Remarks on Complex Sachdev Ye Kitaev model

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Order from Chaos

KITP, Santa Barbara
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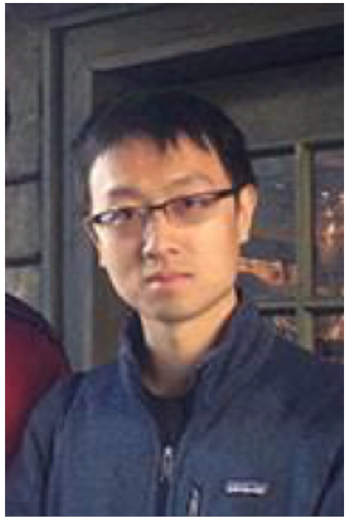


Yingfei Gu
Harvard

Subir Sachdev
Harvard

Talk based on work in progress with
Yingfei Gu and Subir Sachdev

Plan

- Complex SYK model
- Two-point function
- Four-point function
- Correction to h=1 mode of the four-point function

Complex SYK model

- Hamiltonian $$H = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,l=1}^{N} J_{ij;kl} c_i^\dagger c_j^\dagger c_k c_l - \mu \sum_i c_i^\dagger c_i$$

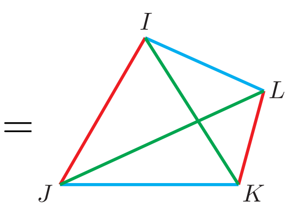
S. Sachdev '15

$$\{c_i^\dagger, c_j\} = \delta_{ij} \qquad \langle |J_{ij,kl}|^2 \rangle = J^2 \qquad J_{ij,kl} = J^*_{kl,ij}$$

R.Davison, W.Fu, A.Georges, Y.Gu, K.Jensen, S.Sachdev '17

- Complex SYK model has the Tensor model counterpart

$$H = \frac{1}{4N^{3/2}} \sum_{I,J,K,L} J_{IJKL} \bar{\psi}_I \bar{\psi}_J \psi_K \psi_L - \mu \sum_I \bar{\psi}_I \psi_I \qquad J_{IJKL} =$$



I. Klebanov, GT'16

- Complex SYK q model

$$H = \sum_{i_1,\dots i_q} J_{i_1 i_2 \dots i_q} c_{i_1}^\dagger c_{i_2}^\dagger \dots c_{i_{q/2}}^\dagger c_{i_{q/2+1}} \dots c_{i_{q-1}} c_{i_q}$$

$$\langle |J_{i_1 i_2 \dots i_q}|^2 \rangle = \frac{J^2 [(q/2)!]^2}{(q/2) N^{q-1}}$$

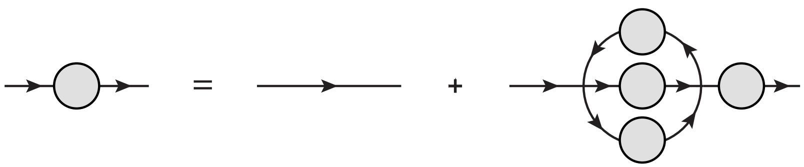
Complex SYK model

- Replica trick $\log Z = \lim_{n\to 0}\frac{1}{n}(Z^n - 1)$ and average over disorder give $\Sigma\, G$ effective action

$$-\frac{\beta F}{N} = \log\det(\partial_\tau - \mu - \Sigma) - \int d\tau_1 d\tau_2 \Big(\Sigma(\tau_1,\tau_2)G(\tau_1,\tau_2) - \frac{J^2}{q}G(\tau_1,\tau_2)^q\Big)$$

- One obtains Schwinger-Dyson equations

$$G(i\omega_n) = \frac{1}{i\omega_n - \mu - \Sigma(i\omega_n)}$$



$$\Sigma(\tau) = -(-1)^{q/2}J^2 G(\tau)^{q/2}G(-\tau)^{q/2-1}$$

- Here we set the chemical potential to zero $\mu = 0$

$$G(\tau) = -G(-\tau)$$

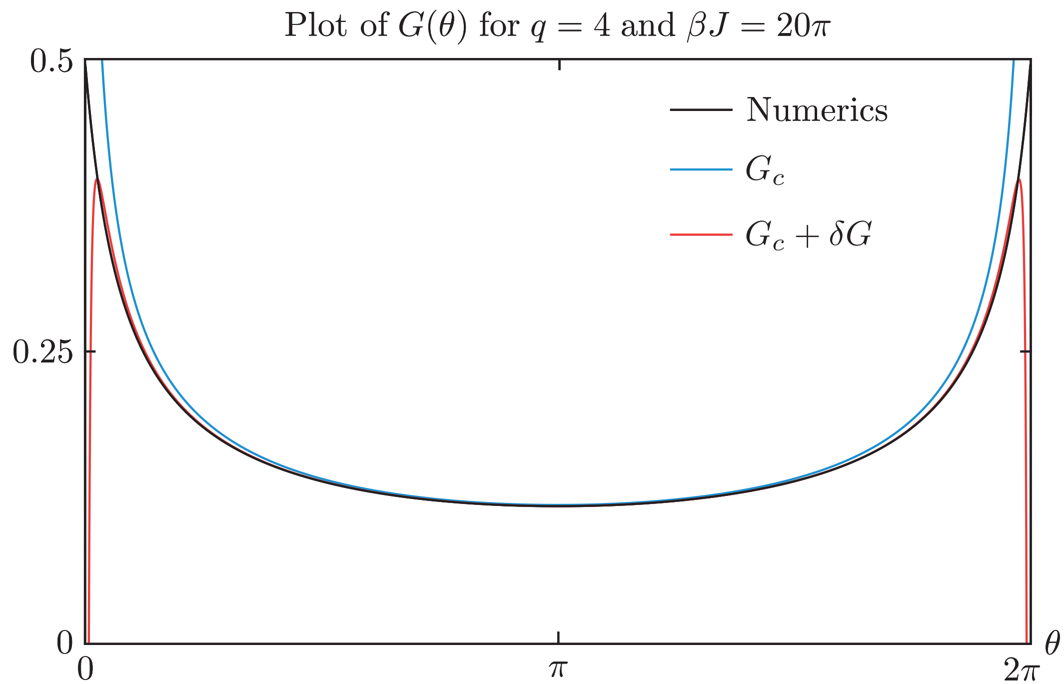
so we arrive at the usual S-D equations

$$G(i\omega_n) = \frac{1}{i\omega_n - \Sigma(i\omega_n)} \qquad \Sigma(\tau) = J^2 G(\tau)^{q-1}$$

