SYK, Harmonic Analysis, and Multipoint Conformal Blocks

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Based on:

V.R, “Multipoint Conformal Blocks in the Comb Channel”, 1810.03244

J. Liu, E. Perlmutter, V.R., D. Simmons-Duffin, “d-dimensional SYK, AdS Loops, and 6j Symbols”, 1808.00612

D. Gross, V.R., “All point correlation functions in SYK”, 1710.08113

Solvable models play an important role in understanding Quantum Field Theory (QFT).

Each solvable, or partly solvable, QFT is simple when viewed in a particular way.

Correspondingly, each model requires its own set of tools in order to solve it. Sometimes, this requires developing new techniques, which then prove useful in other contexts.

In this talk the solvable QFT will be SYK and SYK-like models. The new techniques will include several results in Conformal Field Theory (CFT), in various dimensions.

The talk has two parts.

Part 1: CFT results:

Harmonic analysis on the conformal group

6j symbols

Multichannel conformal blocks.

Part 2: The solution of SYK:

Computation of all-point correlation functions, at large N, in the infrared (conformal limit)

Before discussing SYK, let us recall a few classic examples of solvable QFTs, and the features they have that make them solvable:

1) 2d Integrable models: Sinh-Gordon

2) 4d Integrable model: maximally supersymmetric Yang Mills

3) 2d QCD with quarks ('t Hooft model)

4) 2d CFTs: minimal models

1) 2d Integrable models

Zamolodchikov and Zamolodchikov, ‘79

Solving a QFT means computing the full S matrix.

In particular, we need to compute the $n \rightarrow n$ S matrix, an arbitrarily large amount of work.

For integrable theories, there is no particle production, and the S matrix factorizes into a product of $2 \rightarrow 2$ S matrices.

The $2 \rightarrow 2$ S matrix is computed by solving the constraints of unitarity, crossing, and Yang-Baxter (if there is more than one species)

The simplest example is the Sinh-Gordon model

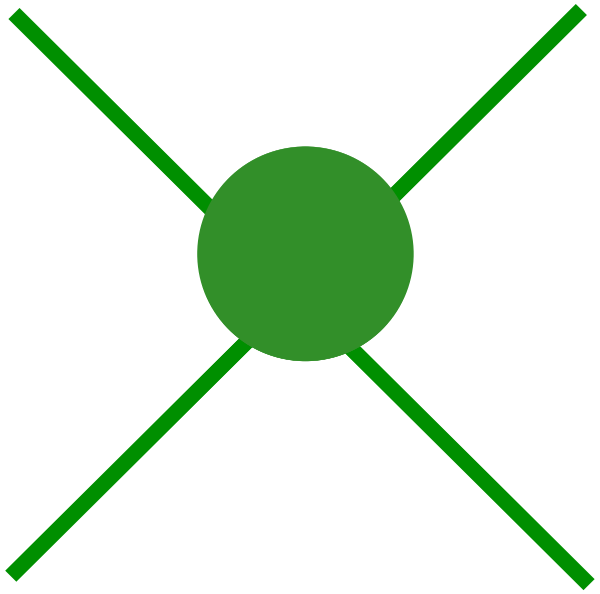
$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{b^2}\left(\cosh(b\phi) - 1\right)$$

$$S(\theta) = \frac{\sinh\theta - i\sin\alpha}{\sinh\theta + i\sin\alpha}$$

$$\alpha = \frac{\pi b^2}{8\pi + b^2}$$

$$\theta = \theta_1 - \theta_2$$

$$E(\theta_i) = m\cosh\theta_i$$



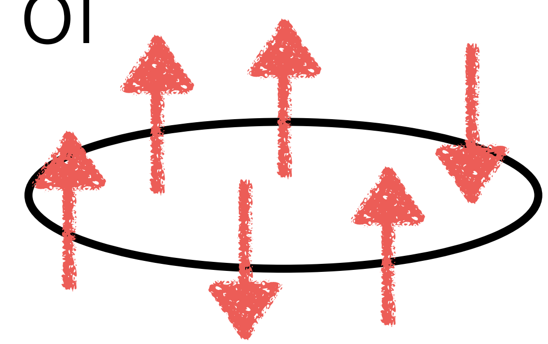
2)

$\mathcal{N}=4$ Super Yang-Mills

Beisert et al., ``Review of AdS/CFT Integrability", 1012.3982

In the simplest context, one wants to compute the dimensions of (single trace) operators.

