da^* W(a_0, a_0^*) \Omega(a(t), a^*(t), t)$$

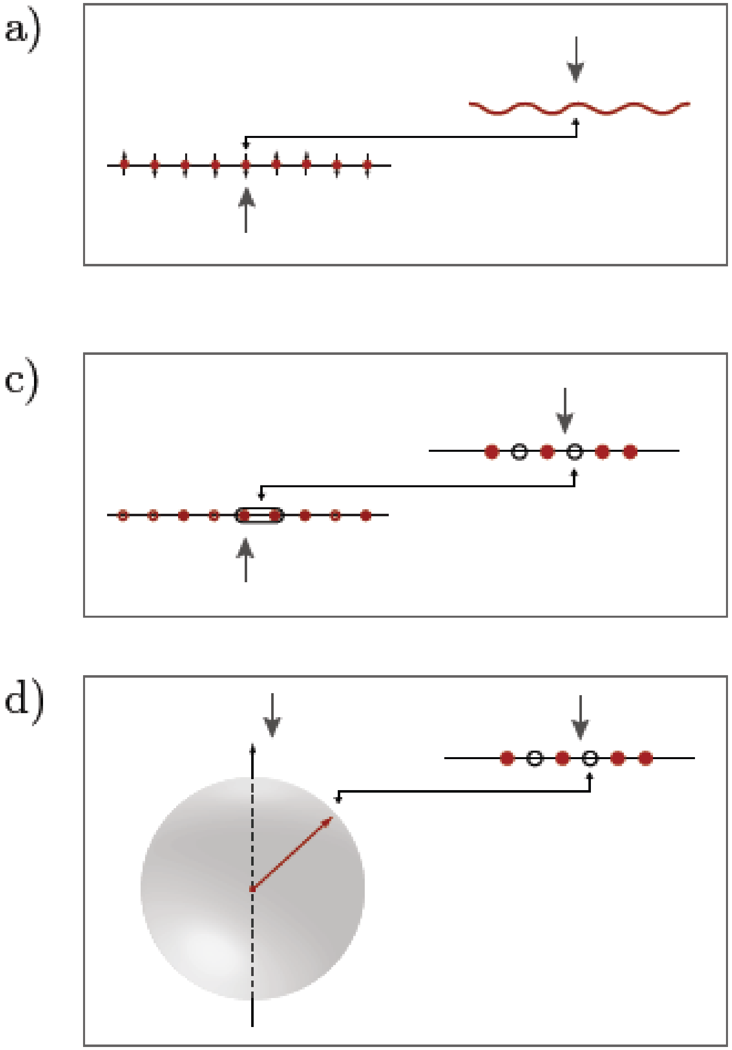
Bosonic complex amplitude

Classical (saddle point) evolution. Fluctuating (quantum) initial conditions

1. Easy to simulate if W is positive
2. Asymptotically exact at short times and near the classical limit
3. Exact for Harmonic systems
4. Quantum corrections can be expressed through quantum jumps.
5. Within accuracy of TWA (up to $\hbar^2$) one can approximate W with a proper Gaussian reproducing leading correlations.
6. TWA (and not the Dirac mean field approximation!)follows from the saddle point approximation of the Schwinger-Keldysh action!

Jaynes-Cummings Model (with A. Altland and V. Gurarie, 2009)

Reduces to Richardson model for BCS at large detuning (simple vector large N)



$$\hat{\mathcal{H}} = -\lambda \hat{b}^\dagger \hat{b} + \frac{g}{\sqrt{2S}} \left(\hat{b}^\dagger \hat{S}^- + \hat{b}\hat{S}^+ \right)$$

Start with polarized spins and empty photon mode. Do LZ sweep

$$\lambda(t) = -vt,\ t \in [-T, T],\ T \to \infty$$

$$W(b^\star, b, S_x, S_y, S_z) \approx 2\mathrm{e}^{-2|b|^2} \frac{1}{\pi S} \mathrm{e}^{-(S_x^2 + S_y^2)/S} \delta(S_z - S)$$

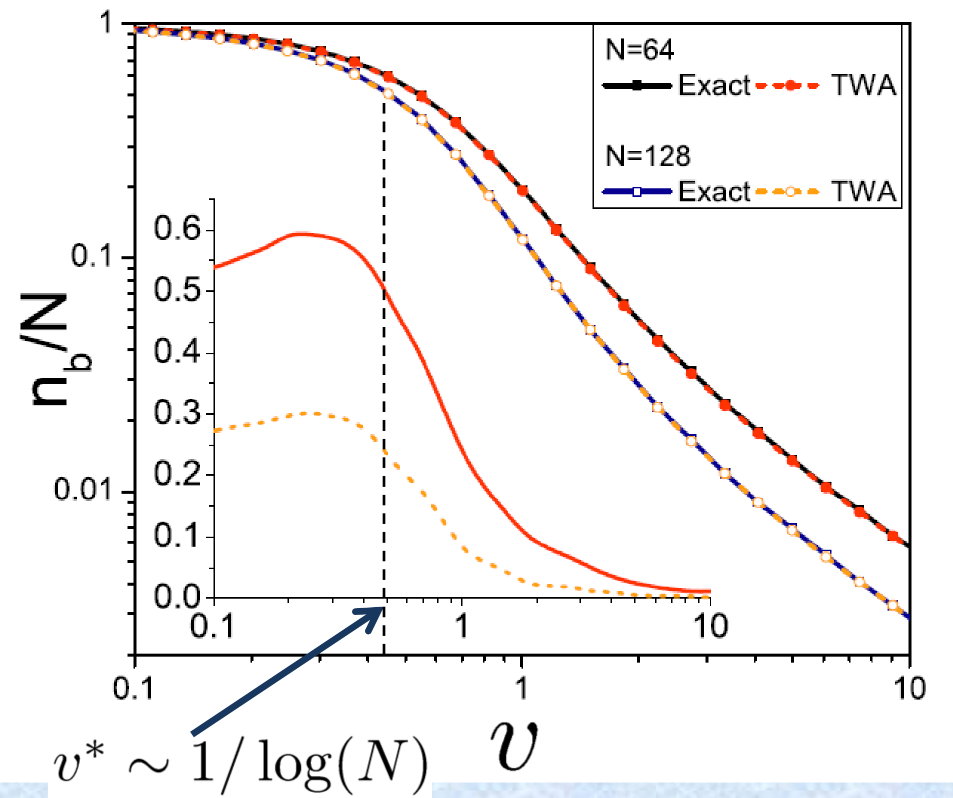
$$i\frac{\partial b}{\partial t} = -\lambda(t) b + \frac{g}{\sqrt{2S}} S^-$$

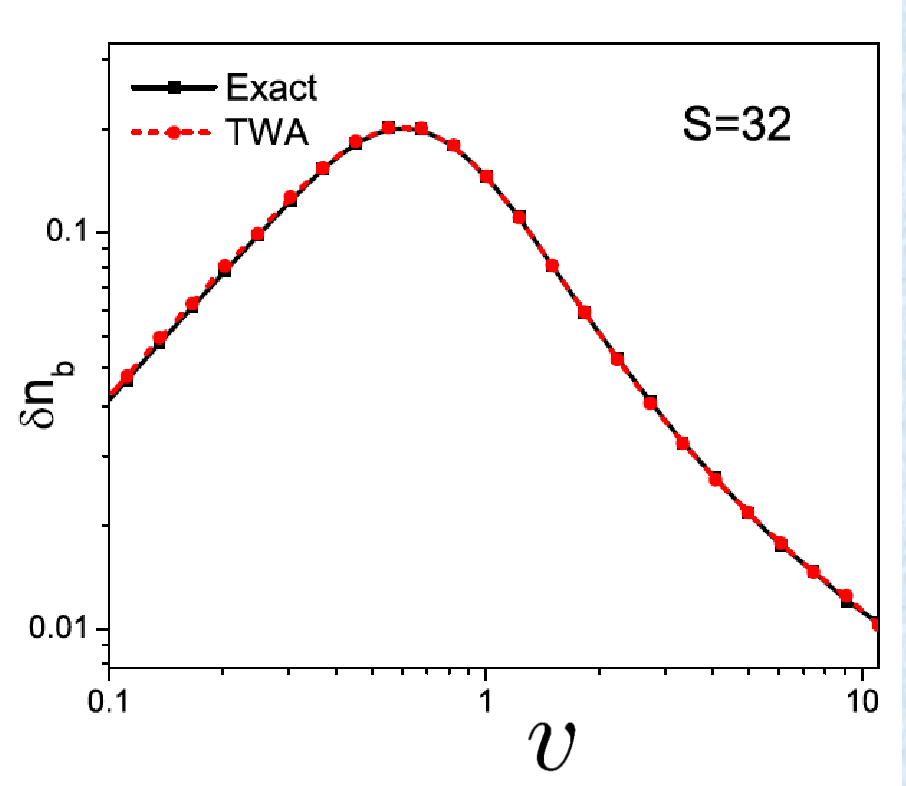
$$\frac{\partial \mathbf{S}}{\partial t} = \frac{2g}{\sqrt{2S}} \mathbf{B} \times \mathbf{S},$$

$$\mathbf{B} = (b^* + b, i(b - b^*), 0))$$

The problem can be solved analytically using adiabatic invariants:

A. Altland, V. Gurarie, T. Kriecherbauer, AP, PRA 79, 042703 (2009) , A.P. Itin, P. Törmä, arXiv:0901.4778.





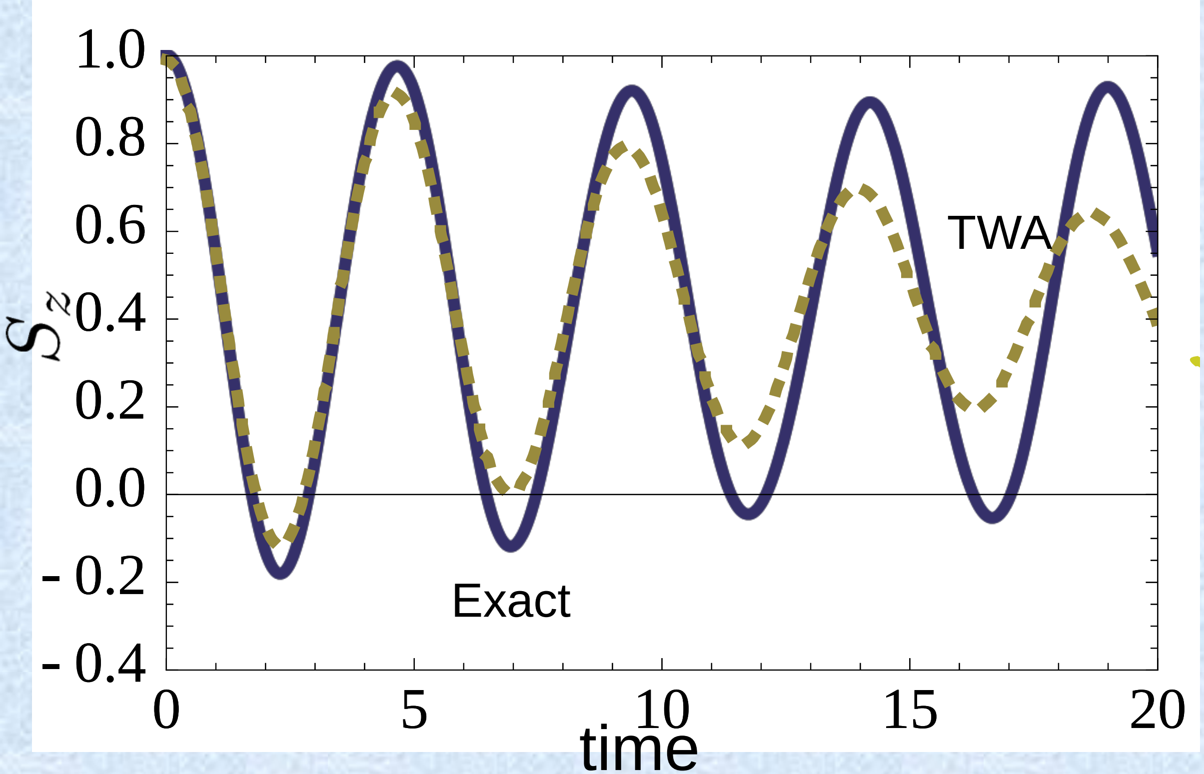
Almost perfect agreement with the exact result in the whole velocity range. Exact analytic solution perfectly agrees with TWA (C. Sun, N. A. Sinitsyn, arXiv:1606.08430)

In Keldysh action the large N-limit gives spurious results: large N and long time (low velocity) limits do not commute:

What if the elementary local degree of freedom (site) has 3 states? E.g. a spin one system.

$$\mathcal{H} = -\mathbf{BS} + \frac{U}{2} S_z^2$$

TWA fails after a short time unless interactions are weak.



$$\mathcal{H} = -S_x - S_z + \frac{1}{2} S_z^2$$

Prepare the spin initially polarized along *z*.

TWA fails. No small parameter to justify it.

Idea: fix TWA introducing additional (hidden) variables

(S. Davidson and A.P., PRL 2015)

Go to SU(3) group. Any 3x3 Hamiltonian is a linear combination of SU(3) generators.

(Mapping taken from M. Kiselev, et. al. EPL (2013) for LZ problem in a 3 level system)

$$X_1 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} X_2 = \begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix} X_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$X_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

....

$$X_8 = \begin{pmatrix} -\frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{2}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{3}} \end{pmatrix}$$

$$X_1 = S_x,\ X_2 = S_y,\ X_3 = S_z,\ X_4 = (S_x)^2 - (S_y)^2,\ X_5 = [S_x, S_y]_+,$$

$$X_6 = [S_x, S_z]_+,\ X_7 = [S_y, S_z]_+,\ X_8 = \frac{1}{\sqrt{3}}\left((S_x)^2 + (S_y)^2 - 2(S_z)^2\right)$$

Schwinger bosons: $X_1 \to a^\dagger_\alpha X_1^{\alpha\beta} a_\beta,\ a^\dagger_1 a_1 + a^\dagger_2 a_2 + a^\dagger_3 a_3 = N = 1$

$$H = \frac{U}{2} S_z^2 - S_z$$

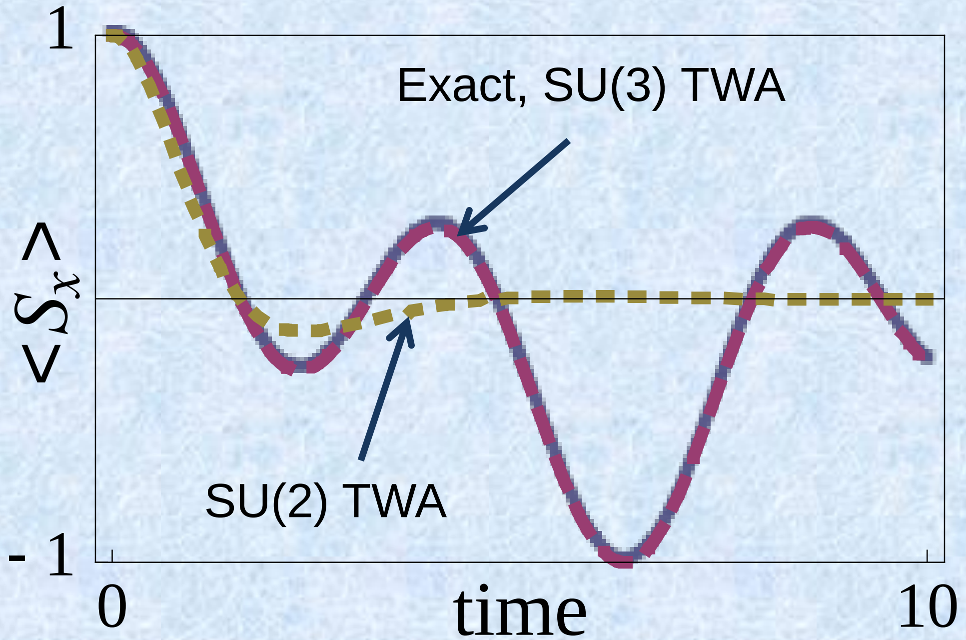
Single site Hamiltonian of Hubbard model: interaction and chemical potential

$$(H_I)_W^{SU(2)} = (U/2)X_3^2 - X_3, \quad (H_I)_W^{SU(3)} = (U/6)(2 - \sqrt{3}X_8) - X_3.$$