S.Sachdev, J.Ye' 93
A.Georges, O.Parcollet, S.Sachdev '01
A.Kitaev '15
J.Polchinski, V.Rosenhaus '16
J.Maldacena, D.Stanford '16
A.Jevicki, K.Suzuki, J.Yoon '16

Two-point function

- In the case of zero chemical potential two-point function is as in SYK model



We have a UV cutoff of order $1/\beta J$

- Leading approx: two-point function of operators with dimension $\Delta = \frac{1}{q}$

$$G_c(\theta_{12}) = \langle \psi_\Delta(\theta_1)\psi_\Delta(\theta_2)\rangle = b\frac{\mathrm{sgn}(\theta_{12})}{|2\sin\frac{\theta_{12}}{2}|^{2\Delta}} \qquad b^q = \frac{1}{\pi J^2}\left(\frac{1}{2}-\frac{1}{q}\right)\tan\left(\frac{\pi}{q}\right) \qquad \mathcal{J} = \sqrt{q}\frac{J}{2^{\frac{q-1}{2}}}$$

- Correction: conformal perturbation theory by operator of dimension $h = -1$

$$\delta G(\theta_{12}) = \frac{\alpha_G}{\beta\mathcal{J}}\int_0^{2\pi} d\theta_0 \langle \psi_\Delta(\theta_1)\psi_\Delta(\theta_2)O_{h=-1}(\theta_0)\rangle = -\frac{\alpha_G}{\beta\mathcal{J}}G_c(\theta_{12})\left(2+\frac{\pi-|\theta_{12}|}{\tan\frac{|\theta_{12}|}{2}}\right)$$

$$\alpha_G|_{q=4} \approx 0.1878$$

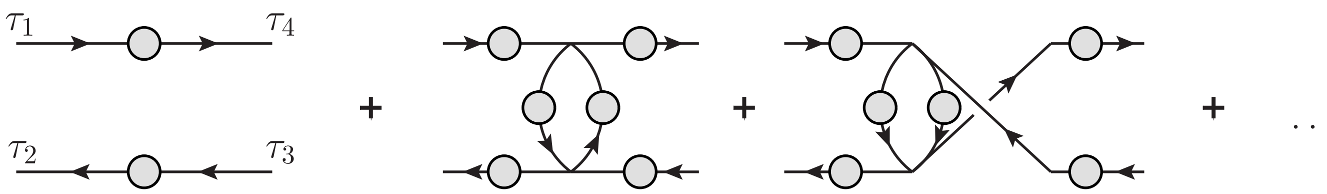
J. Maldacena, D. Stanford' 16
A. Kitaev, J.Suh '17

Four-point function

- 1/N term in the four-point function is given by ladder diagrams

$$\frac{1}{N^2}\sum_{i,j=1}^{N}\langle T(c_i^\dagger(\tau_1)c_i(\tau_2)c_j^\dagger(\tau_3)c_j(\tau_4))\rangle = G(\tau_{12})G(\tau_{34})+\frac{1}{N}\mathcal{F}(\tau_1,\tau_2;\tau_3,\tau_4)+...$$

$$\mathcal{F}=\sum_n \mathcal{F}_n$$



$\mathcal{F}_0$ $\mathcal{F}_1$

- Each new rung in the ladder is generated by Kernel

$$K(\tau_1,\tau_2;\tau_3,\tau_4) = -J^2\Big(\frac{q}{2}G(\tau_{13})G(\tau_{24})G(\tau_{34})^{q-2}-(\frac{q}{2}-1)G(\tau_{14})G(\tau_{23})G(\tau_{34})^{q-2}\Big)$$

- We can decompose the four-point function on symmetric and antisymmetric parts

$$\mathcal{F}(\tau_1,\tau_2;\tau_3,\tau_4) = \mathcal{F}^A(\tau_1,\tau_2;\tau_3,\tau_4)+\mathcal{F}^S(\tau_1,\tau_2;\tau_3,\tau_4) \qquad \mathcal{F}^{A,S}(\tau_1,\tau_2;\tau_3,\tau_4) = \mp\mathcal{F}^{A,S}(\tau_2,\tau_1;\tau_3,\tau_4)$$

$$\mathcal{F} = \frac{1}{1-K^S}\mathcal{F}_0^S + \frac{1}{1-K^A}\mathcal{F}_0^A$$

$$K^S = -J^2 G(\tau_{13})G(\tau_{24})G(\tau_{34})^{q-2} \qquad K^A = -J^2(q-1)G(\tau_{13})G(\tau_{24})G(\tau_{34})^{q-2}$$

$$\mathcal{F}_0^{A,S} = \frac{1}{2}(G(\tau_{14})G(\tau_{23})\mp G(\tau_{24})G(\tau_{13}))$$

Spectrum of Conformal Kernels

- Kernels act on symmetric and antisymmetric functions correspondingly

$$K^S = -J^2 G(\tau_{13}) G(\tau_{24}) G(\tau_{34})^{q-2} \qquad K^A = -J^2 (q-1) G(\tau_{13}) G(\tau_{24}) G(\tau_{34})^{q-2}$$

- There is a Casimir operator, which commutes with conformal kernels

$$[K_c^S, C] = [K_c^A, C] = 0 \qquad G(\theta) \to G_c(\theta) = \langle \psi_\Delta(\theta) \psi_\Delta(0) \rangle$$

- Eigenfunctions of the Casimir

$$\Psi_h^A(\tau_0, \tau_1, \tau_2) = \frac{\mathrm{sgn}(\tau_{12})}{|\tau_{10}|^h |\tau_{20}|^h |\tau_{12}|^{2\Delta - h}} \qquad h = 2, 4, 6, \ldots, \quad h = \frac{1}{2} + is$$

$$\Psi_h^S(\tau_0, \tau_1, \tau_2) = \frac{\mathrm{sgn}(\tau_{01}) \mathrm{sgn}(\tau_{02})}{|\tau_{10}|^h |\tau_{20}|^h |\tau_{12}|^{2\Delta - h}} \qquad h = 1, 3, 5, \ldots, \quad h = \frac{1}{2} + is$$