At weak coupling, this becomes a problem of diagonalizing an (integrable) spin-chain Hamiltonian.



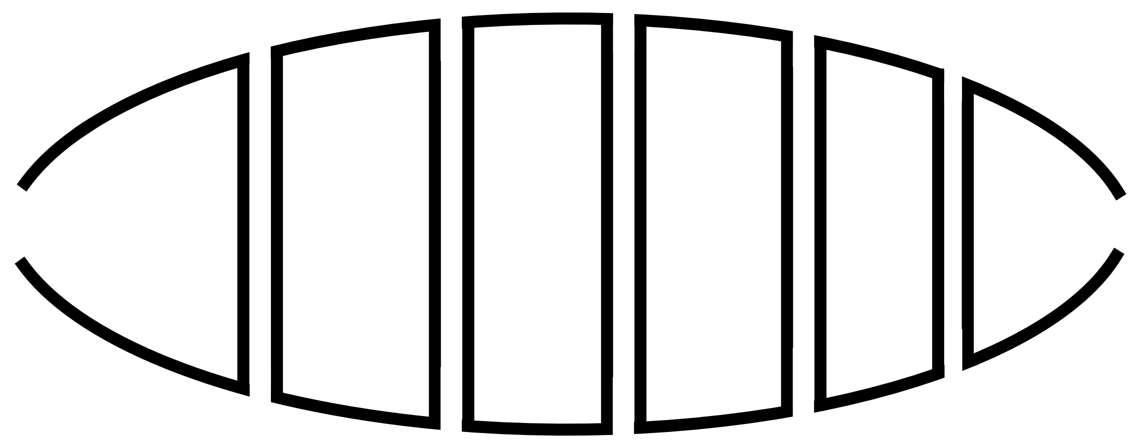
At strong coupling, this is a string in $AdS_5 \times S^5$, which also turns out to be integrable.

At intermediate coupling, one assumes the Hamiltonian, whatever it is, is integrable.

3) 2d QCD at large N ('t Hooft model)

't Hooft, '74;   Callan, Coote, Gross, '76

$$\mathcal{L} = \frac{1}{4} F^{j}_{\mu\nu\, i} F^{\mu\nu\, i}_{j} + \overline{\psi}^{i} \left(i\gamma^{\mu} D_{\mu} - m_{a} \right) \psi_{i}$$



Summing ladder diagrams gives the 't Hooft equation: an integral equation, the solution of which gives the masses of mesons.

Higher-point amplitudes are found by gluing together 4-point amplitudes, and are expressed in terms of the 't Hooft wave function.

4) 2d CFTs: minimal models

Belavin, Polyakov, Zamolodchikov, '84

CFTs in 2d have enhanced symmetry - Virasoro symmetry.

The minimal models have a finite number of Virasoro primaries. The bootstrap equations can be solved explicitly. The correlation functions satisfy differential equations. The models are solved without drawing any Feynman diagrams.

SYK

$$S = \int d\tau \left(\frac{1}{2} \chi_i \partial_\tau \chi_i + \frac{1}{4!} J_{ijkl}\, \chi_i \chi_j \chi_k \chi_l \right)$$

$$\overline{J_{ijkl}^2} = 3! \frac{J^2}{N^3}$$

Sachdev & Ye, '93
Kitaev, '15

Like the 't Hooft model, SYK is solvable, but not integrable.

The infrared of SYK is a near-CFT_1, described by a CFT and the Schwarzian. The infrared of d-dimensional SYK, as well variants of SYK (cSYK), are CFTs. I will be discussing the CFT sector/version.

To solve a QFT, we must find the full S matrix

To solve a CFT, we must find all n-point correlation functions, of the fundamental field. (This is equivalent to finding all 3-point functions, of all operators, but this language is less natural in our context.)

The solvability of SYK will rely on two features:

1) A CFT n-point function is expressed in terms of n-point conformal blocks. This will be Part 1.

2) The Feynman diagrams contributing to the SYK n-point function are built by gluing together four-point functions. This gives simple rules enabling us to, essentially, write down an n-point function once we compute a six-point function. This will be Part 2.