Map interacting SU(2) spin to noninteracting (= linear) SU(3)) spin

TWA, solve SU(3) Bloch equation: $\dot{X}_a = f_{abc} \frac{\partial H}{\partial X_\beta} X_\gamma$

Start from a state polarized along x

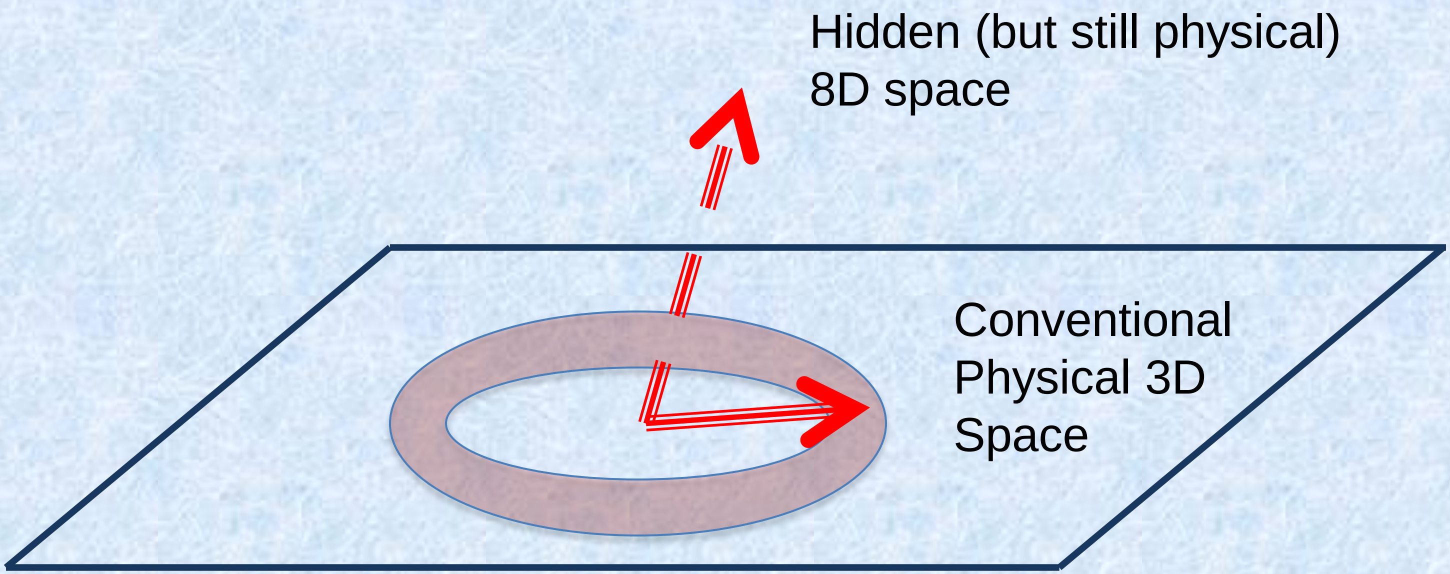


SU(3) TWA – (semi)classical dynamics in 8-dimensional phase space.

Extra variables are like hidden variables.

What did we achieve?

Classical dynamics becomes exact if we go to a higher-dimensional phase space.



If we solve classical equations in 8D space and project to 3D space we are exact (for a single spin one)

Linear (noninteracting) Hamiltonians are also easier to deal with in equilibrium.

Many-body generalization.

Bose Hubbard model in spin 1 representation (E. Altman 2001)

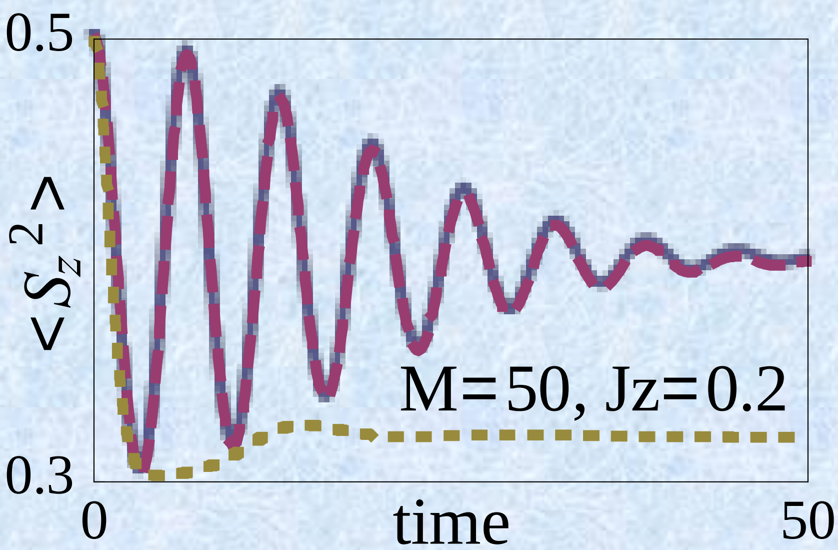
$$H_{eff} = \frac{U}{2}\sum_i (\hat{S}_z^i)^2 - J\bar{n}\sum_{\langle ij\rangle}(\hat{S}_x^i\hat{S}_x^j + \hat{S}_y^i\hat{S}_y^j) - \mu\sum_i \hat{S}_z^i$$

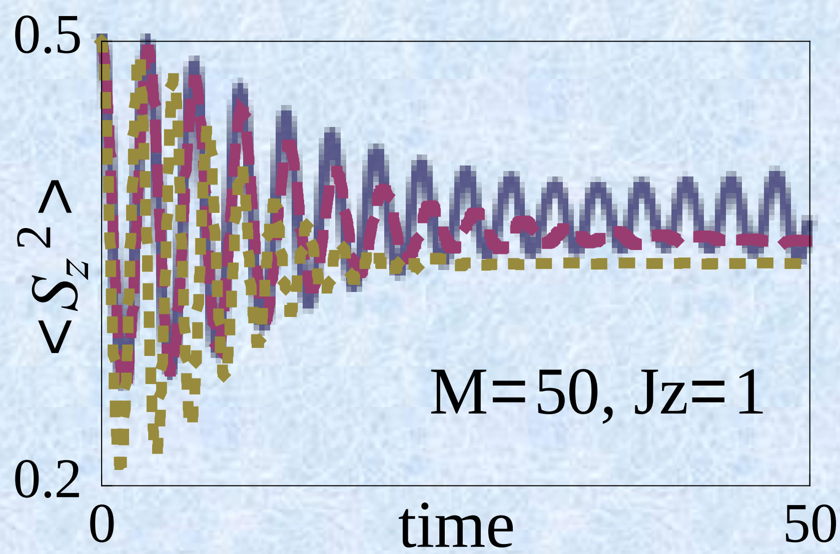
Treat local interactions exactly by mapping to SU(3) spins.
Treat NN interactions semiclassically within TWA.

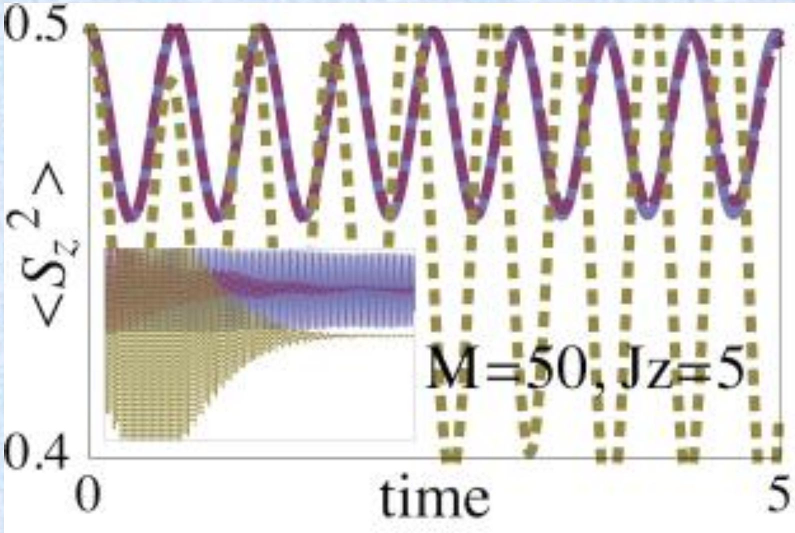
$$H_{eff} = -\frac{\sqrt{3}U}{6}\sum_i X_8^i - \mu\sum_i X_3^i - J\bar{n}\sum_{\langle ij\rangle}(X_1^i X_1^j + X_2^i X_2^j)$$

Small hopping or large dimensionality (connectivity) – expect SU(3) TWA to work much better than SU(2) TWA.

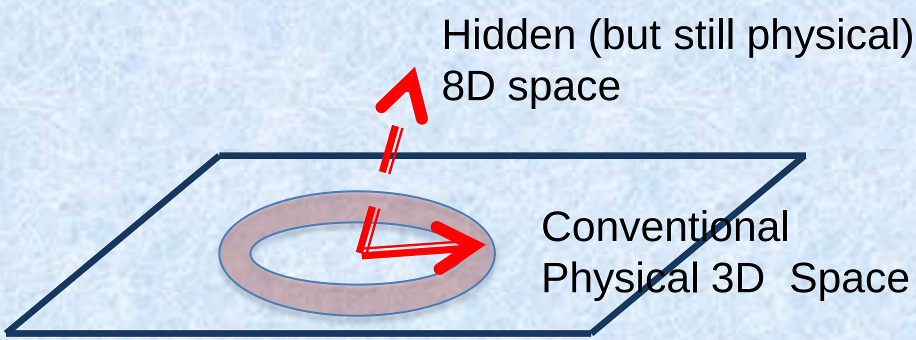
Simulations for fully connected model (z=M - coordination number)





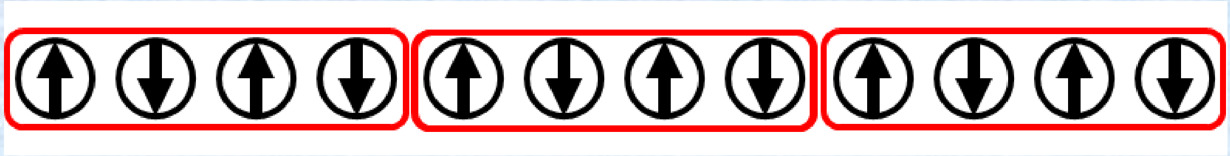


In all these cases improve accuracy of TWA improves by going to SU(3).



8D quantum world = 3D quantum world but it is closer to classical

Cluster TWA (CTWA)



Hilbert space of each cluster is spanned by SU(N) group. N – Hilbert Space Dimension. N=16 in the shown example.

$$H_{\text{cluster}} = \sum_i \sum_{\alpha=1}^{N^2-1} h_\alpha^{(i)} \hat{X}_\alpha^{(i)} \qquad H_{\text{cluster}-\text{cluster}} = \sum_{\langle ij \rangle} \sum_{\alpha\beta} J_{\alpha\beta}^{(ij)} \hat{X}_\alpha^{(i)} \hat{X}_\beta^{(j)}$$

Classical equations of motion

$$\dot{X}_\alpha^{(i)} = f_{\alpha\beta\gamma} \frac{\partial H}{\partial X_\beta^i} X_\gamma^{(i)}, \quad i[X_\alpha^{(i)} X_\beta^{(j)}] = \delta_{ij} f_{\alpha\beta\gamma} X_\gamma^{(i)}$$

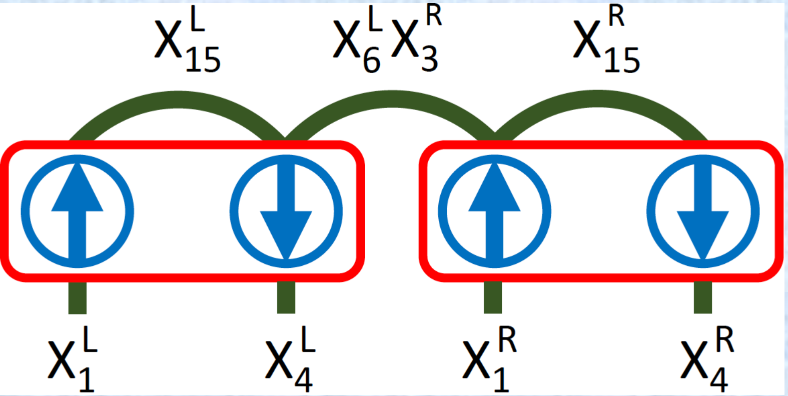
Initial conditions. Choose a Gaussian factorized distribution $W(\vec{X})$

$$\langle \hat{X}_\alpha \rangle = \int D\vec{X} X_\alpha W(\vec{X}), \quad \langle \hat{X}_\alpha \hat{X}_\beta + \hat{X}_\beta \hat{X}_\alpha \rangle \equiv \sum_{\gamma+1}^{N^2} d_{\alpha\beta\gamma} \langle \hat{X}_\gamma \rangle = 2 \int D\vec{X} X_\alpha X_\beta W(\vec{X})$$

This choice can be justified from the short time expansion. Alternative discrete sampling: W. Wooters et. al. 2004; works by A.M. Rey et. al.