- They are eigenfunctions of the conformal kernels

$$\int d\tau_3 d\tau_4 K_c^{A,S}(\tau_1, \tau_2; \tau_3, \tau_4) \Psi_h^{A,S}(\tau_3, \tau_4) = k_c^{A,S}(h) \Psi_h^{A,S}(\tau_3, \tau_4)$$

$$k_c^A(h) = -(q-1) \frac{\Gamma(\frac{3}{2} - \frac{1}{q}) \Gamma(1 - \frac{1}{q}) \Gamma(\frac{h}{2} + \frac{1}{q}) \Gamma(\frac{1}{2} + \frac{1}{q} - \frac{h}{2})}{\Gamma(\frac{1}{2} + \frac{1}{q}) \Gamma(\frac{1}{q}) \Gamma(\frac{3}{2} - \frac{1}{q} - \frac{h}{2}) \Gamma(\frac{h}{2} - \frac{1}{q} + 1)}$$

$$k_c^S(h) = -\frac{\Gamma(\frac{3}{2} - \frac{1}{q}) \Gamma(1 - \frac{1}{q}) \Gamma(\frac{1}{q} - \frac{h}{2}) \Gamma(\frac{h}{2} - \frac{1}{2} + \frac{1}{q})}{\Gamma(\frac{1}{2} + \frac{1}{q}) \Gamma(\frac{1}{q}) \Gamma(\frac{h}{2} + \frac{1}{2} - \frac{1}{q}) \Gamma(1 - \frac{h}{2} - \frac{1}{q})}$$

K. Bulycheva'17
C. Peng, M. Spradlin, A. Volovich'17
S.Giombi, I.Klebanov, GT, unpublished

Back to Four-point function

- Using the basis of eigenfunctions we can write for the four-point function

$$\mathcal{F} = \frac{1}{1-K^S}\mathcal{F}_0^S + \frac{1}{1-K^A}\mathcal{F}_0^A \quad \rightarrow \quad \mathcal{F} = \sum_h \frac{\langle \Psi_h^S | \mathcal{F}_0^S \rangle}{1-k_c^S(h)} \Psi_h^S + \sum_h \frac{\langle \Psi_h^A | \mathcal{F}_0^A \rangle}{1-k_c^A(h)} \Psi_h^A$$

- Old problem: $h=2$ mode: $k_c^A(h=2)=1$ and we get singularity
- New problem: $h=1$ mode: $k_c^S(h=1)=1$ and we get singularity

- Resolution: we should not use conformal kernel, but consider correction

$$K^{A,S} = -J^2\Big(\frac{q}{2} \pm (\frac{q}{2}-1)\Big) G(\theta_{13})G(\theta_{24})G(\theta_{34})^{q-2}$$

$$G(\theta) \to G_c(\theta) + \delta G \qquad \delta G(\theta_{12}) = \frac{\alpha_G}{\beta\mathcal{J}} \int_0^{2\pi} d\theta_0 \langle \psi_\Delta(\theta_1)\psi_\Delta(\theta_2) O_{h=-1}(\theta_0)\rangle$$

- Corrected Kernels have corrected eigenvalues. So we expect roughly

$$k^A = 1 - \frac{\alpha_{K^A}}{\beta\mathcal{J}} + \dots \qquad k^S = 1 - \frac{\alpha_{K^S}}{\beta\mathcal{J}} + \dots$$

where dots denote higher orders in $1/\beta\mathcal{J}$ when $\beta\mathcal{J} \to \infty$

Correction to h=1 and h=2 modes

- Corrected Kernels don't commute with Casimir anymore

$$K^{A,S} = -J^2\Big(\frac{q}{2} \pm (\frac{q}{2} - 1)\Big)G(\theta_{13})G(\theta_{24})G(\theta_{34})^{q-2} \qquad G(\theta) \to G_c(\theta) + \delta G$$

$$[K^{A,S}, C] \neq 0$$

so h is not a good quantum number now. But we still have shift operator

$$D = \partial_{\theta_1} + \partial_{\theta_2} \qquad [K^{A,S}, D] = 0$$

which commutes with the corrected Kernels

- Eigenfunctions of $D = \partial_{\theta_1} + \partial_{\theta_2}$ are $\Psi_n^{A,S}(\theta_1, \theta_2) = e^{-in\frac{\theta_1+\theta_2}{2}}\phi_n^{A,S}(\theta_{12})$

and labeled by integer number n

- In conformal case $h = 2$ and $h = 1$ eigenvalues were degenerate in n but correction lifts this degeneracy, so the answer we expect is

$$k^A(2, n) = 1 - \frac{\alpha_{K^A}|n|}{\beta\mathcal{J}} + \dots \qquad k^S(1, n) = 1 - \frac{\alpha_{K^S}|n|}{\beta\mathcal{J}} + \dots$$

- α_{K^A} was computed $\alpha_{K^A} = -q\frac{dk_c^A(2)}{dh}\alpha_G$. And $S_{\mathrm{Sh}} \propto \alpha_{K^A}\frac{N}{\mathcal{J}}\int_0^\beta d\tau\{\tan\frac{\pi\tau}{\beta}, \tau\}$
- α_{K^S} was not computed. $S_{U(1)} \propto \alpha_{K^S}\frac{N}{\mathcal{J}}\int_0^\beta d\tau(\partial_\tau\phi)^2$

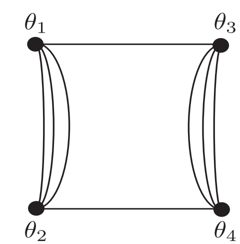
Large q for Antisymmetric Kernel

- We take $q \to \infty$ but keep $\mathcal{J} = \sqrt{q}\frac{J}{2^{\frac{q-1}{2}}}$ fixed. The Green's function is

$$G(\tau) = \frac{1}{2}\mathrm{sgn}(\tau)\Big(1 + \frac{1}{q}g(\tau) + \dots\Big) \qquad e^{g(\tau)} = \frac{\cos^2\frac{\pi v}{2}}{\cos^2(\frac{\pi v}{2} - \frac{\pi v|\tau|}{\beta})} \qquad \beta\mathcal{J} = \frac{\pi v}{\cos\frac{\pi v}{2}}$$

where for large $\beta\mathcal{J}$ one finds $v = 1 - \frac{2}{\beta\mathcal{J}} + \frac{4}{(\beta\mathcal{J})^2} + \dots$

