GGE of 2d CFTs at large c in the thermodynamic limit

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Main results

- first two orders of $1/c$ expansion of the GGE free energy
  - ⋆ non-perturbative in the fugacities $\mu_3, \mu_5, \ldots$
  - ⋆ quantum corrections on gravity side

- mismatch between heavy primary states and the GGE
  - ⋆ breakdown of ergodicity in 2d CFTs

Motivation

- thermalization in 1D systems after a quantum quench
  Cardy, Calabrese
- CFT thermal physics $\Leftrightarrow$ black hole physics
- Eigenstate Thermalization Hypothesis in CFT
- infinite number of conserved charges Q_{2k-1}
  Bazhanov, Lukyanov, Zamolodchikov

Crash-course of 2d CFTs

- Virasoro algebra

  $T(z) = \sum_n \frac{L_n}{z^{n+2}}$, classically $L_n = z^{n+1}\partial_z$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}\delta_{n+m}(n^3 - n).$$

- lowest weight representation

  primary $|\Delta\rangle$, $\quad L_n|\Delta\rangle = 0,\ n > 0$

  descendant states $L_{-m_1} \ldots L_{-m_k}|\Delta\rangle$

- qKdV charges

$$H = Q_1 = \int_0^\ell du\, T(u), \quad Q_3 = \int_0^\ell du\, T(u)^2, \quad \ldots$$

ETH in CFT

- quantization of CFT on a cylinder $\mathbb{S}^1 \times R$
- operator-states correspondence
- $\langle \Delta | \mathcal{O} | \Delta \rangle$ is algebraically related to heavy-heavy-light OPE coefficient
- thermodynamic limit

$$\Delta/\ell^2 - \text{fixed}, \quad \ell \to \infty$$

- Eigenstate Thermalization Hypothesis in CFT

$$\langle \Delta | \mathcal{O}_\delta | \Delta \rangle = a_{\mathcal{O}} \left(\Delta/\ell^2 \right)^{\delta/2}$$

Lashkari, AD, Liu

ETH in CFT

- ETH is inconsistent if applied to all energy eigenstates

$$\begin{aligned} \langle\Delta|\mathcal{O}|\Delta\rangle &= a_{\mathcal{O}}\,\Delta^{\delta/2}/\ell^{\delta} \\ \langle\Delta+n|\mathcal{O}|\Delta+n\rangle &\neq a_{\mathcal{O}}\,(\Delta+n)^{\delta/2}\ell^{\delta} \end{aligned}$$

- for $\mathcal{O}$ from the vacuum conformal family ETH is automatic
- a priory eigenstate expectation value is not related to thermal
- this talk: mismatch at finite c for $\mathcal{O}$ from the vacuum family

$$\langle\Delta|\mathcal{O}|\Delta\rangle \neq \mathrm{Tr}(\mathcal{O}\,e^{-\beta H-\mu_3 Q_3-\dots})/Z$$

Q-charges of a primary state

- Q-charges of *primary* state $|E\rangle$ are completely determined by its dimension E
- only highest power of $T(u)$ contributes to expectation value in the thermodynamic limit

$$\langle E|q_{2k-1}|E\rangle = \ell^{-1}\langle E|Q_{2k-1}|E\rangle = \frac{E^k}{\ell^{2k}}$$

- matching with the GGE

$$-\ell^{-1}\partial_{\mu_{2k-1}} \log Z = (-\ell^{-1}\partial_\beta \log Z)^k$$

- matching of Q's is necessary and sufficient for matching of all $\mathcal{O}$ from the vacuum family

Gibbs free energy

$$Z = \mathrm{Tr}\, e^{-\beta H}, \quad H = \frac{L_0 - c/24}{\ell}$$

- constant in H does not contribute in the thermodynamic limit
- L_0 is degenerate, $L_0|\Delta + n\rangle = (\Delta + n)|\Delta + n\rangle$
- density of primaries is given by Cardy formula

$$Z = \sum_\Delta \sum_n P(n) e^{-\frac{\beta}{l}(\Delta+n)} =$$

$$\int d\Delta\, dn\, e^{\pi\sqrt{\frac{2(c-1)\Delta}{3}} + \pi\sqrt{\frac{2n}{3}} - \frac{\beta}{\ell}(\Delta+n)} =$$

$$\int dE\, e^{\pi\sqrt{\frac{2cE}{3}} - \frac{\beta}{\ell}E} = e^{\frac{c\pi^2\ell}{6\beta^2}}, \qquad E = \frac{c\pi^2\ell^2}{6\beta^2}$$

GGE

$$e^F \equiv Z = \mathrm{Tr}\, e^{-\beta H - \mu_3 Q_3 - \mu_5 Q_5 - \dots}$$

- free energy depends on μ_{2k-1} only in combination

$$t_{2k-1} = \left(\frac{c\,\pi^2}{6\beta^2}\right)^{k-1} \frac{\mu_{2k-1}}{\beta}$$