Part I: some CFT results

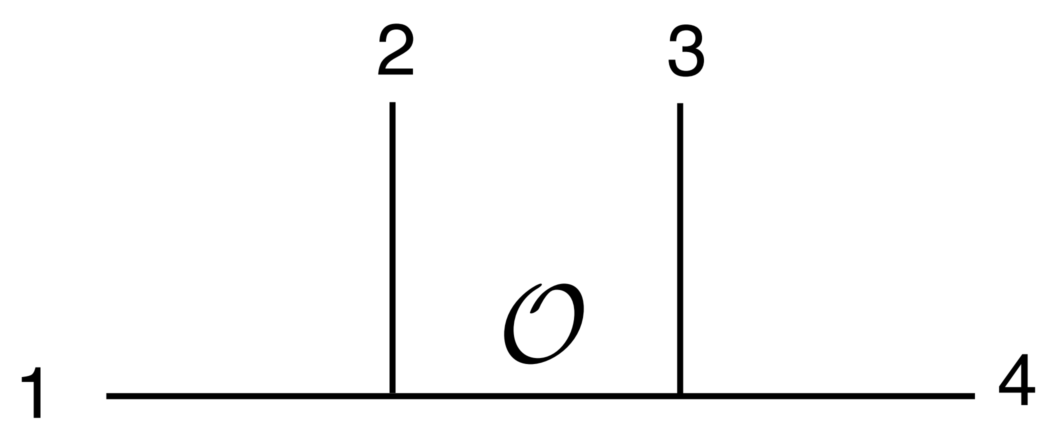
CFT 2-point and 3-point functions are fixed by conformal invariance.

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \frac{1}{|x_{12}|^{2\Delta}}$$

$$\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle = \frac{1}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}}$$

n-point functions are built out of n-point blocks, found by gluing together 3-point functions. For instance, the 4-point conformal partial wave is

$$\Psi^{\Delta_1,\Delta_2,\Delta_3,\Delta_4}_{\Delta,J}(x_i) \ = \int d^d x_0 \, \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}(x_0) \rangle \langle \widetilde{\mathcal{O}}(x_0) \mathcal{O}_3 \mathcal{O}_4 \rangle$$



Each vertex denotes a 3-point function, each line a position, and each internal line is a position that is integrated over.

(A conformal partial wave is a sum of a conformal block and a shadow block.)

Ferrara et al, '72
Simmons-Duffin, '12

Harmonic analysis on the conformal group

We expand a CFT 4-point function in terms of a complete basis of conformal partial waves,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \sum_{J=0}^{\infty} \int_{\frac{d}{2}}^{\frac{d}{2}+i\infty} \frac{d\Delta}{2\pi i} \frac{I_{\Delta,J}}{n_{\Delta,J}} \Psi_{\Delta,J}^{\Delta_1,\Delta_2,\Delta_3,\Delta_4}(x_i)$$

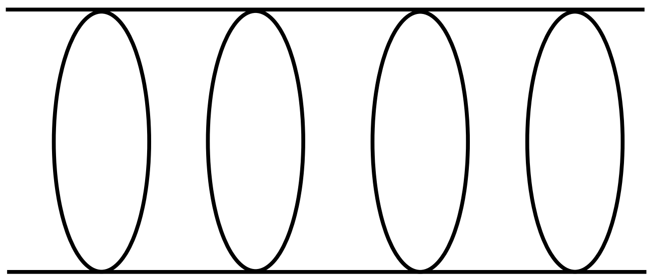
This is analogous to a Fourier expansion, or an expansion in terms of spherical harmonics.