Example: four sites

$$\hat{H} = J\sum_{j=1}^{3}\hat{\sigma}_z^{(j)}\hat{\sigma}_z^{(j+1)} + h_x\sum_{j=1}^{4}\hat{\sigma}_x^{(j)}.$$



$$\hat{X}_0 = \hat{I}^{(1)}\otimes\hat{I}^{(2)}$$
$$\hat{X}_1 = \hat{\sigma}_x^{(1)}\otimes\hat{I}^{(2)},\quad \hat{X}_2 = \hat{\sigma}_y^{(1)}\otimes\hat{I}^{(2)},\quad \hat{X}_3 = \hat{\sigma}_z^{(1)}\otimes\hat{I}^{(2)}$$
$$\hat{X}_4 = \hat{I}^{(1)}\otimes\hat{\sigma}_x^{(2)},\quad \hat{X}_5 = \hat{I}^{(1)}\otimes\hat{\sigma}_y^{(2)},\quad \hat{X}_6 = \hat{I}^{(1)}\otimes\hat{\sigma}_z^{(2)}$$
$$\hat{X}_7 = \hat{\sigma}_x^{(1)}\otimes\hat{\sigma}_x^{(2)},\quad \hat{X}_8 = \hat{\sigma}_x^{(1)}\otimes\hat{\sigma}_y^{(2)},\quad \hat{X}_9 = \hat{\sigma}_x^{(1)}\otimes\hat{\sigma}_z^{(2)}$$
$$\hat{X}_{10} = \hat{\sigma}_y^{(1)}\otimes\hat{\sigma}_x^{(2)},\quad \hat{X}_{11} = \hat{\sigma}_y^{(1)}\otimes\hat{\sigma}_y^{(2)},\quad \hat{X}_{12} = \hat{\sigma}_y^{(1)}\otimes\hat{\sigma}_z^{(2)}$$
$$\hat{X}_{13} = \hat{\sigma}_z^{(1)}\otimes\hat{\sigma}_x^{(2)},\quad \hat{X}_{14} = \hat{\sigma}_z^{(1)}\otimes\hat{\sigma}_y^{(2)},\quad \hat{X}_{15} = \hat{\sigma}_z^{(1)}\otimes\hat{\sigma}_z^{(2)}$$

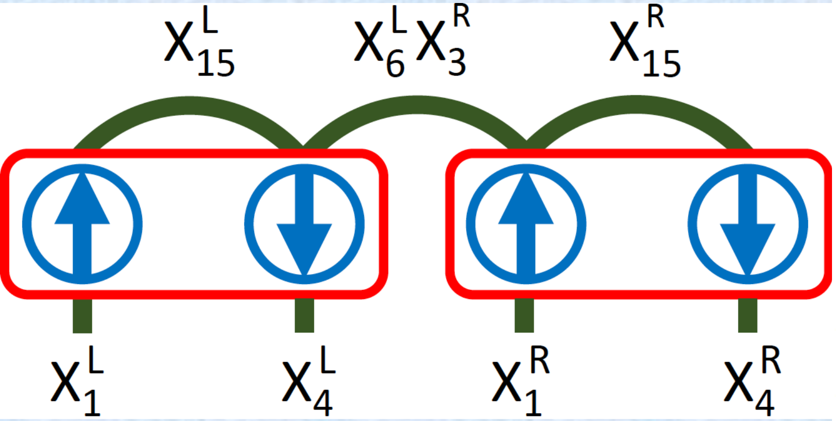
Treat local correlations (entangled degrees of freedom) as independent variables

$$\hat{H} = J\left(\hat{X}^L_{15} + \hat{X}^R_{15} + \hat{X}^L_6\hat{X}^R_3\right) + h_x\left(\hat{X}^L_1 + \hat{X}^L_4 + \hat{X}^R_1 + \hat{X}^R_4\right).$$

$$\langle\psi_0|\hat{X}_3^{L,R}|\psi_0\rangle = 1,\ \langle\psi_0|\hat{X}_6^{L,R}|\psi_0\rangle = \langle\psi_0|\hat{X}_{15}^{L,R}|\psi_0\rangle = -1$$
$$\left\langle\psi_0\right|\left(\hat{X}^L_\alpha\right)^2\left|\psi_0\right\rangle = 1,\quad \alpha\in\{1,\ldots,15\}$$
$$\frac{1}{2}\langle\psi_0|\{\hat{X}^L_\alpha,\hat{X}^L_\beta\}_+|\psi_0\rangle = 1\ (\alpha,\beta)\in[(4,13)\quad(5,14)\quad(6,15)\quad(7,11)]$$
$$\frac{1}{2}\langle\psi_0|\{\hat{X}^L_\alpha,\hat{X}^L_\beta\}_+|\psi_0\rangle = -1\ (\alpha,\beta)\in[(1,9)\quad(2,12)\quad(3,6)\quad(3,15)\quad(8,10)],$$

Some operators are correlated

$$\langle\psi_0|\hat{X}_8^2|\psi_0\rangle = \langle\psi_0|\hat{X}_{10}^2|\psi_0\rangle$$
$$= -\frac{1}{2}\langle\psi_0|\{\hat{X}_8,\hat{X}_{10}\}_+|\psi_0\rangle = 1$$
$$\Rightarrow\quad X_8 = -X_{10}$$



$$\hat{X}_0 = \hat{I}^{(1)} \otimes \hat{I}^{(2)}$$
$$\hat{X}_1 = \hat{\sigma}_x^{(1)} \otimes \hat{I}^{(2)}, \quad \hat{X}_2 = \hat{\sigma}_y^{(1)} \otimes \hat{I}^{(2)}, \quad \hat{X}_3 = \hat{\sigma}_z^{(1)} \otimes \hat{I}^{(2)}$$
$$\hat{X}_4 = \hat{I}^{(1)} \otimes \hat{\sigma}_x^{(2)}, \quad \hat{X}_5 = \hat{I}^{(1)} \otimes \hat{\sigma}_y^{(2)}, \quad \hat{X}_6 = \hat{I}^{(1)} \otimes \hat{\sigma}_z^{(2)}$$
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$$W_L = \frac{1}{Z}\delta(X_3 - 1)\delta(X_6 + 1)\delta(X_{15} + 1)\delta(X_4 - X_{13})\delta(X_5 - X_{14})$$
$$\times\delta(X_1 + X_9)\delta(X_2 + X_{12})\delta(X_7 - X_{11})\delta(X_8 + X_{10}) \exp\left[-\frac{\sum_{\alpha\in\{1,2,4,5,7,8\}} X_\alpha^2}{2}\right],$$

Equations of motion

$$\begin{aligned}
\partial_t X_4^L &= J X_3^R X_5^L + J X_{14}^L \\
\partial_t X_3^L &= -h X_2^L \\
\partial_t X_{12}^L &= -h X_{11}^L + h X_{15}^L - J X_1^L \\
\ldots &= \ldots
\end{aligned}$$

Identical to time-dependent variational principle (TDVP) if ignore fluctuations.
Number of independent variables 2^{N+1} (not 4^N). Need one extra ancilla spin.

Applications to the long time hydrodynamics

Model (motivated by discussions with F. Pollmann): XXZ chain

$$\hat{H} = \sum_i^N \hat{S}_x^i \hat{S}_x^{i+1} + \hat{S}_y^i \hat{S}_y^{i+1} + \Delta \hat{S}_z^i \hat{S}_z^{i+1} + \gamma \sum_i^N \hat{S}_x^i \hat{S}_x^{i+2} + \hat{S}_y^i \hat{S}_y^{i+2} + \Delta \hat{S}_z^i \hat{S}_z^{i+2}.$$

Choose $\Delta = 2, \quad \gamma = 1/2, \quad S_\alpha \equiv \sigma_\alpha$

Describes well Yb_2Pt_2Pb, I. Zaliznyak et. al. unpublished

Central object

$$C_{ij}(t,t') = \frac{1}{Z} \sum_n \mathrm{e}^{-\beta E_n} \langle \psi_n | S_\alpha^i(t) S_\alpha^j(t') | \psi_n \rangle, \quad \alpha = \{x, y, z\}$$

Defines spectral function (dynamic structure factor), spin susceptibilities, diffusion constant, fluctuation-dissipation relation (key indicator of thermalization),...

This work – focus on infinite temperatures

$$C_{ij}(t,t') = \frac{1}{\mathcal{D}} \mathrm{Tr}\left[S_\alpha^i(t) S_\alpha^j(t')\right] = \frac{1}{2\mathcal{D}} \mathrm{Tr}\left[S_\alpha^i(t) S_\alpha^j(t') + S_\alpha^j(t') S_\alpha^i(t)\right]$$

Expected long time behavior

$$C_{ij}(t,t') \equiv C_{ij}(t-t') = \frac{1}{\mathcal{D}}\mathrm{Tr}\left[S^i_\alpha(t)S^j_\alpha(t')\right] \sim \frac{1}{\sqrt{D|t-t'|}}\exp\left[-\frac{|i-j|^2}{2D|t-t'|}\right]$$

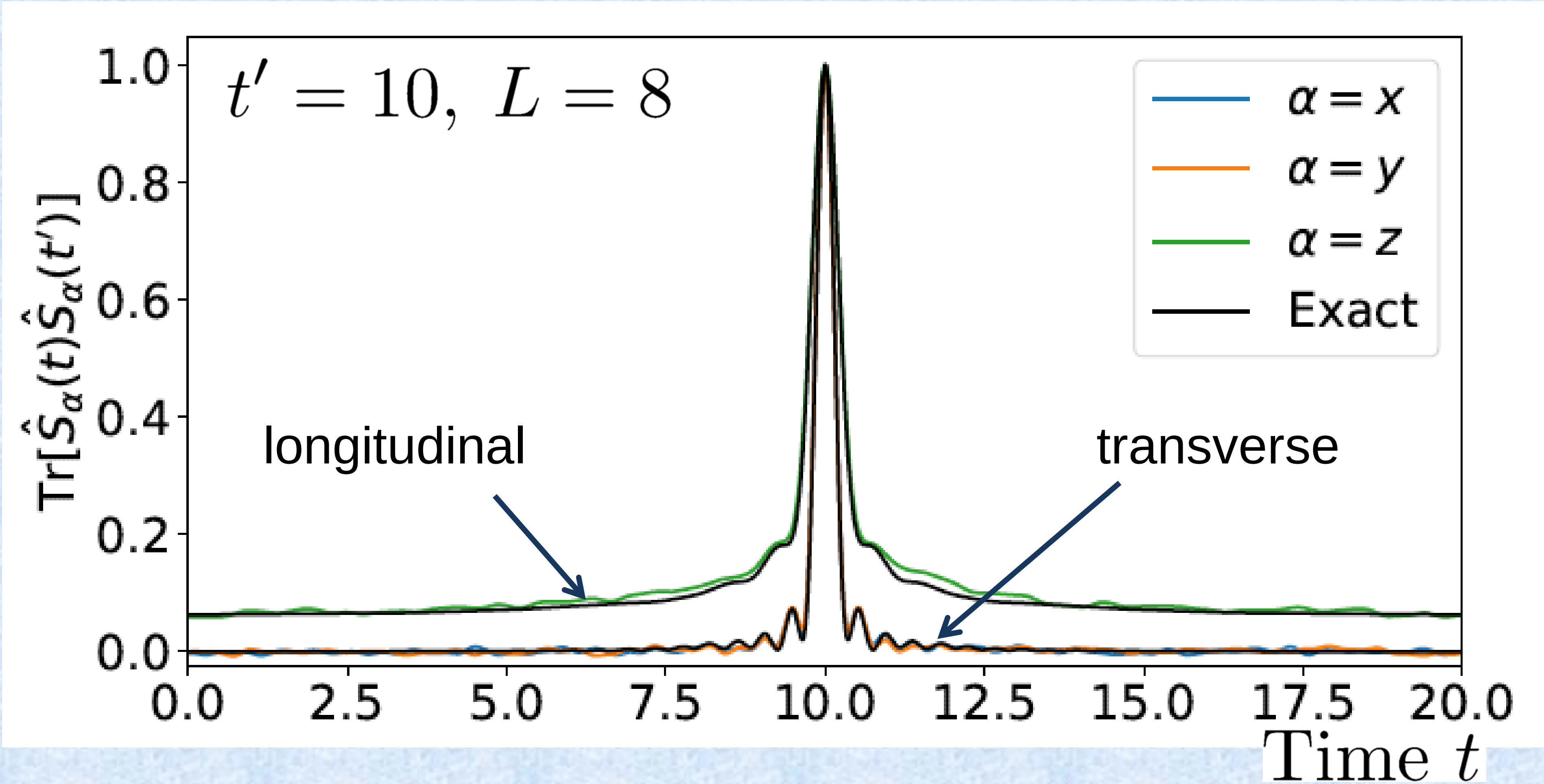
Can be used to extract diffusion constant (D. Luitz and Y. Bar Lev, 2016, 2017)

$$R^2(t) = \frac{\sum_{ij=1}^{N}\frac{N^2}{\pi^2}\sin^2\left(\frac{\pi}{N}(i-j)\right)C_{ij}(t)}{\sum_{ij}C_{ij}(t)} \sim \frac{N^2}{2\pi^2}\left(1-e^{-4Dt\pi^2/N^2}\right)$$

Main challenges: small system sizes amenable to ED can be too small to see asymptotic diffusive behavior.

Approximate methods (DMRG, mean field, TWA, ...) do not preserve time translational invariance, fail at long times.

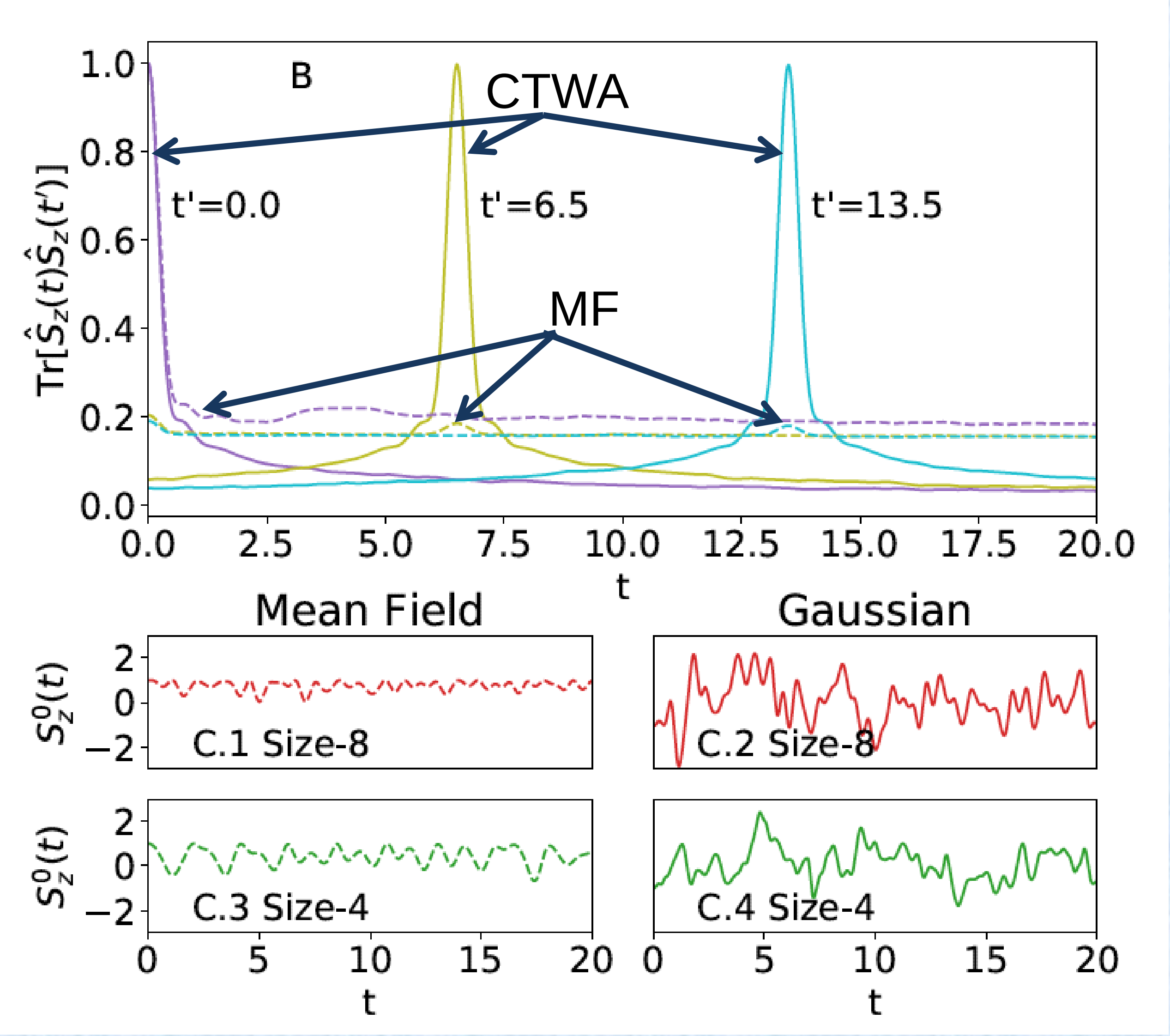
Numerical Results



$$\langle S_z^i(t) S_z^j(t') \rangle \to \frac{\overline{M_z^2}}{N^2} \sim \frac{1}{N}$$