- Now consider symmetric version of antisymmetric kernel



$$\tilde{K}^A(\theta_1, \theta_2; \theta_3, \theta_4) = -(q-1)J^2|G(\theta_{12})|^{\frac{q-2}{2}}G(\theta_{13})G(\theta_{24})|G(\theta_{34})|^{\frac{q-2}{2}}$$

- In the limit $q \to \infty$ it takes the form, which is now independent on q !

$$\tilde{K}^A_{q=\infty}(\theta_1, \theta_2; \theta_3, \theta_4) = -\frac{1}{2}\mathcal{J}^2 e^{\frac{1}{2}g(\theta_{12})}\mathrm{sgn}(\theta_{13})\mathrm{sgn}(\theta_{24})^{\frac{1}{2}g(\theta_{34})}$$

- Eigenvalues and eigenvectors now depend only on $\beta\mathcal{J}$

$$\int d\theta_3 d\theta_4 \tilde{K}^A_{q=\infty}(\theta_1, \theta_2; \theta_3, \theta_4)\Psi(\theta_3, \theta_4) = k^A_{q=\infty}\Psi(\theta_1, \theta_2)$$

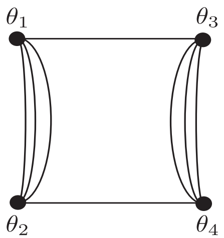
- It is possible to find correction to the $h = 2$ mode

$$k^A_{q=\infty} = 1 - \frac{3|n|}{\beta\mathcal{J}} + \frac{7|n|^2}{(\beta\mathcal{J})^2} + \dots$$

Maldacena, D.Stanford '16

Large q doesn't work for Symmetric Kernel

- Now consider symmetric version of symmetric kernel



$$\tilde{K}^S(\theta_1,\theta_2;\theta_3,\theta_4) = -J^2|G(\theta_{12})|^{\frac{q-2}{2}}G(\theta_{13})G(\theta_{24})|G(\theta_{34})|^{\frac{q-2}{2}}$$

doesn't have prefactor $(q-1)$

- In the limit $q \to \infty$ it takes the form

$$\tilde{K}^S_{q=\infty}(\theta_1,\theta_2;\theta_3,\theta_4) = -\frac{1}{2q}\mathcal{J}^2 e^{\frac{1}{2}g(\theta_{12})}\text{sgn}(\theta_{13})\text{sgn}(\theta_{24})^{\frac{1}{2}g(\theta_{34})}$$

- We find for the eigenvalues

$$\int d\theta_3 d\theta_4 \tilde{K}^S_{q=\infty}(\theta_1,\theta_2;\theta_3,\theta_4)\Psi(\theta_3,\theta_4) = k^S_{q=\infty}\Psi(\theta_1,\theta_2) \qquad k^S_{q=\infty} \propto \frac{1}{q}$$

- Large $\beta\mathcal{J}$ and large q limits don't commute

example $k^S = \dfrac{1}{1+\frac{q}{\beta\mathcal{J}}}$

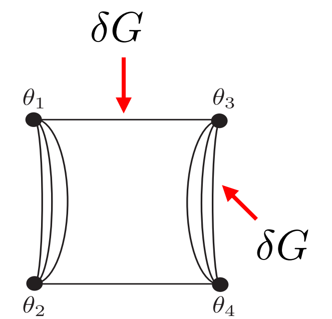
Direct approach to h=1 eigenvalues

J.Maldacena, D.Stanford '16

- Consider correction as a perturbation of conformal kernel

$$\delta\tilde{K}^S = -J^2|G(\theta_{12})|^{\frac{q-2}{2}}G(\theta_{13})G(\theta_{24})|G(\theta_{34})|^{\frac{q-2}{2}} - \tilde{K}^S_c(\theta_1,\theta_2;\theta_3,\theta_4)$$

$$G(\theta)\to G_c(\theta)+\delta G \qquad\qquad G(\theta)\to G_c(\theta)$$



$$\delta\tilde{K}^S = \delta\tilde{K}^S_{rung} + \delta\tilde{K}^S_{rail}$$

$$\delta\tilde{K}^S_{rung} = (q-2)\tilde{K}^S_c(\theta_1,\theta_2;\theta_3,\theta_4)f_0(\theta_{12}) \qquad \delta\tilde{K}^S_{rail} = 2\tilde{K}^S_c(\theta_1,\theta_2;\theta_3,\theta_4)f_0(\theta_{13})$$

$$f_0(\theta) = \frac{\delta G}{G_c(\theta)} = -\frac{\alpha_G}{\beta\mathcal{J}}\left(2-\frac{\pi-|\theta|}{\tan\frac{|\theta|}{2}}\right)$$

- First order perturbation theory for energy levels (w.f. are unperturbed harmonics of Casimir eigenfunctions)

$$\delta k^S_{rung} = \langle\Psi_{1,n}|\delta\tilde{K}^S_{rung}|\Psi_{1,n}\rangle \qquad \delta k^S_{rail} = \langle\Psi_{1,n}|\delta\tilde{K}^S_{rail}|\Psi_{1,n}\rangle$$

$$\Psi_{1,n}(\theta_1,\theta_2) = \int_0^{2\pi} d\theta_0 e^{-in\theta_0}\Psi^S_{h=1}(\theta_0,\theta_1,\theta_2) = \frac{e^{-in\frac{\theta_1+\theta_2}{2}}}{2\pi|n|^{1/2}}\frac{\sin\frac{n\theta_{12}}{2}}{\sin\frac{\theta_{12}}{2}}$$

- It is easy to compute rung correction and get divergence!

$$\delta k^S_{rung} \propto \langle\Psi_{1,n}|f_0(\theta_{12})|\Psi_{1,n}\rangle \propto \int d\theta_1 d\theta_2 \frac{\sin^2\frac{n\theta_{12}}{2}}{\sin^2\frac{\theta_{12}}{2}}\left(2+\frac{\pi-|\theta_{12}|}{\tan\frac{|\theta_{12}|}{2}}\right) \sim \int_{1/\beta J}\frac{d\theta_{12}}{|\theta_{12}|} \sim \log\beta J$$

$$k^S(h=1,n) = 1 - \frac{\alpha_{K^S}\log(\beta\mathcal{J})}{\beta\mathcal{J}}|n| + \ldots \qquad ???$$