Dobrev et al., ’77

Gadde, ’17;
Karateev, Kravchuk, Simmons-Duffin ‘18

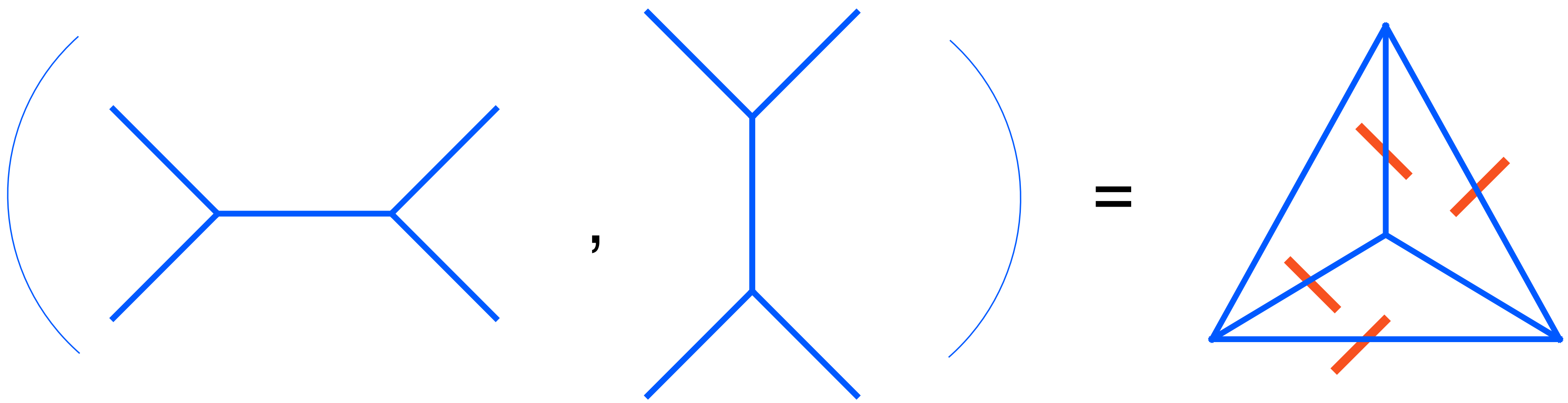
The basic formalism of harmonic analysis was introduced a long time ago, but was not widely used.

SYK forces it upon us. The 4-point function is a sum of ladder diagrams. This sum becomes a trivial geometric sum when written in the above form (the conformal blocks are the eigenvectors that diagonalize the kernel that adds rungs to the ladders).



Kitaev, '15
Polchinski & V.R, '16
Maldacena & Stanford, '16

Within CFT, one sometimes wishes to expand a 4-point function in various channels. The harmonic analysis formalism provides a way to do this. One makes use of the crossing kernel: the overlap between the s-channel and the t-channel conformal partial waves



In fact, this quantity is the 6j symbol for the Euclidean conformal group, SO(d+1,1).

We computed it in dimensions d=1, 2, 4, making use of Caron-Huot's Lorentzian inversion formula.

J. Liu, E. Perlmutter, V.R., D. Simmons-Duffin

The 6j symbol is probably most familiar in the context of spins in quantum mechanics. We can translate between the two:

SU(2)	**SO(d+1,1)**
angular momentum J	dimension Δ, spin J
z-component angular momentum	position x
Clebsch-Gordan	3-pt function $\langle \mathcal{O}_{\Delta_1,J_1}(x_1)\mathcal{O}_{\Delta_2,J_2}(x_2)\mathcal{O}_{\Delta_3,J_3}(x_3)\rangle$

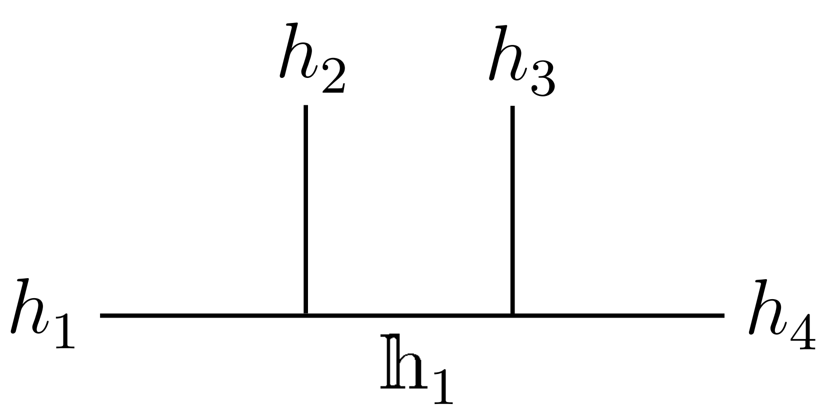
So far, we have discussed CFT 4-point functions. We said that these can be expanded in terms of a complete basis of conformal blocks (analogous to Fourier expansion), and some group-theoretic quantities like the 6j symbol give us control over the expansion.

We will need these results for computing correlation functions in SYK.