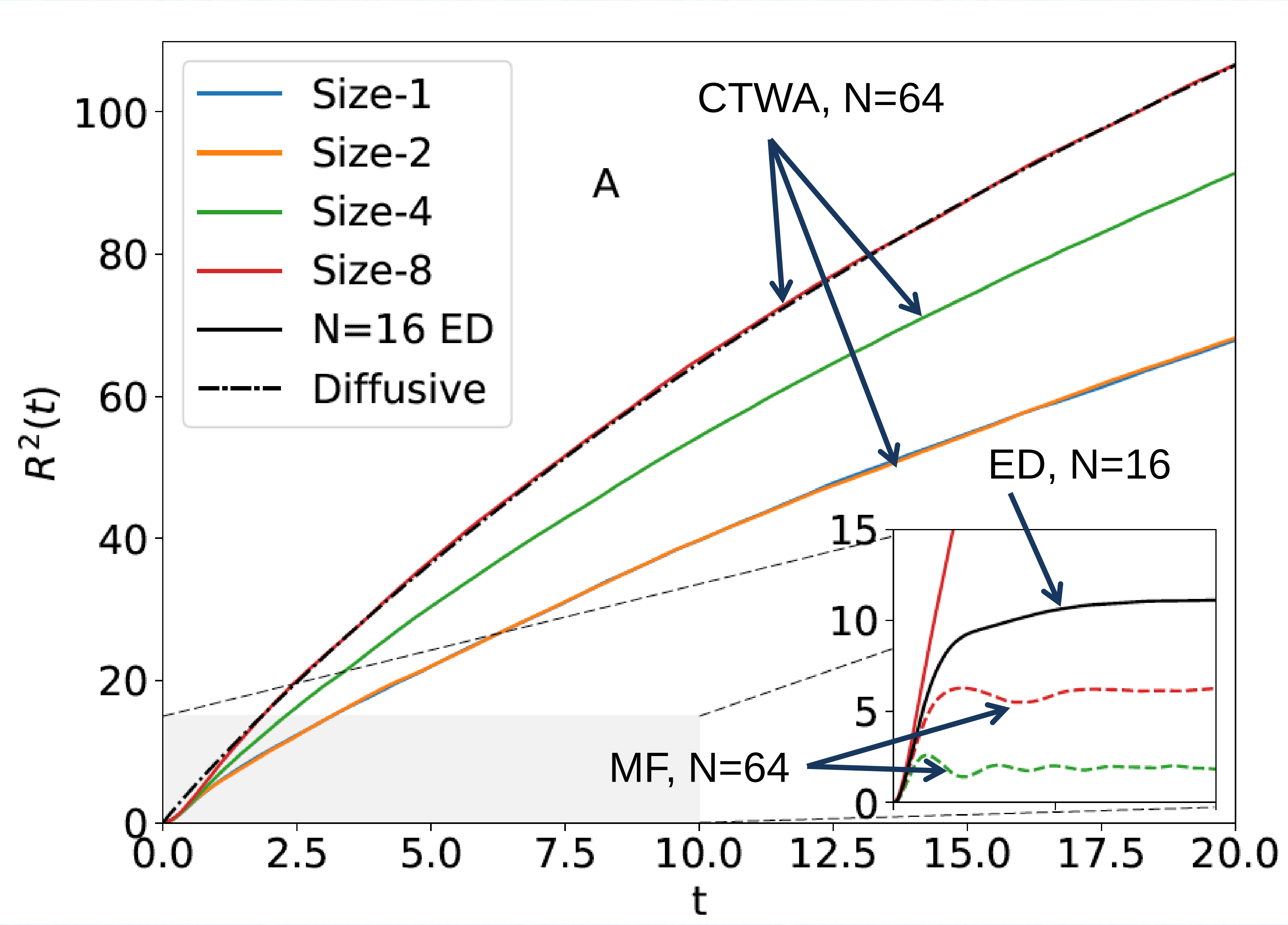
Follows from conservation of Z-magnetization

Longitudinal correlations, comparison with mean-field dynamics



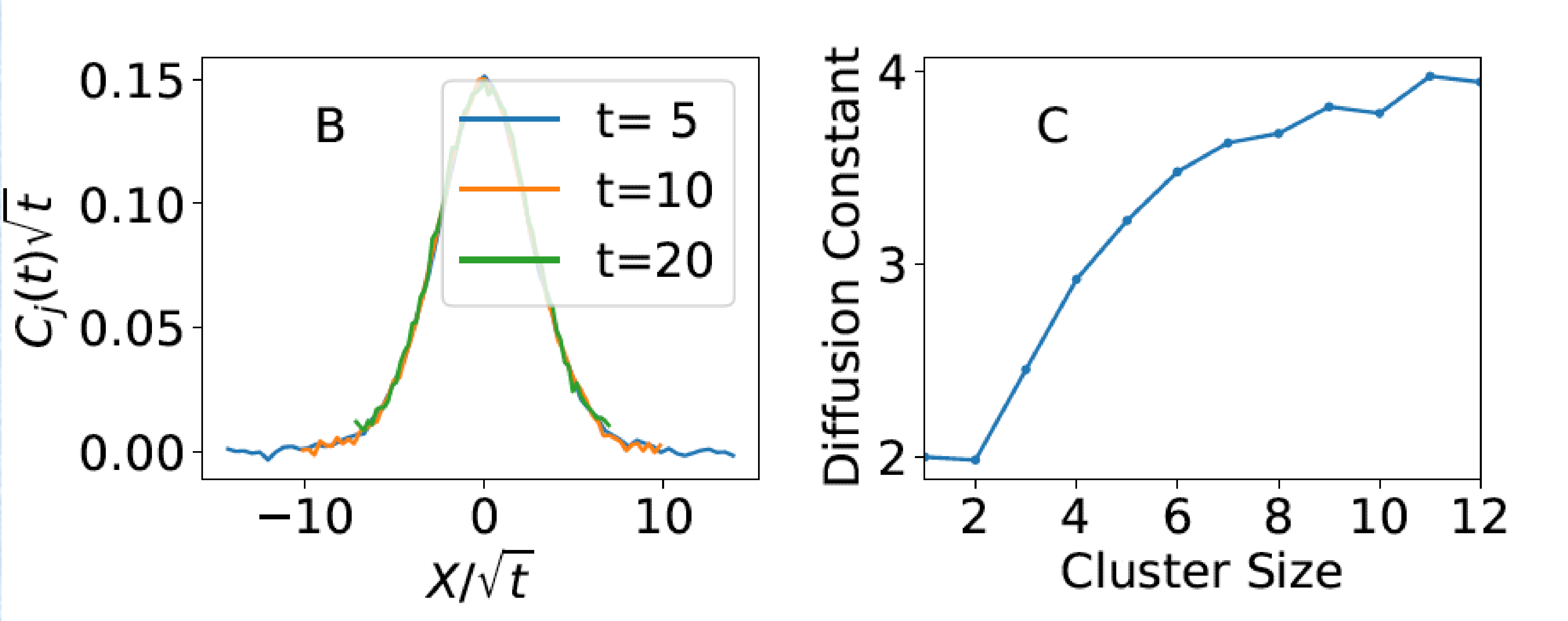
CTWA respects time-translation invariance: correct noise. MF fails, increasing cluster size makes things even worse due to ETH. Non-equilibrium initial state: MF is expected to fail completely.

Extracting diffusion constant



MF fails, ED gives a wrong diffusion constant

Excellent convergence to diffusive profile for all cluster sizes

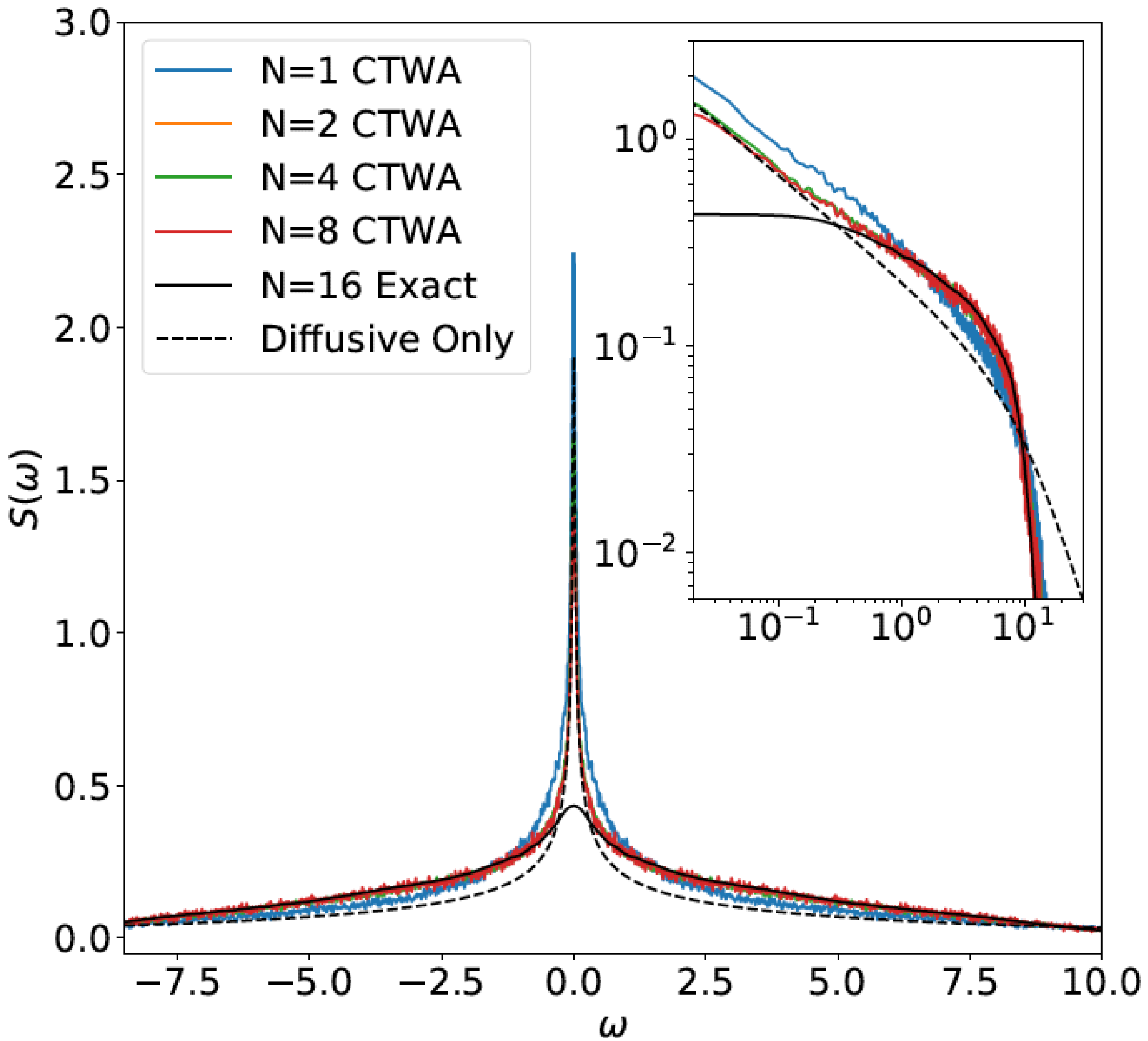


Very slow saturation of the diffusive constant with the cluster size (strong quantum renormalization).

This does not happen if we remove Z-conservation law. Classical dynamics gives nearly exact diffusion constant. MF works much better.

Can reproduce well the whole dynamical structure factor

$$S(k,\omega) = \sum_{ij} \int_{-\infty}^{\infty} dt e^{i\omega t + ik(i-j)} C_{ij}(t), \quad S(\omega) = \frac{1}{N} \sum_k S(k,\omega) = \frac{2\pi}{N} \sum_i \int_{-\infty}^{\infty} dt e^{i\omega t} C_{ii}(t)$$



Small frequency tail

$$S(\omega) \propto \frac{1}{\sqrt{D\omega}}$$

indicates asymptotic diffusive behavior. Only visible for N>32.

High frequency (exponential) asymptotes are quantum and can not be recovered from hydrodynamic approaches.

CTWA captures both!

Fermions. No obvious classical limit.

Main idea: use bilinear strings as dynamical variables. Non-locality is crucial

$$\hat{E}^{\alpha}_{\beta} = \hat{c}^{\dagger}_{\alpha}\hat{c}_{\beta} - \frac{1}{2}\delta_{\alpha\beta}, \quad \hat{E}_{\alpha\beta} = \hat{c}_{\alpha}\hat{c}_{\beta}, \quad \hat{E}^{\alpha\beta} = \hat{c}^{\dagger}_{\alpha}\hat{c}^{\dagger}_{\beta}.$$

Group structure

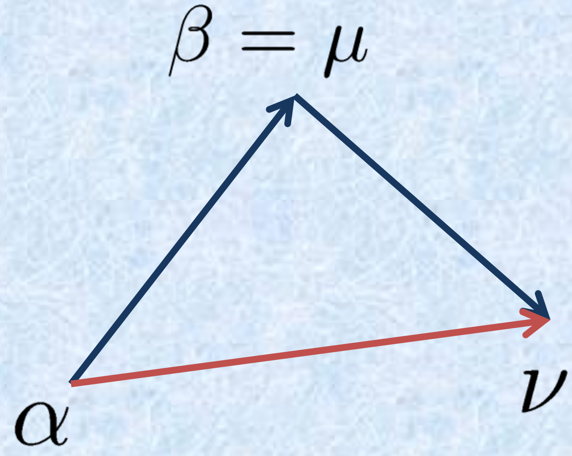
$$U(N) = \{\hat{E}^{\alpha}_{\beta}\}$$

$$SO(2N) = \{\hat{E}^{\alpha}_{\beta}, \hat{E}_{\alpha\beta}, \hat{E}^{\alpha\beta}\}$$

Treat string variables as SO(2N) nonlocal spin degrees of freedom. Phase space dimensionality ~ $2N^2$ (instead of 2N).

Non-interacting system. Hamiltonian is linear. TWA is exact.

Poisson brackets (commutation relations). Encode locality



$$[\hat{E}^{\alpha}_{\beta}, \hat{E}^{\mu}_{\nu}]_- = \delta_{\beta\mu}\hat{E}^{\alpha}_{\nu} - \delta_{\alpha\nu}\hat{E}^{\mu}_{\beta},$$

$$[\hat{E}^{\alpha}_{\beta}, \hat{E}_{\mu\nu}]_- = \delta_{\alpha\nu}\hat{E}_{\beta\mu} - \delta_{\alpha\mu}\hat{E}_{\beta\nu},$$

$$[\hat{E}^{\alpha\beta}, \hat{E}_{\mu\nu}]_- = \delta_{\alpha\nu}\hat{E}^{\beta}_{\mu} + \delta_{\beta\mu}\hat{E}^{\alpha}_{\delta} - \delta_{\alpha\mu}\hat{E}^{\beta}_{\nu} - \delta_{\beta\nu}\hat{E}^{\alpha}_{\mu},$$

$$[\hat{E}_{\alpha\beta}, \hat{E}_{\mu\nu}]_- = 0, \quad [\hat{E}^{\alpha\beta}, \hat{E}^{\mu\nu}]_- = 0.$$

New non-local phase space variables

$$\rho_{\alpha\beta} = \left(\hat{E}^{\alpha}_{\beta}\right)_W, \ \tau_{\alpha\beta} = \left(\hat{E}_{\alpha\beta}\right)_W, \ \tau^*_{\alpha\beta} = -\left(\hat{E}^{\alpha\beta}\right)_W,$$

$$\rho_{\alpha\beta} = \rho^*_{\beta\alpha}, \quad \tau_{\alpha\beta} = -\tau_{\beta\alpha}$$

These variables satisfy canonical Poisson bracket relations, e.g.

$$\{\rho_{\alpha\beta}, \rho_{\mu\nu}\} = \delta_{\beta\mu}\rho_{\alpha\nu} - \delta_{\alpha\nu}\rho_{\mu\beta}$$

Equations of motion

$$\dot{\rho}_{\alpha\beta} = \{\rho_{\alpha\beta}, H_W\}, \quad \dot{\tau}_{\alpha\beta} = \{\tau_{\alpha\beta}, H_W\}$$

Initial conditions: exact Wigner function is too complicated. Use the best Gaussian (alternatively discrete sampling A. M. Rey group).

$$\langle\hat{\rho}_{\alpha\beta}\rangle = \int D\rho D\tau \rho_{\alpha\beta} W(\rho,\tau), \quad \langle\rho_{\alpha\beta}\rho_{\mu\nu} + \rho_{\mu\nu}\rho_{\alpha\beta}\rangle = 2\int D\rho D\tau \rho_{\alpha\beta}\rho_{\mu\nu} W(\rho,\tau),$$

Example: initial free Fermi sea, indexes –momentum modes

$$\langle\rho_{\alpha\beta}\rangle = \delta_{\alpha\beta}(n_\alpha - 1/2),\ \langle\tau_{\alpha\beta}\rangle = 0, \qquad \rho_{\alpha\beta} \leftrightarrow c^\dagger_\alpha c_\beta - 1/2, \quad \tau_{\alpha\beta} \leftrightarrow c_\alpha c_\beta$$

$$\left\langle\rho^*_{\alpha\beta}\rho_{\mu\nu}\right\rangle_c = \frac{1}{2}\delta_{\alpha\mu}\delta_{\beta\nu}\left(n_\alpha + n_\beta - 2n_\alpha n_\beta\right), \qquad (6)$$

$$\langle\tau^*_{\alpha\beta}\tau_{\mu\nu}\rangle_c = \frac{1}{2}\left(\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\beta\mu}\delta_{\alpha\nu}\right)\left(1 + 2n_\alpha n_\beta - n_\alpha - n_\beta\right)$$

Normal variables: no fluctuations at zero or unit filling.
Superconducting variables – always fluctuate.