Large n limit

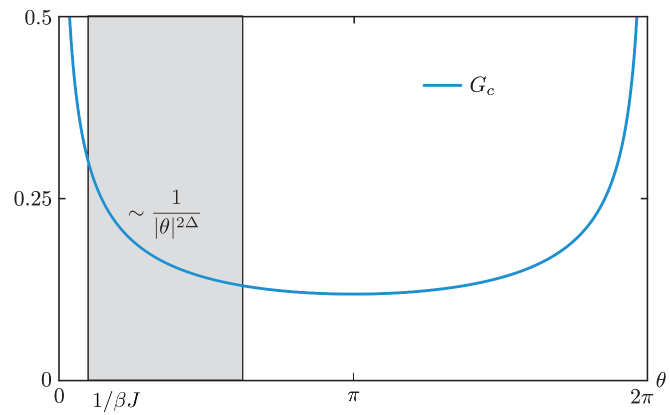
J.Maldacena, D.Stanford '16

- Consider eigenfunctions in h=1 sector. They have oscillating terms

$$\Psi_{1,n}(\theta_1,\theta_2) = \frac{1}{2\pi|n|^{1/2}} e^{-in\frac{\theta_1+\theta_2}{2}} \frac{\sin\frac{n\theta_{12}}{2}}{\sin\frac{\theta_{12}}{2}}$$

with frequency n .

- If we take the limit $n \to +\infty$ we expect everything dominated by small angles



- To utilize this idea let us redefine circle variable to $\theta = \dfrac{2\pi\tau}{\beta}$ and take limit $n \to +\infty$ but keep $\beta = \pi n$, then we find

$$\theta = \frac{2\tau}{n} \qquad \int_{-\pi}^{+\pi} d\theta \to \frac{2}{n}\int_{-\infty}^{+\infty} d\tau$$

Large n limit

- Wave function, kernel and correction at the limit $n \to +\infty$ take the form

$$\Psi_{1,n}(\theta_1,\theta_2) = \frac{e^{-in\frac{\theta_1+\theta_2}{2}}}{2\pi|n|^{1/2}} \frac{\sin\frac{n\theta_{12}}{2}}{\sin\frac{\theta_{12}}{2}} \to \frac{n^{1/2}}{2\pi} e^{-i(\tau_1+\tau_2)} \frac{\sin\tau_{12}}{\tau_{12}} + \dots$$

$$\tilde{K}_c^S(\theta_1,\theta_2;\theta_3,\theta_4) \to -\frac{n^2}{4\alpha_0(q-1)} \frac{\mathrm{sgn}(\tau_{13})\mathrm{sgn}(\tau_{24})}{|\tau_{12}|^{1-2\Delta}|\tau_{13}|^{2\Delta}|\tau_{24}|^{2\Delta}|\tau_{34}|^{1-2\Delta}} + \dots$$

$1/n$ corrections

$$f_0(\theta) = -\frac{\alpha_G}{\beta\mathcal{J}}\left(2 - \frac{\pi-|\theta|}{\tan\frac{|\theta|}{2}}\right) \to -\frac{\alpha_G}{\beta\mathcal{J}}\frac{\pi n}{|\tau|}$$

- Integrals transform from complicated expressions to a simple power law form

$$\delta k^S_{rung} = (q-2)\int_{\theta_1,\dots,\theta_4} \Psi^*_{1,n}(\theta_1,\theta_2)\tilde{K}_c^S(\theta_1,\theta_2,\theta_3,\theta_4) f_0(\theta_{12})\Psi_{1,n}(\theta_3,\theta_4)$$

$$\delta k^S_{rail} = 2\int_{\theta_1,\dots,\theta_4} \Psi^*_{1,n}(\theta_1,\theta_2)\tilde{K}_c^S(\theta_1,\theta_2,\theta_3,\theta_4) f_0(\theta_{13})\Psi_{1,n}(\theta_3,\theta_4)$$

- In large n limit these horrible integrals reduce to quite simple expressions

$$\delta k^S_{rung} \to n\frac{q-2}{\alpha_0(q-1)}\frac{\alpha_G}{\beta\mathcal{J}}\int_{\tau_2,\tau_3,\tau_4} \frac{e^{i(\tau_2-\tau_3-\tau_4)}\sin(\tau_2)\mathrm{sgn}(\tau_{13})\mathrm{sgn}(\tau_{24})\sin(\tau_{34})}{|\tau_2|^{3-2\Delta}|\tau_3|^{2\Delta}|\tau_{24}|^{2\Delta}|\tau_{34}|^{2-2\Delta}} + \dots$$

$$\delta k^S_{rail} \to n\frac{2}{\alpha_0(q-1)}\frac{\alpha_G}{\beta\mathcal{J}}\int_{\tau_2,\tau_3,\tau_4} \frac{e^{i(\tau_2-\tau_3-\tau_4)}\sin(\tau_2)\mathrm{sgn}(\tau_{13})\mathrm{sgn}(\tau_{24})\sin(\tau_{34})}{|\tau_2|^{2-2\Delta}|\tau_3|^{2\Delta+1}|\tau_{24}|^{2\Delta}|\tau_{34}|^{2-2\Delta}} + \dots$$

linear in n as expected

Computation of integrals

- Consider for example the rung integral

$$\delta k^S_{rung} \to n \frac{q-2}{\alpha_0(q-1)} \frac{\alpha_G}{\beta \mathcal{J}} \int_{\tau_2,\tau_3,\tau_4} \frac{e^{i(\tau_2-\tau_3-\tau_4)} \sin(\tau_2) \mathrm{sgn}(\tau_{13}) \mathrm{sgn}(\tau_{24}) \sin(\tau_{34})}{|\tau_2|^{3-2\Delta} |\tau_3|^{2\Delta} |\tau_{24}|^{2\Delta} |\tau_{34}|^{2-2\Delta}} + \dots$$

- Take Fourier transform of all 4 terms

$$\frac{\mathrm{sgn}(\tau)}{|\tau|^\alpha} = f_\alpha \int \frac{d\omega}{2\pi} e^{-i\omega\tau} \frac{\mathrm{sgn}(\omega)}{|\omega|^{1-\alpha}} \qquad f_\alpha = 2i \cos(\frac{\pi\alpha}{2}) \Gamma(1-\alpha)$$

- Integrate out all τ and get sum of integrals like

$$\int \frac{d\omega}{2\pi} \frac{1}{|\omega|^\alpha |\omega+1|^\beta} \qquad \int \frac{d\omega}{2\pi} \frac{\mathrm{sgn}(\omega)\mathrm{sgn}(\omega+1)}{|\omega|^\alpha |\omega+1|^\beta}$$