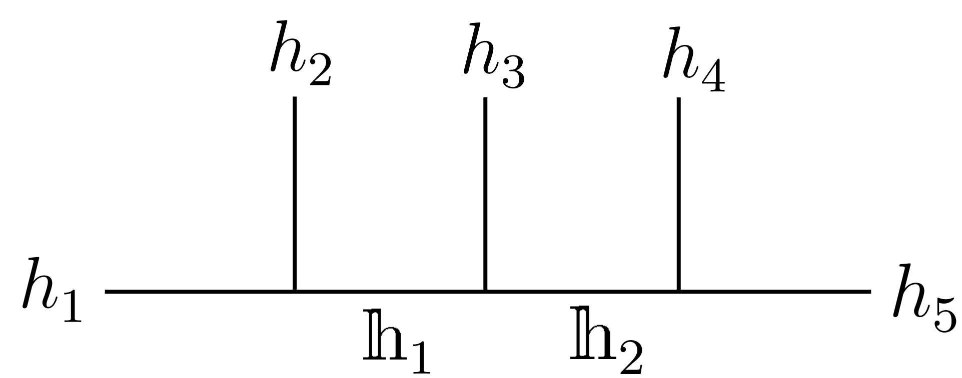
But before that we need to discuss n-point blocks.

n-point conformal partial waves are found in the same way as the 4-point partial wave: by gluing together 3-point functions.

Let me focus on dimension d=1, and label the dimensions of operators by h.



4-point



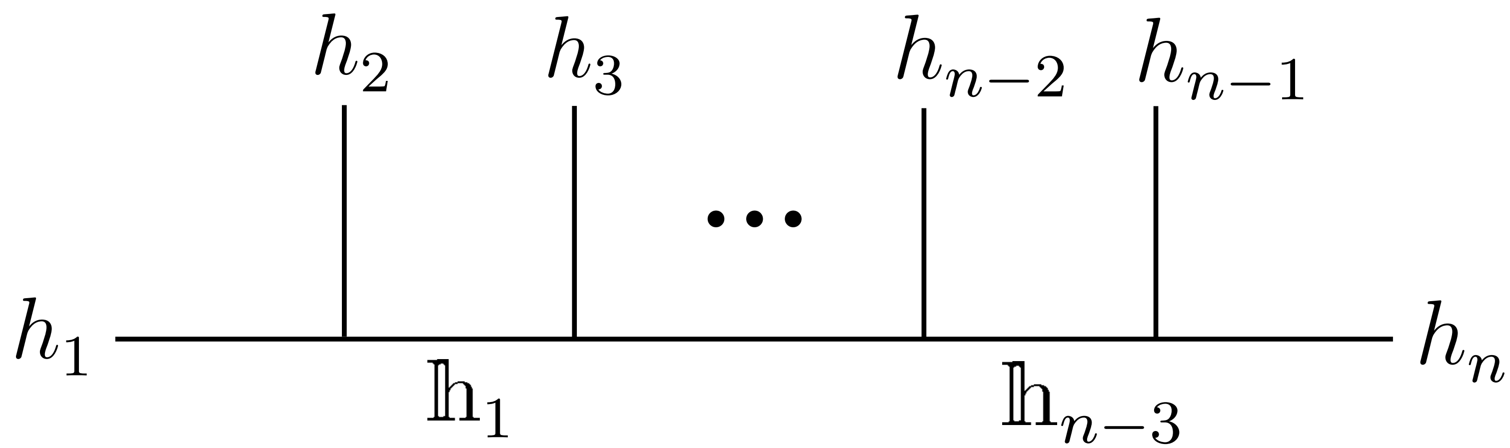
5-point

Recall the form of the 4-point block

$$g_{\mathbb{h}_1}^{h_1,h_2,h_3,h_4} = \chi_1^{\mathbb{h}_1}\, {}_2F_1\left[\begin{matrix}\mathbb{h}_1+h_{12},\ \mathbb{h}_1-h_{34} \\ 2\mathbb{h}_1\end{matrix};\chi_1\right] \qquad \chi_1 = \frac{z_{12}z_{34}}{z_{13}z_{24}}$$

This is the standard hypergeometric function of one variable, ${}_2F_1$

We computed the n-point block, for any n, in the following channel, which we call the comb channel.



The result is in terms of an n-3 variable hypergeometric function, which we call the comb function.