SYK model, many unusual properties

$$\hat{H}_{SYK} = \frac{1}{(2N)^{3/2}} \sum_{ijkl} J_{ij;kl} \hat{c}_i^\dagger \hat{c}_j^\dagger \hat{c}_k \hat{c}_l.$$

$$J_{ij;kl} = -J_{ij;lk} = -J_{ji;kl}$$

$$J_{ij;kl} = J^{\leftarrow}_{kl;ij},$$

$$\overline{|J_{ij;kl}|^2} = J^2.$$

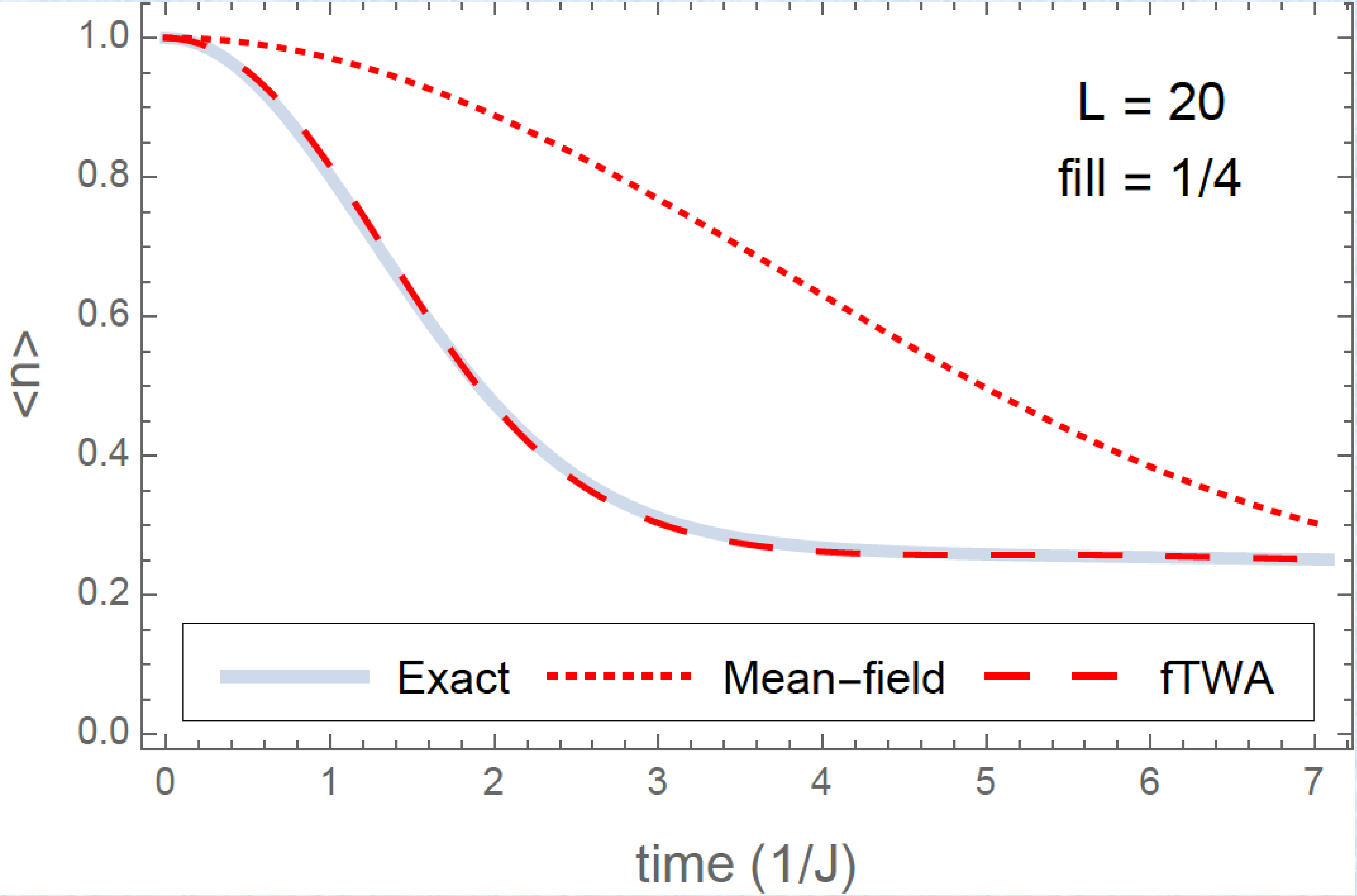
No obvious classical limit even at large N.

Use representation of the Hamiltonian through the superconducting variables

$$\mathcal{H}_\tau = \frac{1}{(2N)^{3/2}} \sum_{ijkl} J_{ij;kl} \left(\tau_{ij}^* \tau_{kl} + \frac{1}{2}\rho_{jk}\delta_{il} - \frac{1}{2}\rho_{jl}\delta_{ik} + \frac{1}{2}\rho_{il}\delta_{kj} - \frac{1}{2}\rho_{ik}\delta_{jl} \right)$$

Start from the same initial conditions as for the expansion example

Magnetization decay for a quench to SYK Hamiltonian



Error goes down as $1/N^2$ (M. Schmidt, et. al. 2018). SYK model realizes a classical matrix model (L. G. Yaffe, Rev. Mod. Phys. 54, 407 (1982).)

TWA vs Mean field for SYK

Mean field – a random superconductor

$$\hat{H} = \frac{1}{\sqrt{2N}} \sum_{ij} (\Delta_{ij}(t) \hat{c}_i^\dagger \hat{c}_j^\dagger + h.c.)$$

$$\Delta_{ij}(t) = \frac{1}{2N} \sum_{kl} J_{ijkl} \langle \hat{c}_k \hat{c}_l \rangle_t = \frac{1}{2N} \sum_{kl} J_{ijkl} \tau_{kl}(t)$$

$$\mathrm{i}\frac{d\rho_{\alpha\beta}}{dt} = -\frac{2}{\sqrt{2N}} \sum_k \Delta_{k\alpha}(t)^* \tau_{\beta k} - [\alpha \leftrightarrow \beta]^*$$

$$\mathrm{i}\frac{d\tau_{\alpha\beta}}{dt} = \frac{2}{\sqrt{2N}} \sum_j \Delta_{\alpha j}(t) \rho_{j\beta} - [\alpha \leftrightarrow \beta]$$

$$\rho_{\alpha\beta}(0) = \langle \hat{c}_\alpha^\dagger \hat{c}_\beta \rangle_{t=0} - \frac{\delta_{\alpha\beta}}{2}$$

$$\tau_{\alpha\beta}(0) = \langle \hat{c}_\alpha \hat{c}_\beta \rangle_{t=0} = 0$$

Does not really work: $\tau_{\alpha\beta}(t) \equiv 0$ due to the particle number conservation

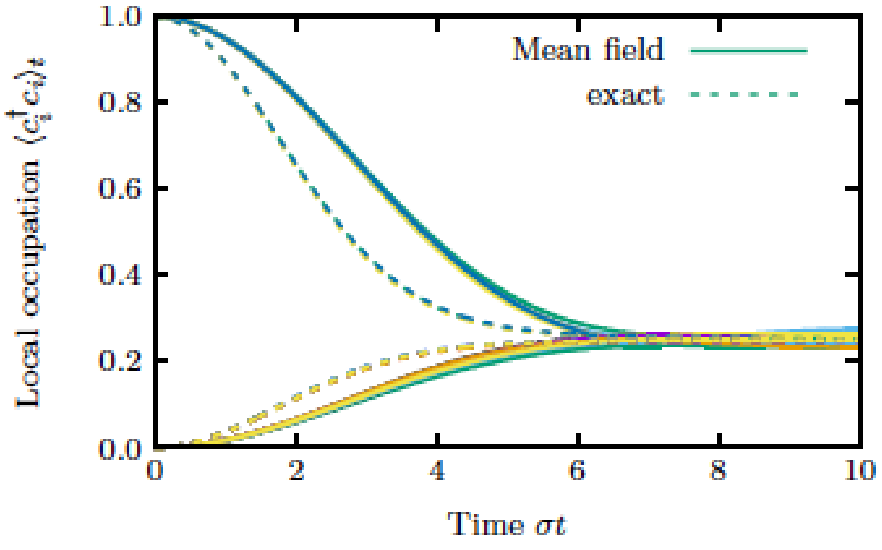
Essentially TWA (for the SYK model): introduce a fictitious ensemble of wave functions: $|\Psi_w\rangle$. These wave functions can be thought as Schwinger bosons of the corresponding SO(2N) group.

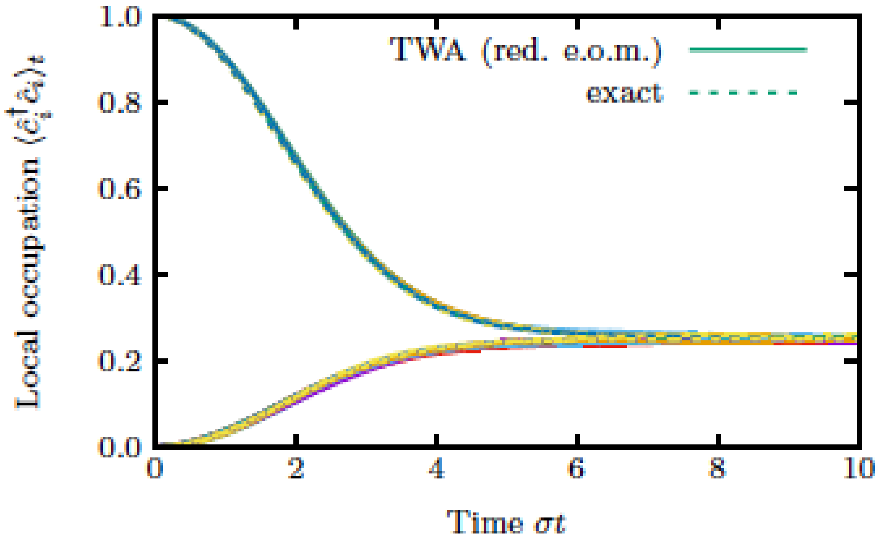
Require

$$0 = \langle \hat{c}_\beta^\dagger \hat{c}_\alpha^\dagger \rangle_0 = \overline{\langle \Psi_w | \hat{c}_\beta^\dagger \hat{c}_\alpha^\dagger | \Psi_w \rangle}, \quad \langle \hat{c}_\beta^\dagger \hat{c}_\alpha^\dagger \hat{c}_\gamma \hat{c}_\beta \rangle_0 = \overline{\langle \Psi_w | \hat{c}_\beta^\dagger \hat{c}_\alpha^\dagger \ \hat{c}_\gamma \hat{c}_\beta | \Psi_w \rangle}$$

Run mean field for each fictitious wave function. Average in the end.

Apply this procedure to SYK, obtain TWA





- TWA – a consistent way of implementing fluctuating meanfield approximation.
- Wave functions = Schwinger bosons for the basis operators.
- No issues with large entanglement = classical mutual information (checked numerically it works)
- No need to invent artificial large N-parameters, need to invent good phase space variables. N = dimensionality of the group.

Echo dynamics related to OTOC

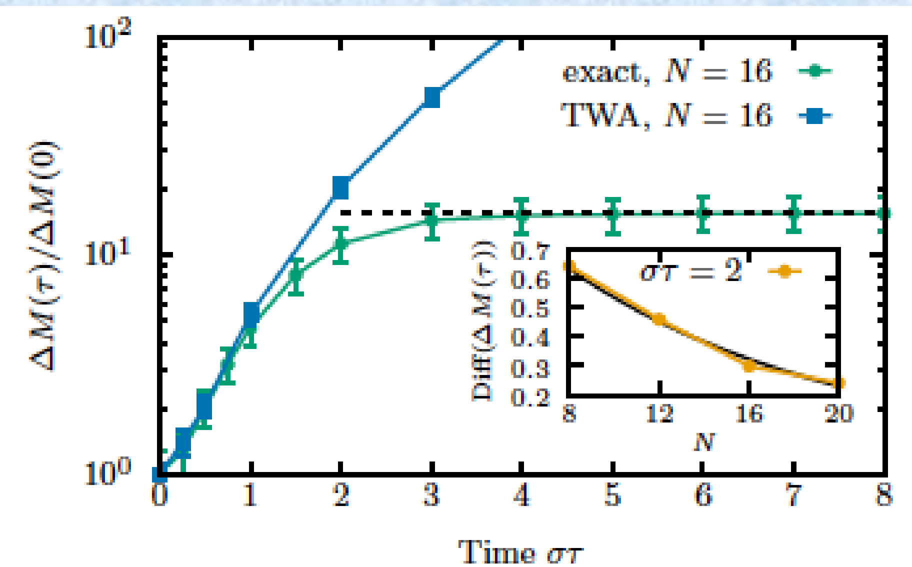
M. Schmitt, D. Sels, S. Kehrein, A.P. (2018).