- For some powers of α and β these integrals are divergent as $\Gamma(0)$ so we need regularization

- The simplest naïve way to regularize these integrals is to shift a power in f_0

$$\frac{\delta G(\theta)}{G_c(\theta)} = f_0(\theta) = -\frac{\alpha_G}{\beta \mathcal{J}} \left(2 + \frac{\pi - |\theta|}{\tan \frac{|\theta|}{2}} \right) \to -\frac{\alpha_G}{\beta \mathcal{J}} \frac{\pi n}{|\tau|^{1-\epsilon}}$$

Regularization

Results

- Using regularization and taking the limit $\epsilon \to 0$ we find

$$\delta k^S_{rung} = \frac{n\alpha_G}{\beta\mathcal{J}}\Big(\frac{2(q-2)}{\epsilon} + \text{rung finite part}\Big)$$

$$\delta k^S_{rail} = \frac{n\alpha_G}{\beta\mathcal{J}}\Big(-\frac{2(q-2)}{\epsilon} + \text{rail finite part}\Big)$$

- Divergencies exactly cancel! There is no $\log(\beta J)$ term!

- We finally find

$$k^S(1,n) = 1 + \delta k^S_{rung} + \delta k^S_{rail} = 1 - \alpha_{K^S}\frac{|n|}{\beta\mathcal{J}}$$

$$\alpha_{K^S} = -2(q-1)\frac{dk_c^A(2)}{dh}\alpha_G$$

wrong!

$$\alpha_{K^A} = -q\frac{dk_c^A(2)}{dh}\alpha_G$$

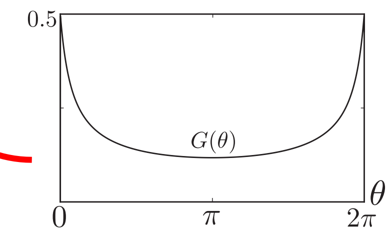
$$k_c^A(h) = -(q-1)\frac{\Gamma(\frac{3}{2}-\frac{1}{q})\Gamma(1-\frac{1}{q})\Gamma(\frac{h}{2}+\frac{1}{q})\Gamma(\frac{1}{2}+\frac{1}{q}-\frac{h}{2})}{\Gamma(\frac{1}{2}+\frac{1}{q})\Gamma(\frac{1}{q})\Gamma(\frac{3}{2}-\frac{1}{q}-\frac{h}{2})\Gamma(\frac{h}{2}-\frac{1}{q}+1)}$$

- Unfortunately this is a wrong answer!

Numerics for Kernel eigenvalues

- Consider symmetric or antisymmetric kernel made of exact numerical $G(\theta)$

$$\tilde{K}^{A,S}(\theta_1,\theta_2;\theta_3,\theta_4) = -\Big(\frac{q}{2} \pm (\frac{q}{2}-1)\Big) J^2 |G(\theta_{12})|^{\frac{q-2}{2}} G(\theta_{13}) G(\theta_{24}) |G(\theta_{34})|^{\frac{q-2}{2}}$$



- The kernel commutes with operator $D = \partial_{\theta_1} + \partial_{\theta_2}$

$$[\tilde{K}^{A,S}, D] = 0$$

- Look for eigenfunctions of the kernel, which are eigfunctions of D

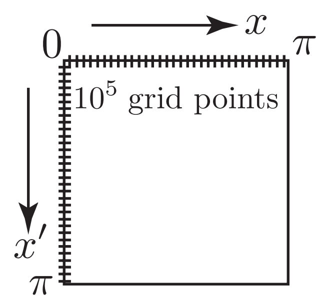
$$\psi^{A,S}_{h,n}(\theta_1,\theta_2) = e^{in\frac{\theta_1+\theta_2}{2}} \phi^{A,S}_{h,n}(\theta_{12}) \qquad y = \frac{\theta_1+\theta_2}{2} \qquad x = \theta_{12}$$

- Projecting on a given sector n we obtain a simple formula for the kernel

$$\tilde{K}^{A,S}(x,x') = -8\pi \Big(\frac{q}{2} \pm \big(\frac{q}{2}-1\big)\Big) J^2 |G(x')|^{\frac{q-2}{2}} |G(x')|^{\frac{q-2}{2}} \times$$
$$\times \int_0^\pi dy \cos(ny) \Big(G(y+\frac{x-x'}{2}) G(y-\frac{x-x'}{2} \mp G(y+\frac{x+x'}{2}) G(y-\frac{x+x'}{2})) \Big)$$

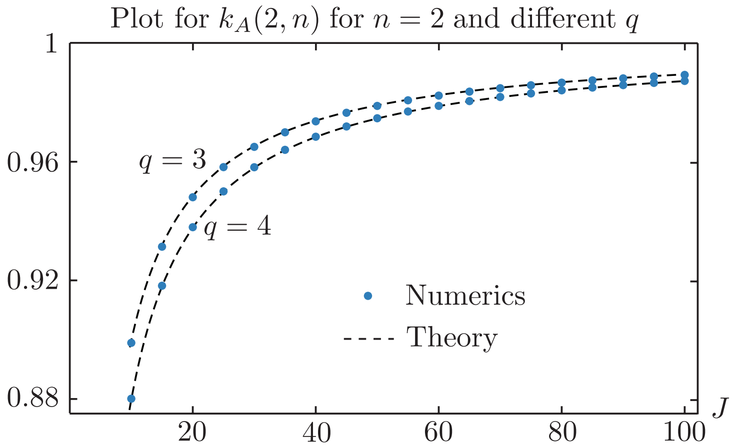
- Find eigenvalues of discretized matrix