- free energy is a polynomial in μ_{2k-1}, c

$$\begin{aligned} F &= \frac{c\pi^2\ell^2}{6\beta^2}\left(f_0 + \frac{f_1}{c} + \frac{f_2}{c^2} + \dots\right) \\ f_0 &= 1 - t_3 + 4t_3^2 + \dots \\ f_1 &= -\frac{22}{5}t_3 + \dots \end{aligned}$$

Structure of Q_{2k-1}

$$\ell^3 Q_3 = L_0^2 - \frac{c+2}{12}L_0 + \frac{c(5c+22)}{2880} + 2\sum_{n=1}^{\infty} L_{-n}L_n$$

$$\ell^5 Q_5 = L_0^3 - \frac{c+4}{8}L_0^2 + \frac{(c+2)(3c+20)}{576}L_0 - \frac{c(3c+14)(7c+68)}{290304} +$$

$$\sum : L_{n_1}L_{n_2}L_{n_3} : + \sum_{n=1}^{\infty}\left(\frac{(c+11)}{6}n^2 - \frac{4+c}{4}\right)L_{-n}L_n + \frac{3}{2}\sum_{r=1}^{+\infty} L_{1-2r}L_{2r-1}$$

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- only L_0^k terms contributes extensively in the $c \to \infty$ limit

$$\int dE \, e^{\pi \sqrt{\frac{2cE}{3}} - \frac{\beta}{\ell} E - \frac{\mu_3}{\ell^3} E^2 - \frac{\mu_5}{\ell^5} E^3 - \dots} = e^{\frac{c\pi^2 \ell}{6\beta^2}} f_0$$

GGE at infinite c

- saddle point approximation is exact

$$
\begin{aligned}
s_0 &= 2\sqrt{e} - e - t_3 e^2 - t_5 e^3 - \dots \\
f_0 &= s_0(e), \qquad \left.\frac{\partial s_0}{\partial e}\right|_e = 0
\end{aligned}
$$

- GGE e.v. exactly match primary ones for any μ_{2k-1}

$$E^* = \frac{c\pi^2 \ell}{6\beta} e, \quad \Delta^* = \frac{(c-1)\pi^2 \ell}{6\beta} e, \quad n^* = \frac{\pi^2 \ell}{6\beta} e$$

$$\ell^{-1} \langle Q_{2k-1} \rangle_{\mathrm{GGE}} = (E^*/\ell^2)^k$$

$$(f_0 + 3t_3 \partial_{t_3} f_0 + 5t_5 \partial_{t_5} f_0 + \dots)^k + \partial_{t_{2k-1}} f_0 = 0$$

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- all qKdV charges split into two

$$Q_{2k-1} = \hat{Q}_{2k-1}(L_0) + \tilde{Q}_{2k-1}$$

- $\tilde{Q}_{2k-1}$ acting on $|\Delta + n\rangle$ is a polynomial in c, Δ

$\tilde{Q}$ restricted to $|\Delta + n\rangle$

- $\tilde{Q}_3$ is not more than linear in c

$$\tilde{Q}_3 = c\,\tilde{Q}_3^c + \Delta\,\tilde{Q}_3^\Delta + \tilde{Q}_3^{(0)} = \frac{2}{\ell^3}\sum_{n=1}^{\infty} L_{-n}L_n$$

- $\tilde{Q}_5$ is not more than quadratic in c

$$\tilde{Q}_5 = c^2\tilde{Q}_5^{cc} + c\,\Delta\tilde{Q}_5^{c\Delta} + \Delta^2\tilde{Q}_5^{\Delta\Delta} + c\,\tilde{Q}_5^c + \Delta\tilde{Q}_5^\Delta + \tilde{Q}_5^{(0)}$$

- all leading order in c matrices are lower-triangular!

$$\ell^3\tilde{Q}_3^c|m_i,\Delta\rangle = \lambda|m_i,\Delta\rangle + \ldots, \qquad \lambda = \frac{1}{6}\left(\sum_i m_i^3 - m_i\right)$$

$$\ell^3\tilde{Q}_3^\Delta|m_i,\Delta\rangle = \nu|m_i,\Delta\rangle + \ldots, \qquad \nu = 4\,n$$

Only μ_3 turned on

- expectation value of $\tilde{Q}_3$

$$
\begin{aligned}
\ell^3 \langle \tilde{Q}_3 \rangle_{\Delta,n} &= \frac{1}{P(n)} \sum_{\{m_i\}=n} \frac{c}{6} \left(\sum_i m_i^3 - m_i \right) + 4\Delta n + O(c^0) \\
&= \frac{2c}{5} n^2 + 4\Delta n + 4n^2 + O(\ell^3)
\end{aligned}
$$

- Δ^* is fixed by leading (infinite) order in c, $\Delta^* = \frac{c\pi^2 \ell}{6\beta} e$
- n^* is determined by summing over Young tableaux

$$
e^{\frac{\pi^2 \ell}{6\beta}} f_1 = e^{-\frac{\pi}{2}\sqrt{\frac{\Delta^*}{6c}}} \sum_{\{m_i\}} e^{-\frac{\beta}{\ell} n - 6\frac{\mu_3}{\ell^3} n\Delta^* - \frac{\mu_3}{\ell^3}\frac{c}{6}\sum_i m_i^3}
$$