VR, ‘18

$$G^{h_1,\ldots,h_n}_{\mathbb{h}_1,\ldots,\mathbb{h}_{n-3}}(z_1,\ldots,z_n) = \mathcal{L}^{h_1,\ldots,h_n}(z_1,\ldots,z_n)\; g^{h_1,\ldots,h_n}_{\mathbb{h}_1,\ldots,\mathbb{h}_{n-3}}(\chi_1,\ldots,\chi_{n-3})\,, \qquad \chi_i = \frac{z_{i,i+1}\, z_{i+2,i+3}}{z_{i,i+2}\, z_{i+1,i+3}}$$

$$z_{ij} \equiv z_{i,j} \equiv z_i - z_j$$

$$\mathcal{L}^{h_1,\ldots,h_n}(z_1,\ldots,z_n) = \left(\frac{z_{23}}{z_{12}z_{13}}\right)^{h_1} \left(\frac{z_{n-2,n-1}}{z_{n-2,n}\, z_{n-1,n}}\right)^{h_n} \prod_{i=1}^{n-2} \left(\frac{z_{i,i+2}}{z_{i,i+1}\, z_{i+1,i+2}}\right)^{h_{i+1}}$$

$$g^{h_1,\ldots,h_n}_{\mathbb{h}_1,\ldots,\mathbb{h}_{n-3}}(\chi_1,\ldots,\chi_{n-3}) = \prod_{i=1}^{n-3} \chi_i^{\mathbb{h}_i}$$

$$F_K\left[\begin{matrix} h_1+\mathbb{h}_1-h_2,\ \mathbb{h}_1+\mathbb{h}_2-h_3,\ \ldots,\ \mathbb{h}_{n-4}+\mathbb{h}_{n-3}-h_{n-2},\ \mathbb{h}_{n-3}+h_n-h_{n-1} \\ 2\mathbb{h}_1,\ \ldots\ 2\mathbb{h}_{n-3} \end{matrix};\chi_1,\ldots,\chi_{n-3}\right]$$

$$F_K\left[\begin{matrix} a_1,\ b_1,\ \ldots,\ b_{k-1},\ a_2 \\ c_1,\ \ldots,\ c_k \end{matrix};x_1,\ldots,x_k\right]$$

$$= \sum_{n_1,\ldots,n_k=0}^{\infty} \frac{(a_1)_{n_1}(b_1)_{n_1+n_2}(b_2)_{n_2+n_3}\cdots(b_{k-1})_{n_{k-1}+n_k}(a_2)_{n_k}}{(c_1)_{n_1}\cdots(c_k)_{n_k}} \frac{x_1^{n_1}}{n_1!}\cdots\frac{x_k^{n_k}}{n_k!}$$

$$(a)_n \equiv \frac{\Gamma(a+n)}{\Gamma(a)}$$

VR, ‘18

The result in d=2 immediately follows, and is just a product of two d=1 blocks. (These are global blocks, not Virasoro).

For n-point blocks with n > 5 there are channels other than the comb channel. This is work in progress. The answer may not be as simple.

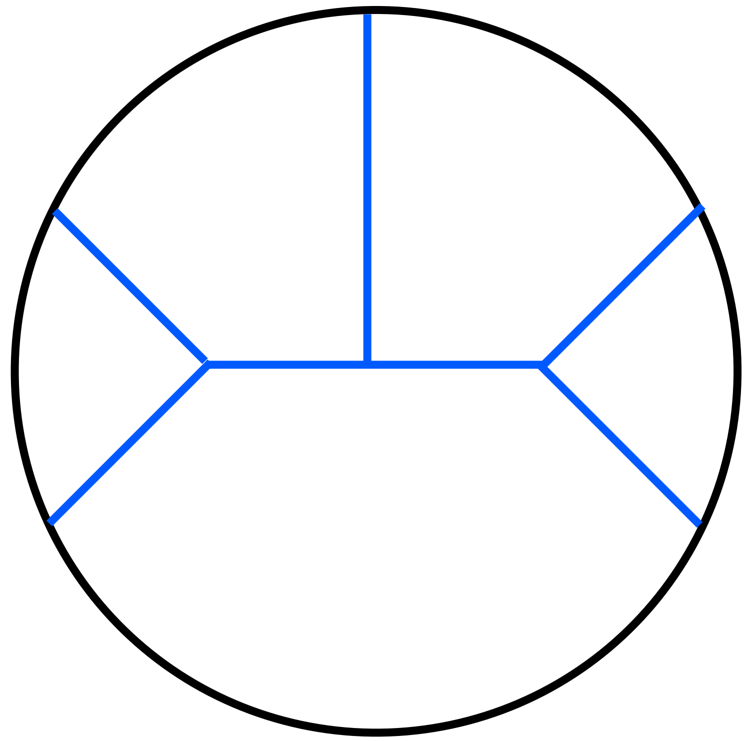
In dimensions d>2, there are more cross ratios than in 2 dimensions. For the 5-point block there are 5 cross-ratio. The result is significantly more involved.