$$\tilde{K}^{A,S}(x,x') =$$



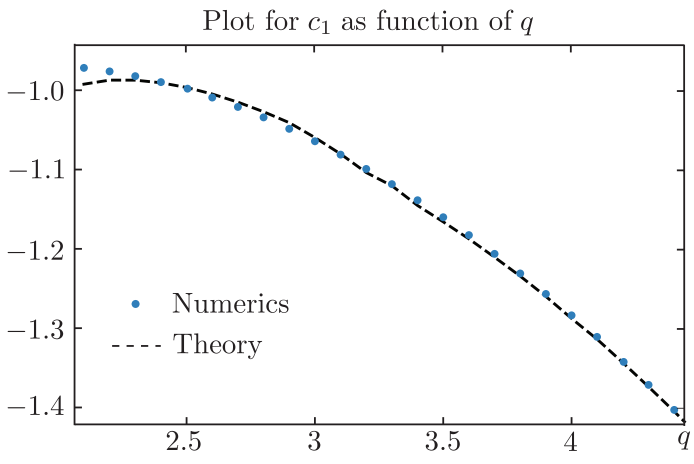
J.Maldacena, D.Stanford '16

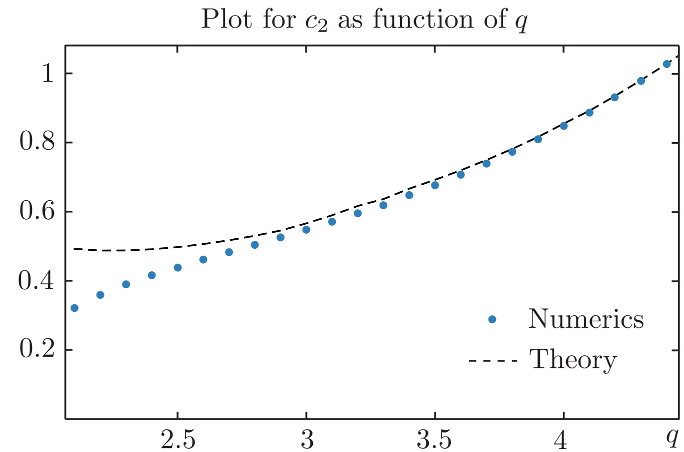
Numerics for Antisymmetric Kernel



$$k^A(2, n) = 1 + \frac{dk_c^A(2)}{dh} \frac{q\alpha_G|n|}{\beta\mathcal{J}} + \frac{1}{2} \frac{d^2k_c^A(2)}{dh^2} \left(\frac{q\alpha_G|n|}{\beta\mathcal{J}}\right)^2 + \dots$$

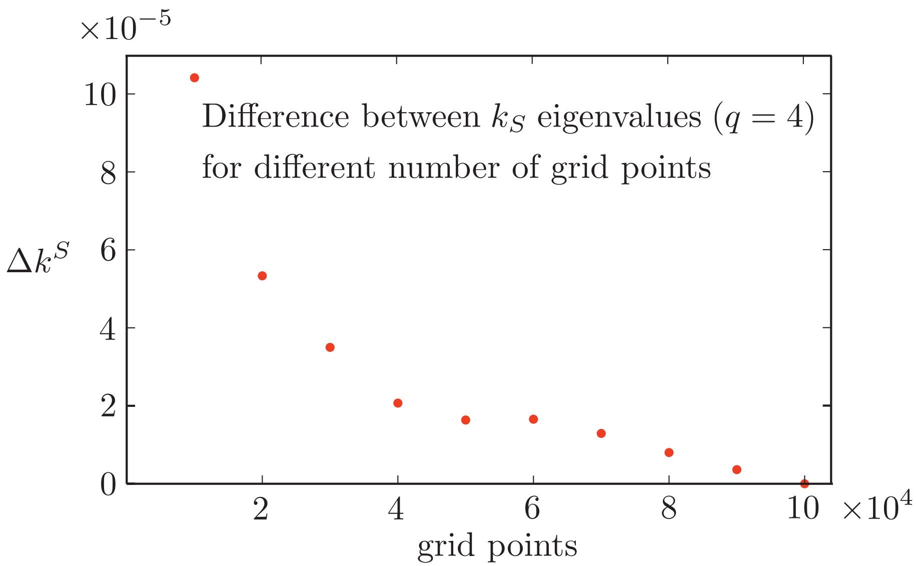
$$k_c^A(h) = -(q-1) \frac{\Gamma(\frac{3}{2} - \frac{1}{q})\Gamma(1 - \frac{1}{q})\Gamma(\frac{h}{2} + \frac{1}{q})\Gamma(\frac{1}{2} + \frac{1}{q} - \frac{h}{2})}{\Gamma(\frac{1}{2} + \frac{1}{q})\Gamma(\frac{1}{q})\Gamma(\frac{3}{2} - \frac{1}{q} - \frac{h}{2})\Gamma(\frac{h}{2} - \frac{1}{q} + 1)}$$