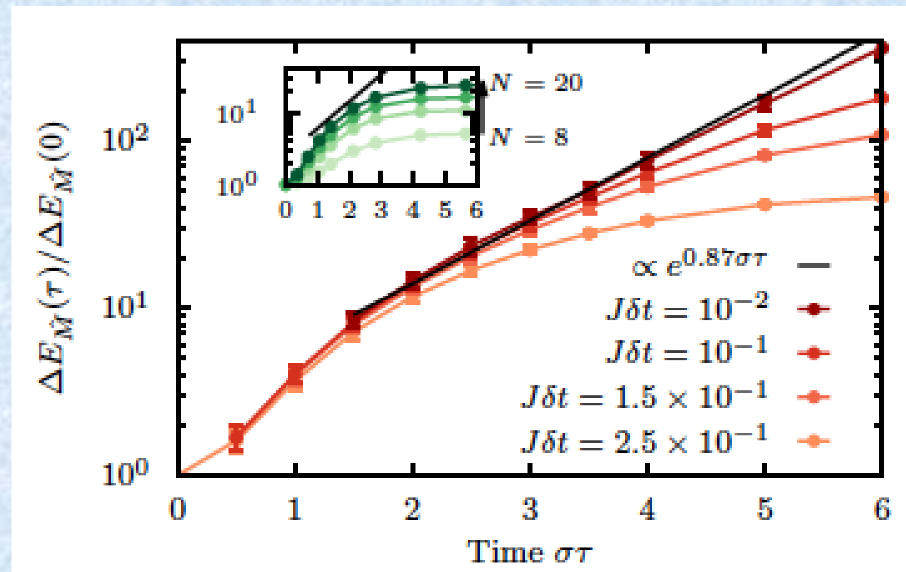
$$E_{\hat{O}}(\tau) = \langle\psi_0|\hat{U}_E^{\delta t}(\tau)^\dagger \hat{O} \hat{U}_E^{\delta t}(\tau)|\psi_0\rangle \quad \hat{U}_E^{\delta t}(\tau) = \mathrm{e}^{i\hat{H}t}\mathrm{e}^{-i\hat{H}_p\delta t}\mathrm{e}^{-i\hat{H}t}$$

$$\hat{O}|\psi_0\rangle = \mu|\psi_0\rangle \quad \Rightarrow \quad E_{\hat{O}}(\tau) = \mu + \frac{\delta t^2}{2}\langle\psi_0|[\hat{H}_p(\tau),[\hat{H}_p(\tau),\hat{O}]]|\psi_0\rangle + \mathcal{O}(\delta t^3)$$

Echo can be used to probe OTOC and scales as in classical systems

$E_{\hat{O}}(\tau) \sim \exp[2\lambda\tau]$ (also B. V. Fine et. al. Phys. Rev. E 89, 012923, 2014)



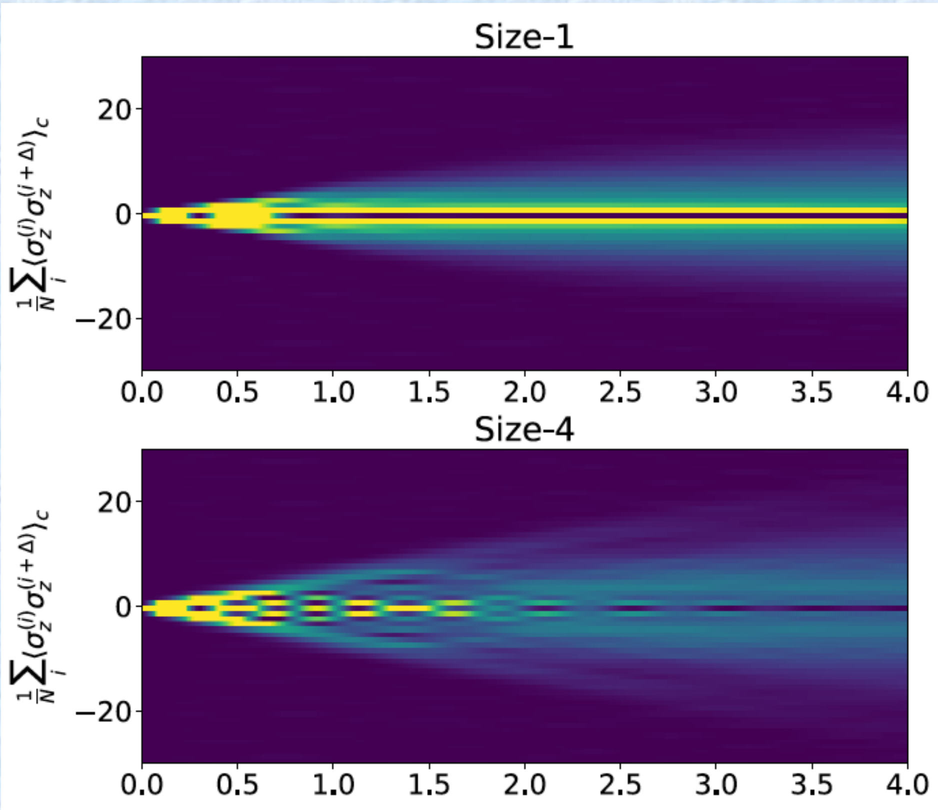


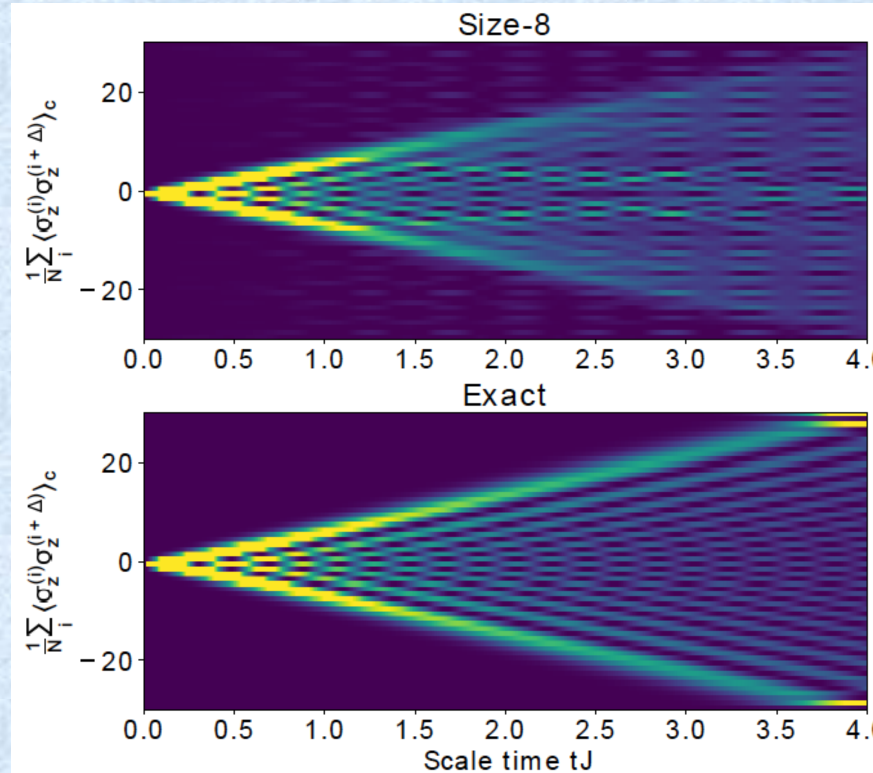
Accurate short-time echo growth

The (semi-classical) Lyapunov exponent agrees with a fully quantum infinite temperature predictions: B. Kobrin, C. Olund, D. Stanford, J. Moore, and N. Yao, Talk given at APS March Meeting 2018. Different results: T. Scaffidi, E. Altman, 2018

Non-local correlations: cluster vs. fermion TWA for XY chain

$$\hat{H} = \sum_{i=0}^{64} \hat{\sigma}_x^{(i)} \hat{\sigma}_x^{(i+1)} + \hat{\sigma}_y^{(i)} \hat{\sigma}_y^{(i+1)}.$$





Accuracy of TWA depends on the choice of basis operators= phase space variables!

Integrability is seeing as emerging asymptotically from CTWA with increasing cluster size.

Conclusions

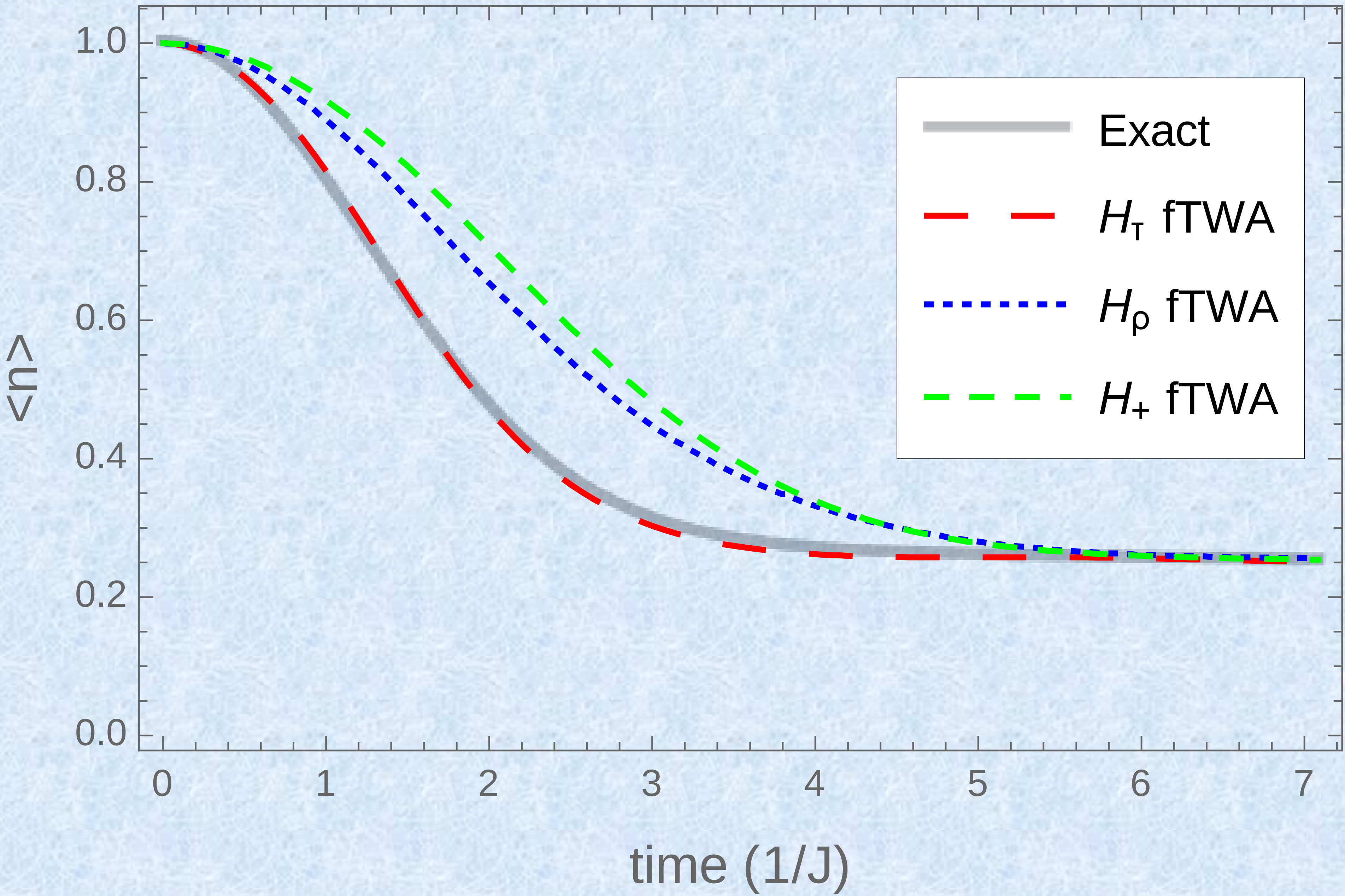
Can incorporate (short-distance) quantum fluctuations into TWA by adding more degrees of freedom.

CTWA - cluster degrees of freedom; fTWA – fermionic bilinears as degrees of freedom. In general need a closed set of commutation relations to define Poisson brackets..

TWA as a fluctuating mean field. Fluctuations in initial conditions are crucial for recovering non-equal time correlation functions and correct hydrodynamic behavior.

Can dramatically improve accuracy of TWA by using better degrees of freedom.

Comparison with the normal variable representation



Example: fermion expansion.

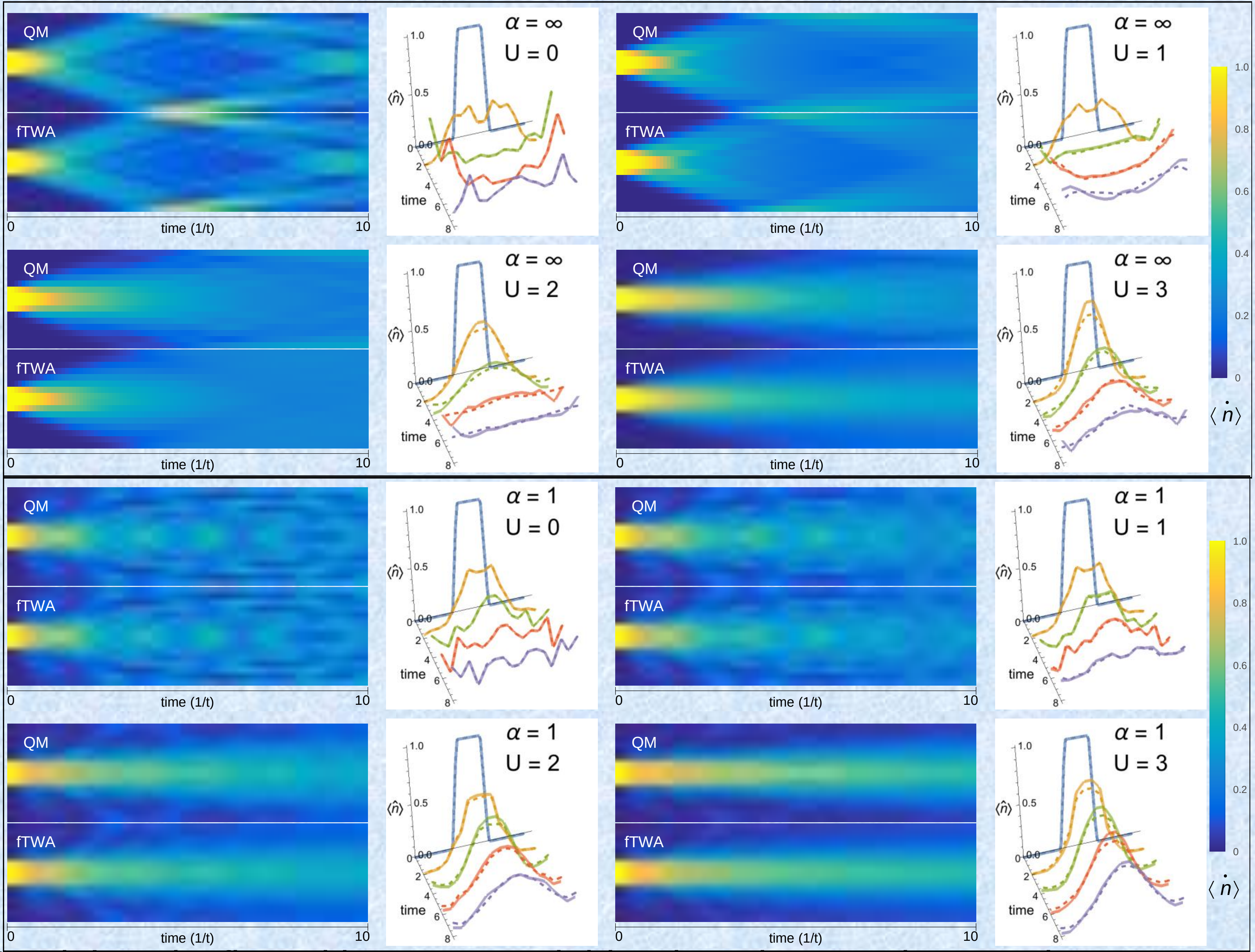
$$\hat{H}_{\mathrm{Hub}} = -\sum_{ij\sigma} J_{ij}\left(\hat{c}^{\dagger}_{i\sigma}\hat{c}_{j\sigma} + \mathrm{h.c}\right) + U\sum_{i}\hat{n}_{i\uparrow}\hat{n}_{i\downarrow}, \qquad J_{ij} = \frac{t}{|i-j|^{\alpha}}.$$