Part II: SYK

In Part I, we saw that a CFT n-point function can be expanded in a complete basis of n-point conformal blocks. The n-point conformal blocks can in turn be found by gluing together 3-point functions (the shadow formalism).

The next question is when would actually be able to compute a CFT n-point function?

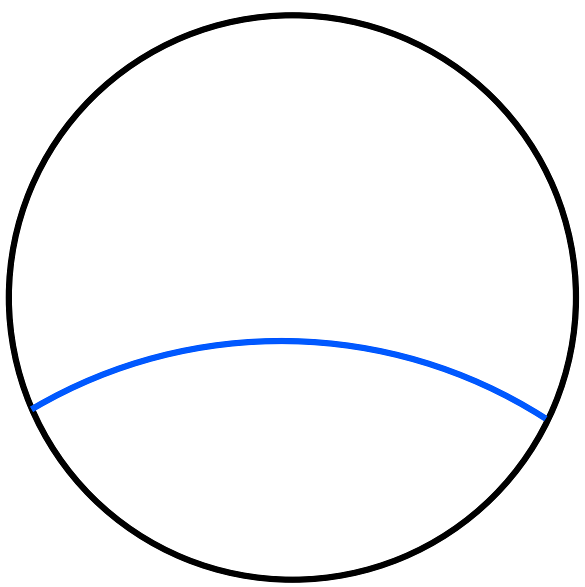
An obvious case is QFT in AdS

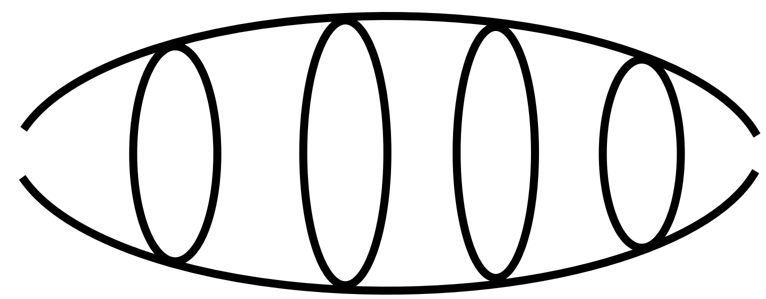


These Witten diagrams give CFT correlation functions. If we were to compute the above diagram, it could be expressed in terms of the 5-point blocks we found.

However, we would like an actual CFT computation.

We could imagine the bulk particle as a composite of two particles on the boundary.

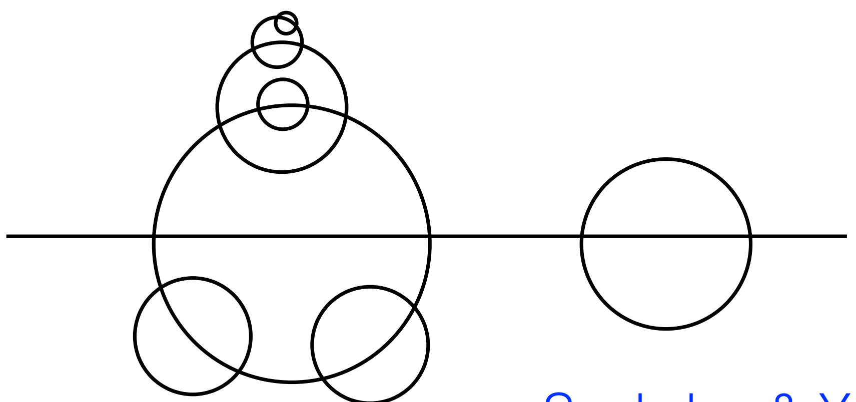




The Witten diagram for the two point function translates into a CFT 4-point function of fundamentals (2-point function of bilinears)