Summing over Young tableaux

- free boson representation, r_k

$$5 = 1 + 1 + 3$$

$$m_1 = 1, m_2 = 1, m_3 = 3, \qquad r_1 = 2, r_2 = 0, r_3 = 1, r_4 = 0, \ldots$$

$$n = \sum_i m_i = \sum_k r_k\, k, \qquad \sum_i m_i^3 = \sum_k r_k\, k^3$$

- free boson partition functions

$$Z = \sum_{\{m_i\}} e^{-\frac{x}{\ell} n - \frac{y}{\ell^3} \sum_i m_i^3} = \prod_k \sum_{r_k} e^{-\left(\frac{x}{\ell} k + \frac{y}{\ell^3} k^3\right) r_k} =$$

$$= \exp\left\{ - \sum_k \log\left(1 - e^{-\frac{x}{\ell} k - \frac{y}{\ell^3} k^3}\right)\right\}$$

Nonperturbative f_1 as a function μ_3

- exact answer in the thermodynamic limit

$$f_1 = -\sqrt{e} - \frac{6}{\pi^2} \int_0^\infty dk \log\left(1 - e^{-(1+6t_3 e)k - t_3 k^3/\pi^2}\right)$$

- knowing 1pt function $\langle Q_3 \rangle_{\Delta,n}$ or equivalently $\mathrm{Tr}\left(q^{L_0} Q_3\right)_\Delta$ over a particular Verma module fixes full non-perturbative answer

Turning on μ_5

- eigenvalues of $\ell^3 \tilde{Q}_5^{cc} = \frac{1}{12}\left(\sum_i \frac{m_i^5}{6} - \frac{5m_i^3}{12} - \frac{m_i}{4}\right)$
- eigenvalues of $\ell^3 \tilde{Q}_5^{c\Delta} = \sum_i \frac{5}{6} m_i^3 - m_i$
- eigenvalues of $\ell^3 \tilde{Q}_5^{\Delta\Delta} = 12\, n$

- non-perturbative answer for $f_1(t_3, t_5)$

$$
\begin{aligned}
f_1 &= -\sqrt{e} - \\
&\quad \frac{6}{\pi^2} \int_0^\infty dk \log\left(1 - e^{-(1+6t_3 e+15 t_5 e^2)k-(t_3+5t_5 e)k^3/\pi^2 - \frac{1}{2} t_5 k^5/\pi^4}\right)
\end{aligned}
$$

Mismatch of GGE and primary state

- discrepancy between GGE and primary eigenstate is given by the e.v. of $\tilde{Q}$

$$\ell^{-1}\left(\langle Q_3\rangle_{\mathrm{GGE}} - \langle E|Q_3|E\rangle\right) = \ell^{-1}\langle\tilde{Q}_3\rangle_{\mathrm{GGE}}$$

- at leading order in c, $\tilde{Q}_3$ can be substituted by its lower-triangular part and GGE average can be substituted by an average over Young tableuax in a sector with fixed Δ^*, n^*

$$\tilde{Z}\ell^3\langle\tilde{Q}_3\rangle_{\mathrm{GGE}} = \sum_{\{m_i\}=n^*}\left(\frac{c}{6}\sum m_i^3 + 4\Delta^* n^*\right) e^{-\frac{\mu_3}{\ell^3}\left(\frac{c}{6}\sum_i m_i^3 + 4\Delta^* n^*\right) - \frac{\mu_5}{\ell^5}\left(\frac{c^2}{72}\sum_i m_i^5 + \dots\right)} + O(c^0)$$

Mismatch of GGE and primary state

- e.v. of $\tilde{Q}_3$ is strictly positive unless $n^* = 0$

$$\ell^{-1}\langle\tilde{Q}_3\rangle_{\mathrm{GGE}} = \frac{a(t_{2k-1})c\,(n^*)^2 + 4\Delta^* n^*}{\ell^4} + O(c^0) > 0$$

- the mismatch between primaries and GGE has to vanish if $n^* = 0$, i.e. if primaries dominate the GGE; we have shown it vanishes if *and only if*
- effective descendant level $\mathrm{n} = n^*/\ell^2$ can't vanish because of $P(n)$

$$L = \pi\sqrt{\frac{2\mathrm{n}}{3}} - \left(\mathrm{n}\left(\beta + 6\mu_3\Delta^*/\ell^2 + \dots\right) + \mathrm{n}^2\left(2c\mu_3/5 + \dots\right) + \dots\right)$$

Conclusions

- In the thermodynamic GGE simplifies. First $1/c$ correction to free energy reduces to a sum over Young tableaux, which can be calculated explicitly

  we hope to extend our results to all qKdV charges to obtain $f_1(t_3, t_5, t_7, \dots)$

- Beyond infinite c (primary) eigenstate does not match GGE no matter the choice of qKdV fugacities. This signals breakdown of ergodicity in 2d CFTs: there are initial configurations that are not described by GGE upon equilibration