In 1D the model maps to XXZ chain with power law XY interactions

Use Bopp operators to get the Weyl symbol

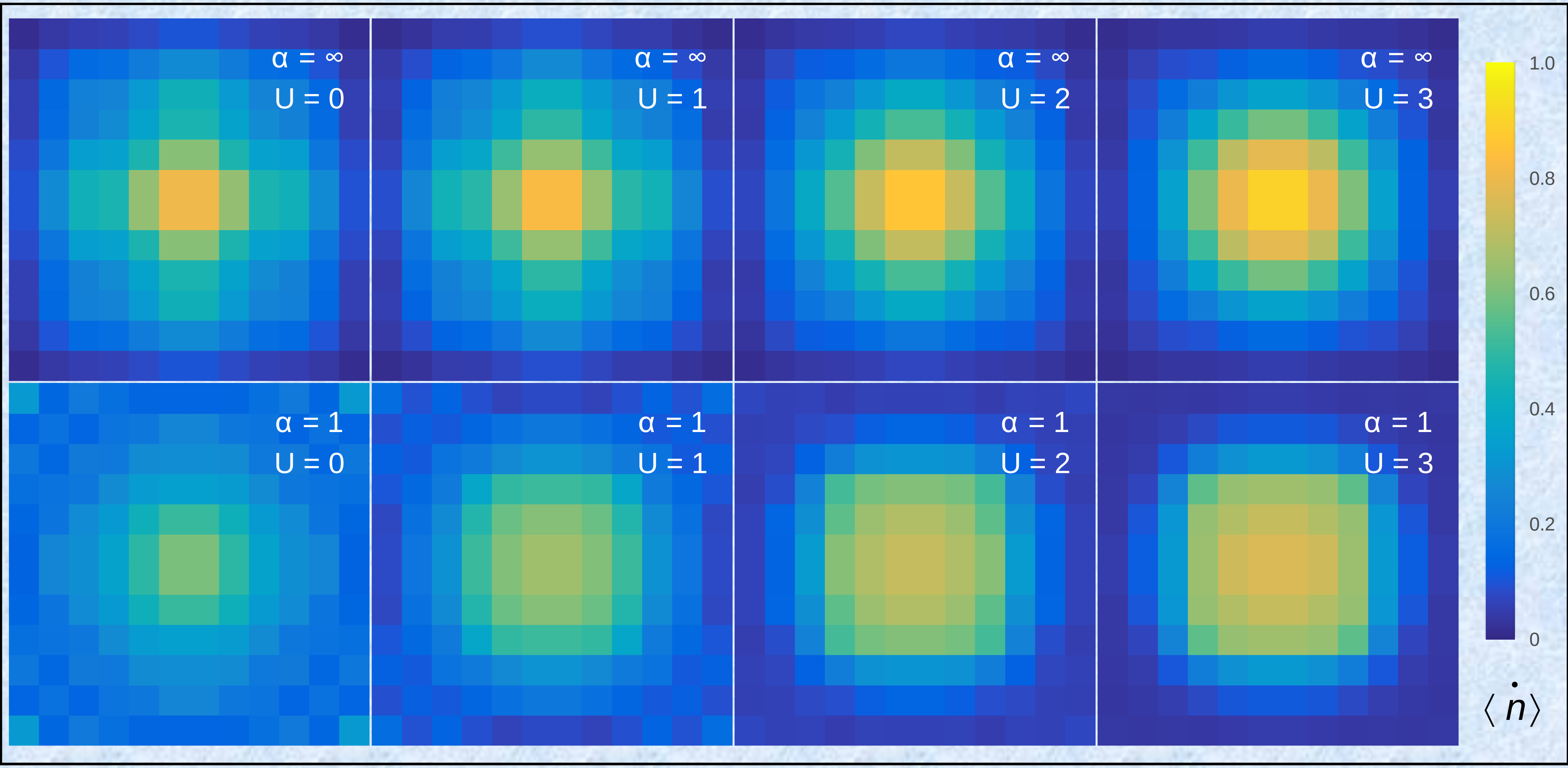
$$\mathcal{H}_{\mathrm{Hub}} = -\sum_{ij\sigma} J_{ij}\left(\rho_{i\sigma,j\sigma} + \mathrm{h.c}\right) + U\sum_{i}\left(\rho_{i\uparrow,i\uparrow} + 1/2\right)\left(\rho_{i\downarrow,i\downarrow} + 1/2\right).$$

Start from Gaussian initial conditions describing expectation values and fluctuations of the bilenears in the initial state.



Model works fine with nearest neighbor hopping. Works even better with long range hopping.

Fermion expansion in 2D



Schwinger boson TWA

Schwinger bosons: $\hat{X}_\alpha \to a_n^\dagger X_\alpha^{nm} a_m,\ \sum_n a_n^\dagger a_n = 1$

Need to solve D=2^N equations

$$i\dot{a}_n^{(i)} = \frac{\partial H}{\partial a_n^{*\,(i)}}$$

Can almost satisfy initial conditions with the Gaussian state. Works very well.

Reduction from D^2 operators to D Schwinger bosons is like reduction from the density matrix to the wave function.

Make a product ansatz $|\psi\rangle = \prod_j |\psi_j\rangle$

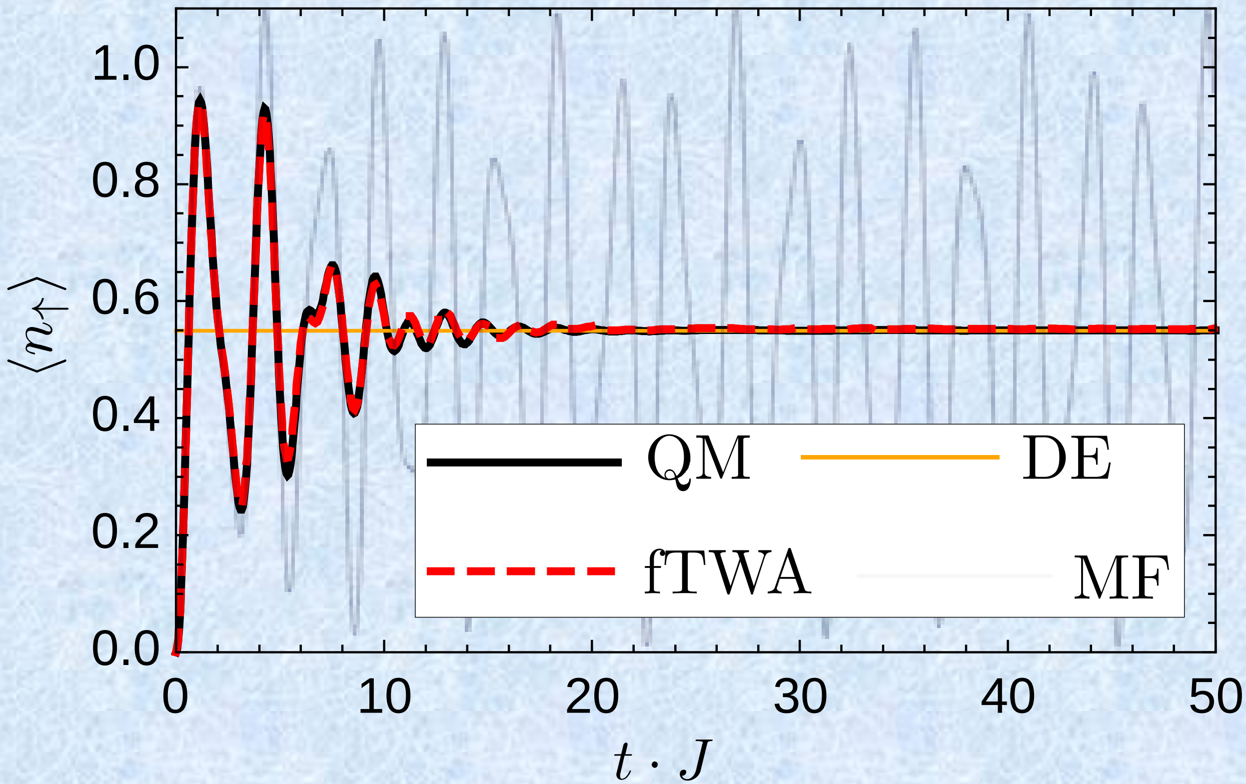
Dirac mean field equations $\langle\psi|i\partial_t|\psi\rangle = \langle\psi|H|\psi\rangle$ are identical to classical equations. TWA is like a statistical mixture of many mean fields. This does make a difference!

Application: two channel model (cartoon for gauge theories)

$$H = \mu_B \sum_j b_j^\dagger b_j - J \sum_{\sigma,\langle ij\rangle} \left(c_{\sigma i}^\dagger c_{\sigma j} + h.c.\right) + g \sum_j \left(b_j c_{\uparrow j}^\dagger c_{\downarrow j}^\dagger + h.c.\right)$$

Large positive (negative) μ_B – attractive (repulsive) Hubbard model

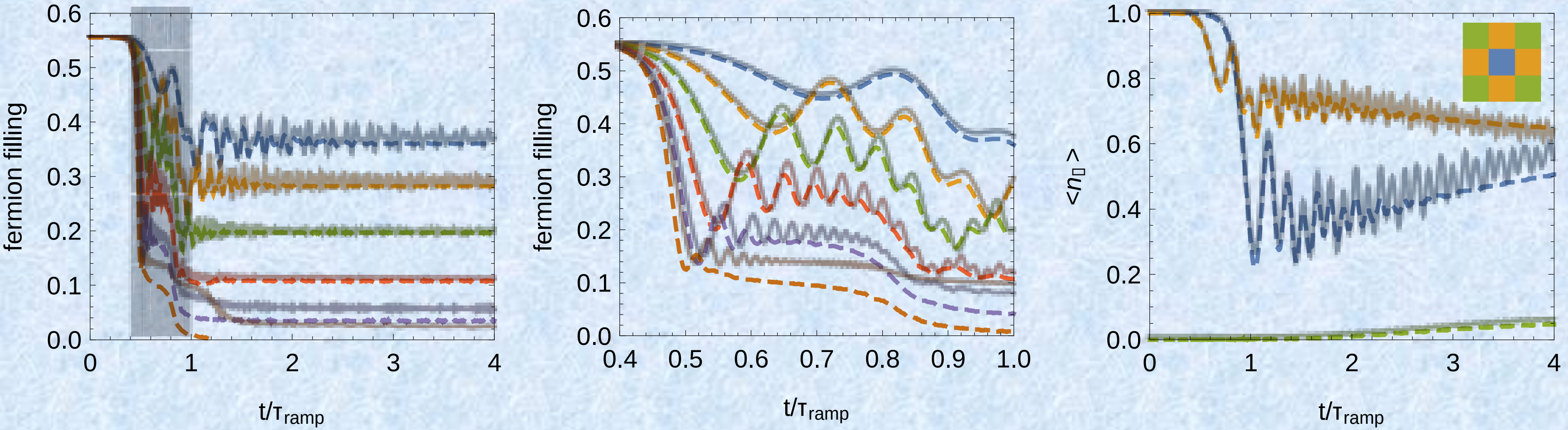
Two-site model, near mean-field regime. Fermion vacuum, coherent state for bosons with N=9 per site. Quench to $\mu_B = 1,\ g = 1/3$



- MF – only short times
- fTWA nearly exact including long time limit (but no revivals)
- Hilbert space is sufficiently large to thermalize.

Same model. Initially no bosons, half filling of fermions. No obvious small parameter 3x3 system.

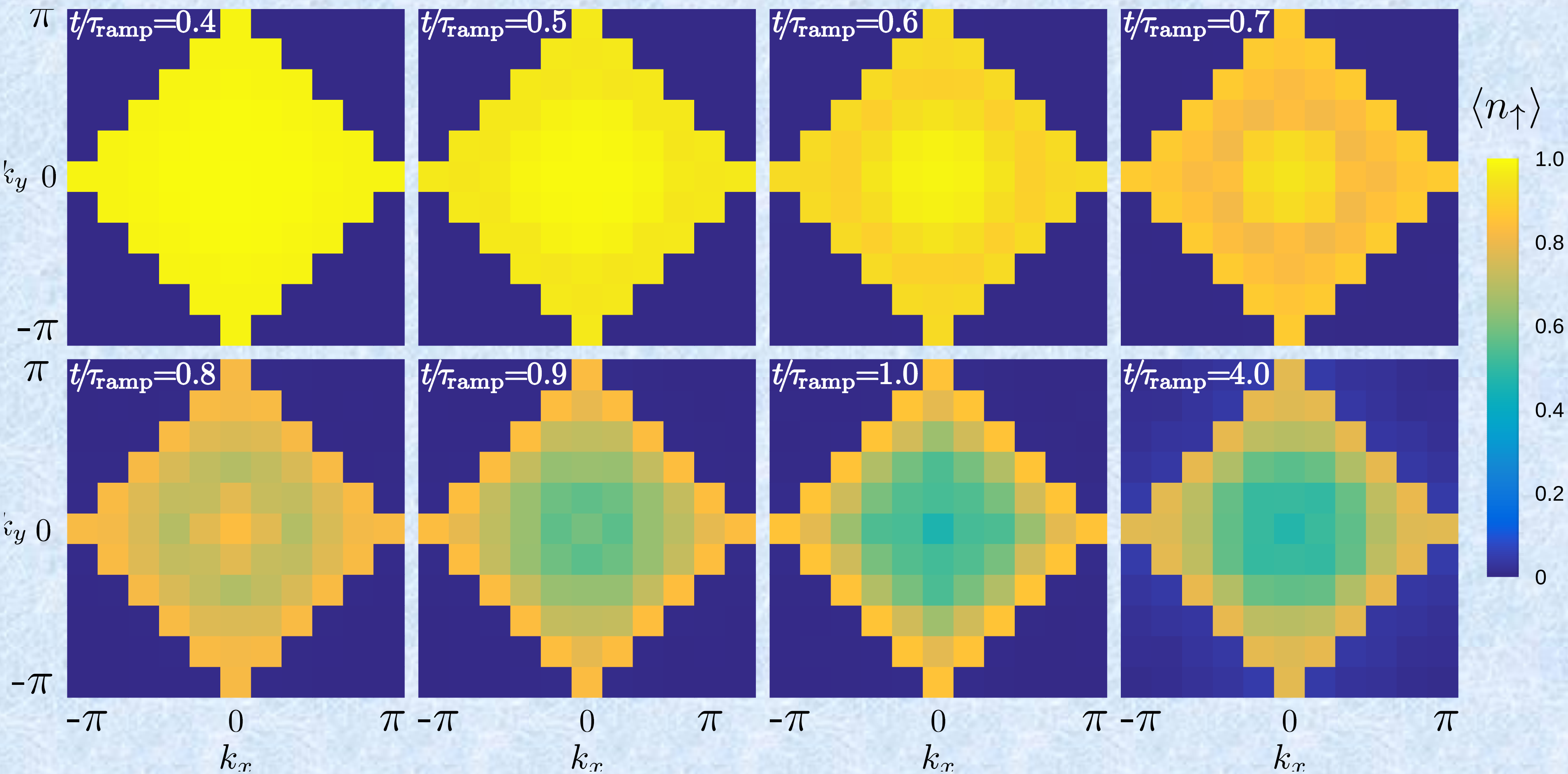
$$\mu_B(t) = -10(1 - e^{-(t/\tau_{\text{ramp}})^2}) \text{ and } g(t) = 1 - e^{-(t/\tau_{\text{ramp}})^2}$$



(a) (b) (c)

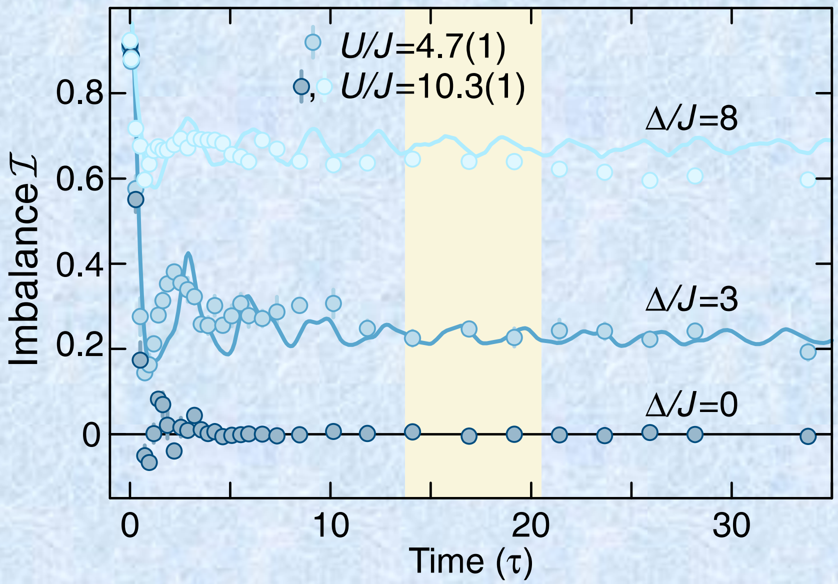
fTWA works very well except for very slow ramps. Can not predict correctly strongly-correlated GS. Works very well for short and intermediate time ramps.