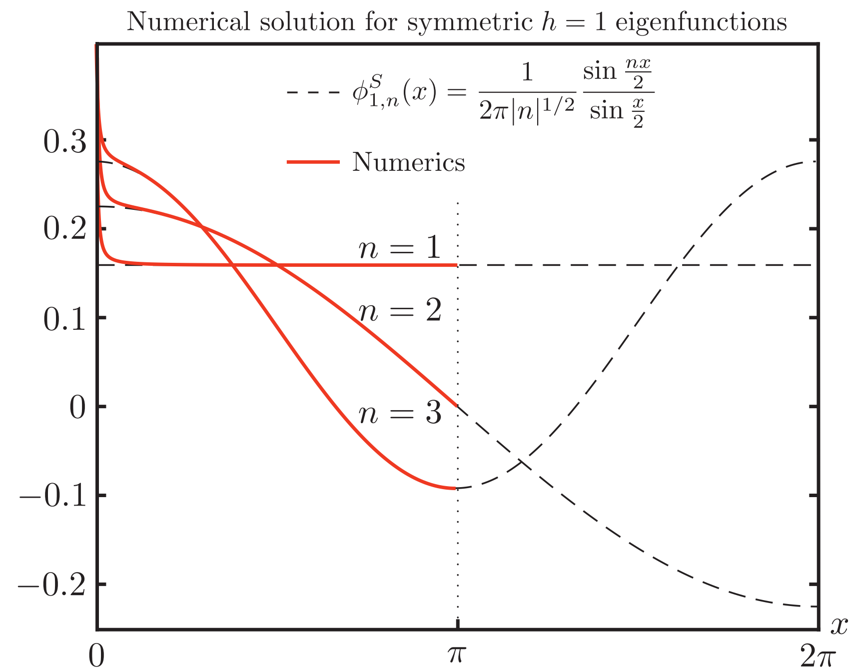


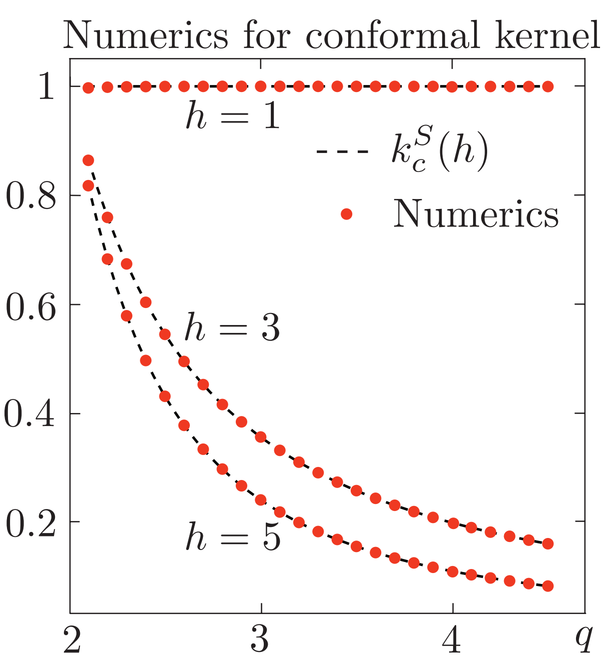


$$k^A(2, n = 2) = 1 + \frac{c_1}{\beta J} + \frac{c_2}{(\beta J)^2} + \dots$$

Sanity checks for Symmetric Kernel





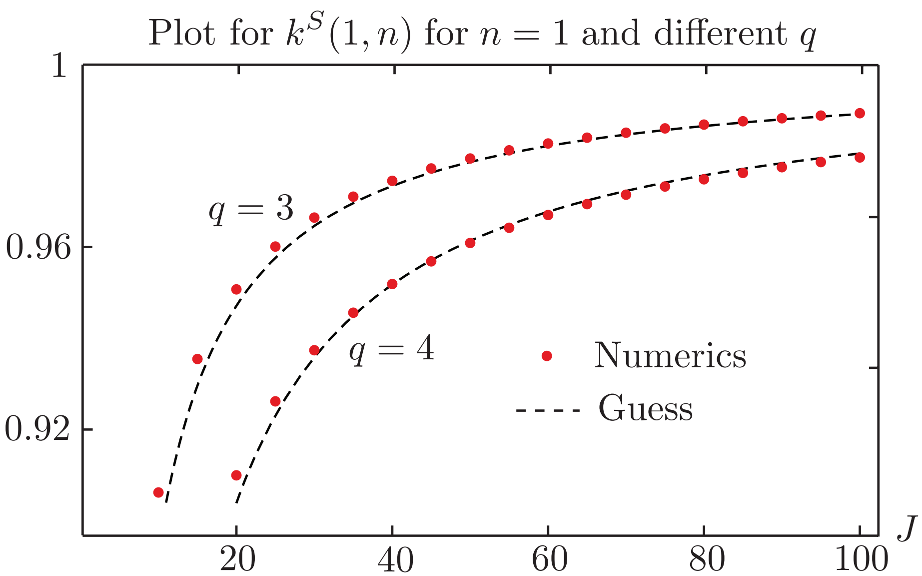


$$\tilde{K}^S(\theta_1, \theta_2; \theta_3, \theta_4) = -J^2 |G(\theta_{12})|^{\frac{q-2}{2}} G(\theta_{13}) G(\theta_{24}) |G(\theta_{34})|^{\frac{q-2}{2}}$$

$$G_c(\theta) = b \frac{\mathrm{sgn}(\theta)}{|2 \sin \frac{\theta}{2}|^{2\Delta}}$$

$$k^S_c(h) = -\frac{\Gamma(\frac{3}{2} - \frac{1}{q})\Gamma(1 - \frac{1}{q})\Gamma(\frac{1}{q} - \frac{h}{2})\Gamma(\frac{h}{2} - \frac{1}{2} + \frac{1}{q})}{\Gamma(\frac{1}{2} + \frac{1}{q})\Gamma(\frac{1}{q})\Gamma(\frac{h}{2} + \frac{1}{2} - \frac{1}{q})\Gamma(1 - \frac{h}{2} - \frac{1}{q})}$$

Numerics for Symmetric Kernel

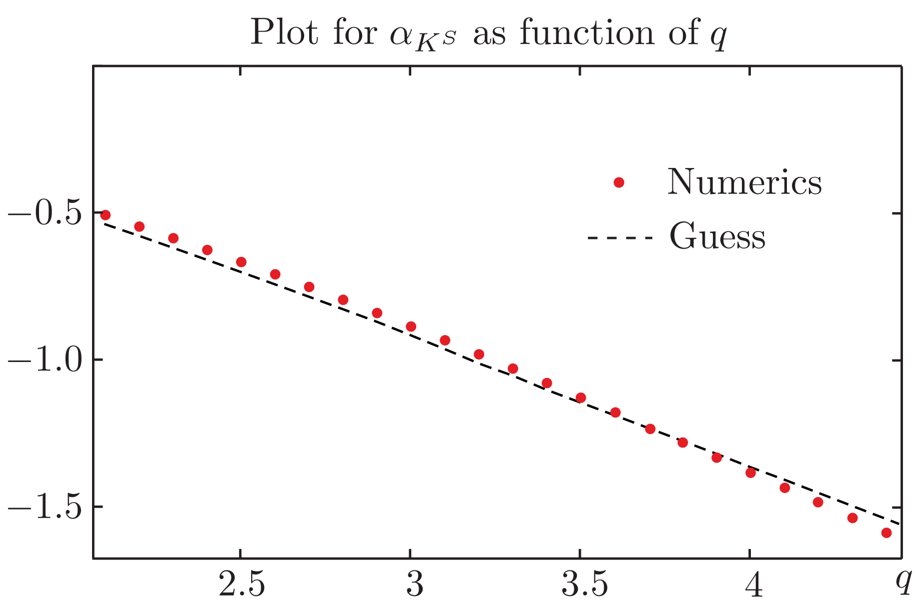


- Guess

$$k^S(1,n) = 1 + \frac{dk_c^A(2)}{dh}\frac{q(q-1)\alpha_G|n|}{\beta\mathcal{J}} + \dots$$

- Wrong result

$$k^S(1,n) = 1 + \frac{dk_c^A(2)}{dh}\frac{2(q-1)\alpha_G|n|}{\beta\mathcal{J}} + \dots$$



$$k^S(1,n) = 1 - \frac{\alpha_{K^S}|n|}{\beta\mathcal{J}} + \dots$$

- Guess

$$\alpha_{K^S} = -\frac{dk_c^A(2)}{dh}q(q-1)\alpha_G$$

Thank you for your attention!