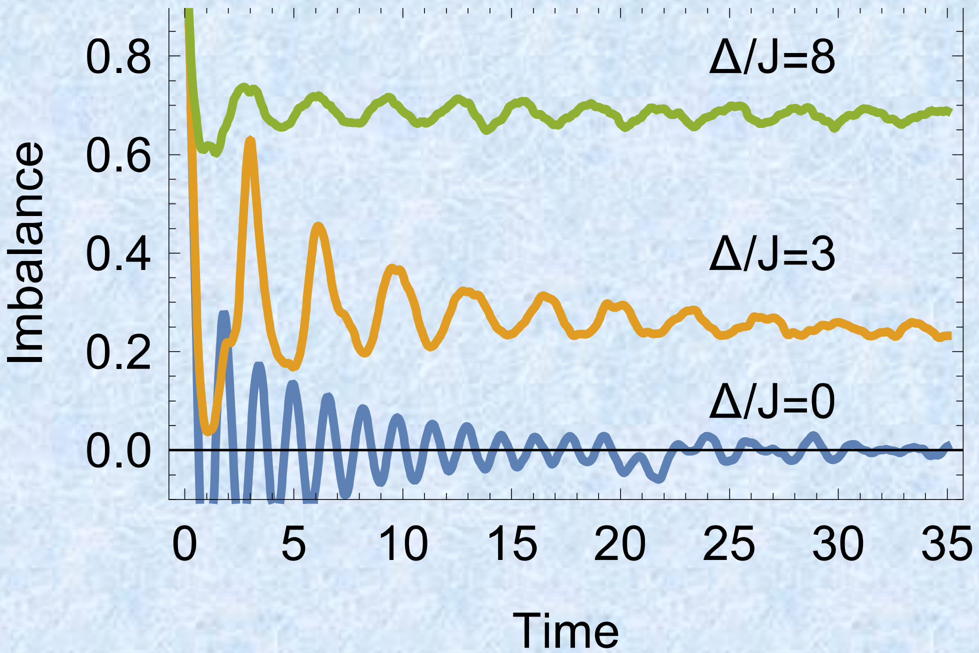
Same as in the previous slide but for 10x10 lattice



Emergence of a very unusual (ring-type) state of fermions.

Application to MBL experiment (M. Schreiber et. al.). Same parameters, same number of dublons. L=40



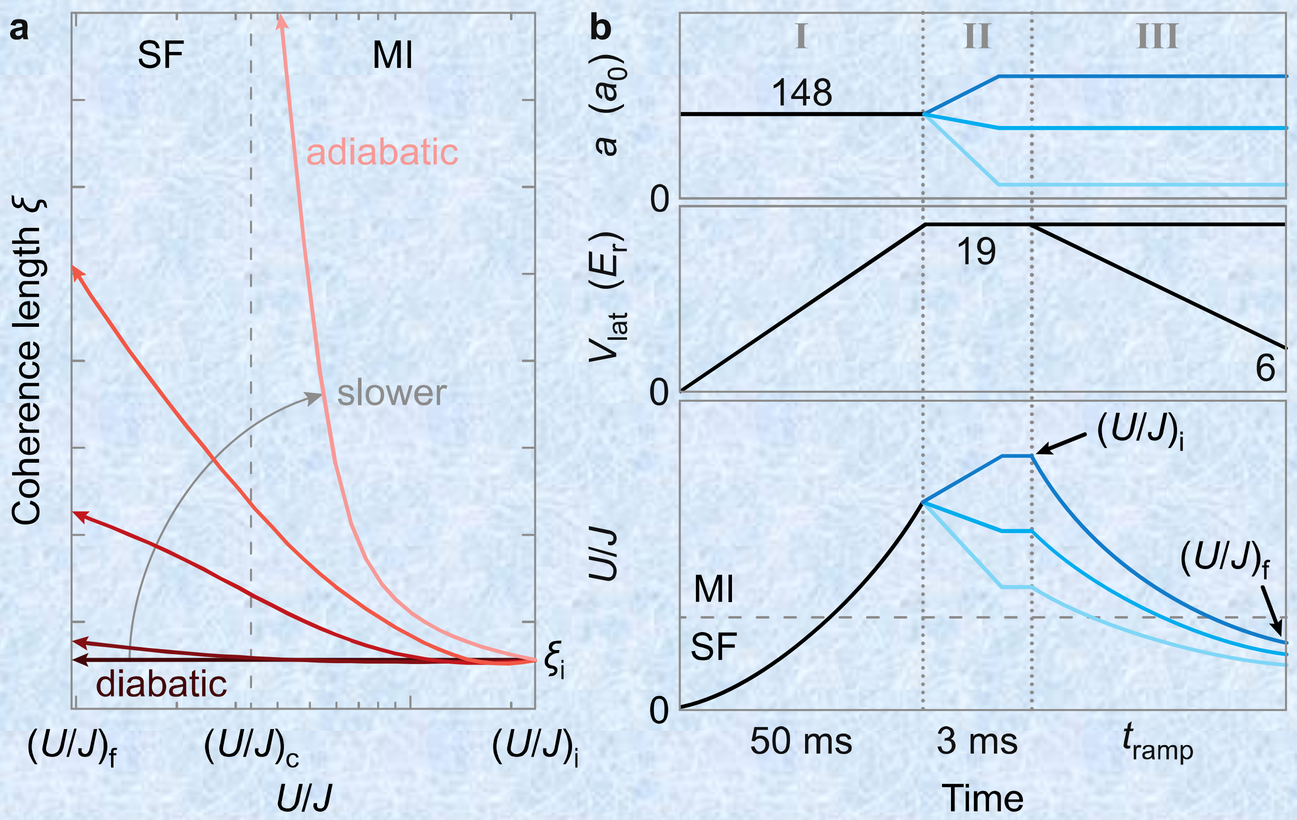


$$\hat{H} = -J \sum_{<ij>\sigma} \left(\hat{c}^{\dagger}_{i\sigma}\hat{c}_{j\sigma} + \text{h.c}\right) + \sum_i \mu_B \hat{b}^{\dagger}_i \hat{b}_i + g\sum_i \left(\hat{b}_i \hat{c}^{\dagger}_{\uparrow i}\hat{c}^{\dagger}_{\downarrow i} + \text{h.c.}\right) + \Delta \sum_{i\sigma} \cos\left(2\pi\beta i + \phi\right) \hat{c}^{\dagger}_{i\sigma}\hat{c}_{i\sigma}$$

$$g = 2,\ \mu_b = 10 \Rightarrow U \approx 0.4$$

fTWA works qualitatively well for at least intermediate times and better than CTWA. Long times – tendency to decay.

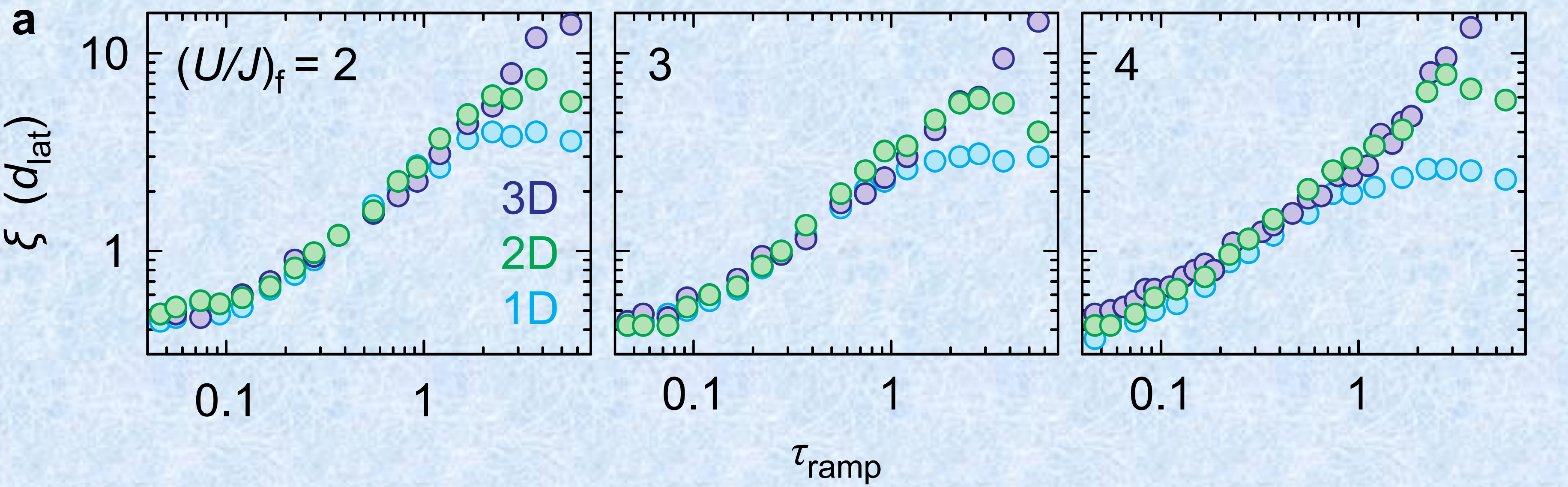
Slow Ramps from IN to SF

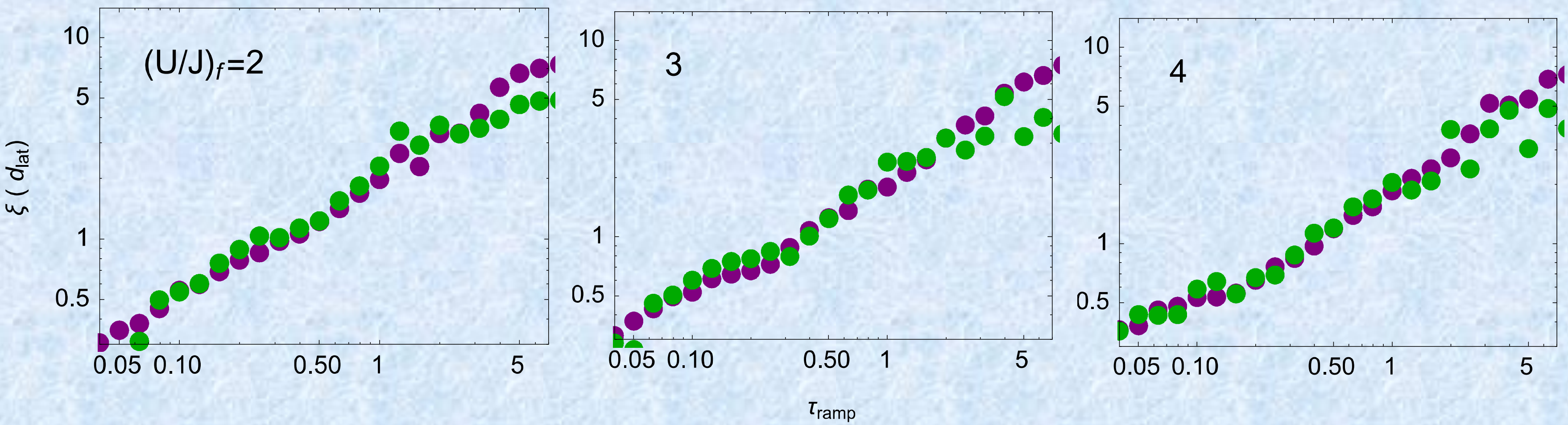


S. Braun, … I. Bloch, U. Schneider, J. Eisert, PNAS 2015

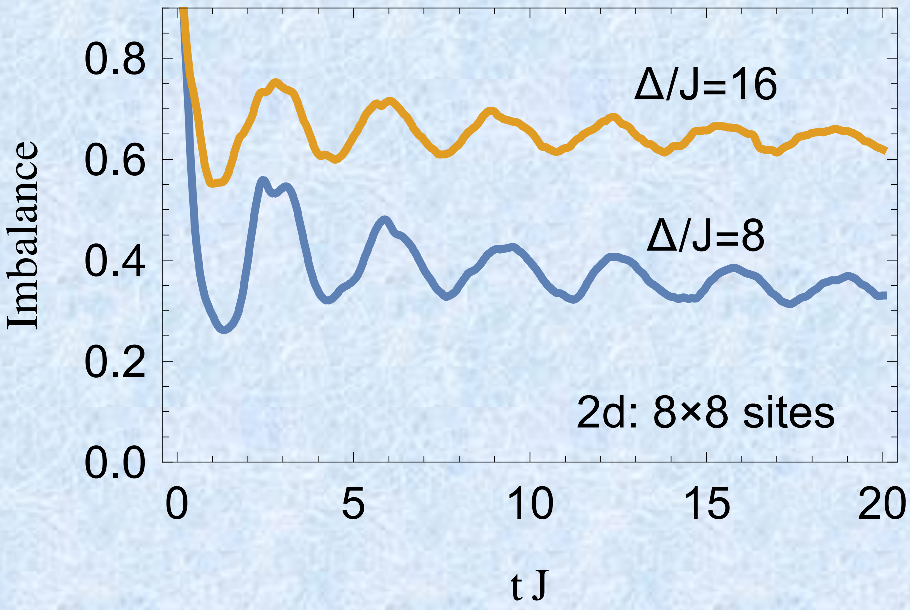
Check correlation length in the SF state as a function of ramp rate

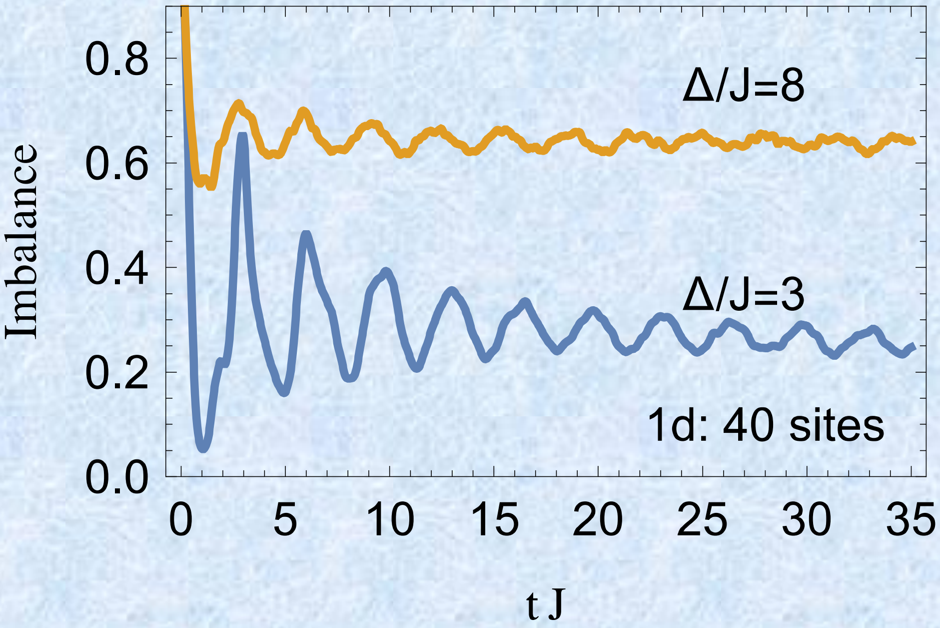
Experiment vs. SU(3) TWA





2D simulation (uncorrelated disorder), 8x8 lattice (quick run)





Reliable for the time scales shown.