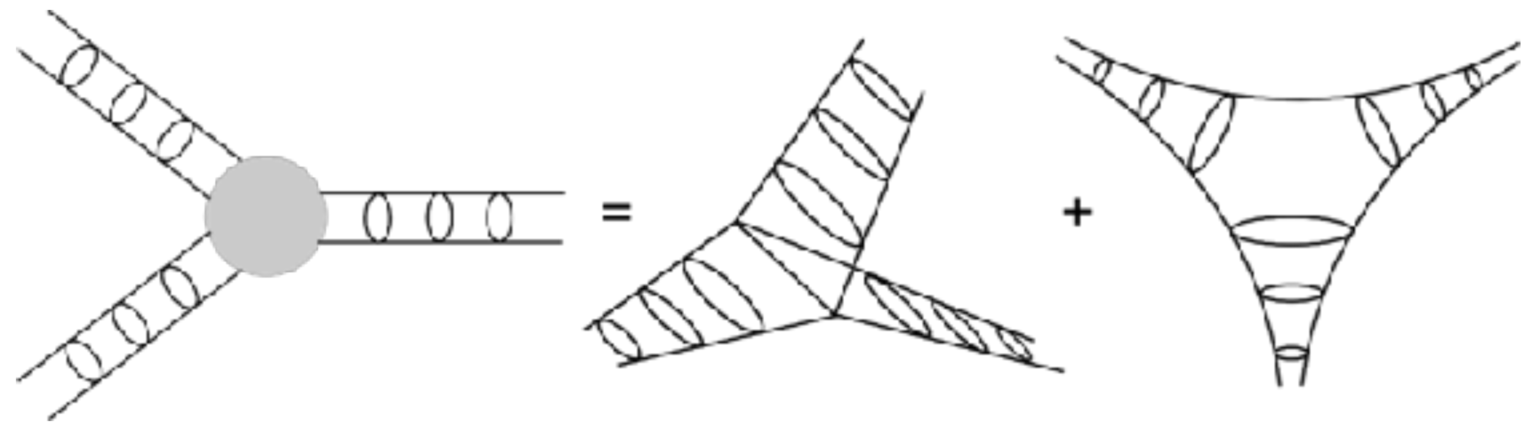
We could imagine there being a CFT in which higher-point correlation functions are built out of four-point functions. Something that parallels tree level Witten diagrams in the bulk.

We can compute all large N correlation functions in SYK by summing all Feynman diagrams.

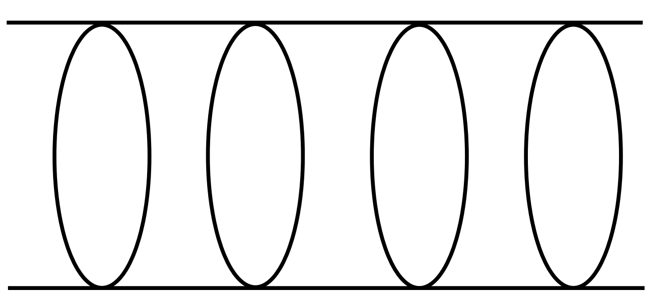


Sachdev & Ye

2-pt: Melons -> Conformal in IR

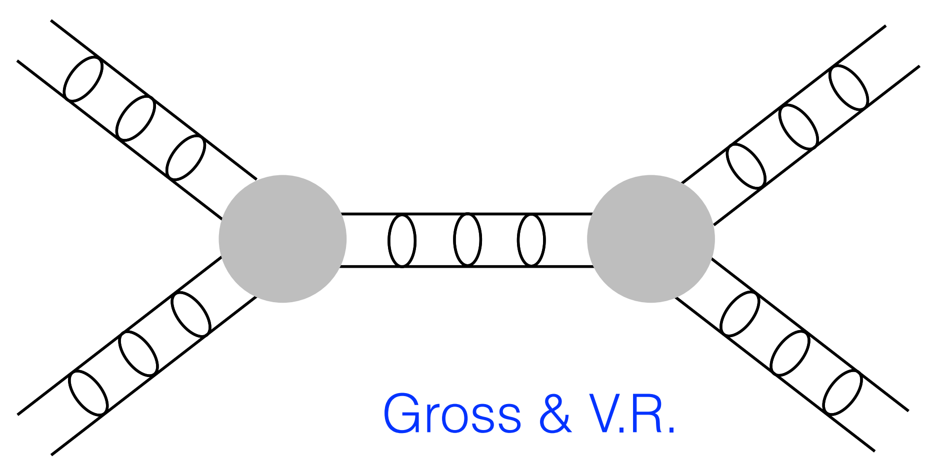


6-pt: glue three 4-pt functions



Kitaev; Polchinski & V.R.; Maldacena & Stanford

4-pt: Ladders: geometric sum



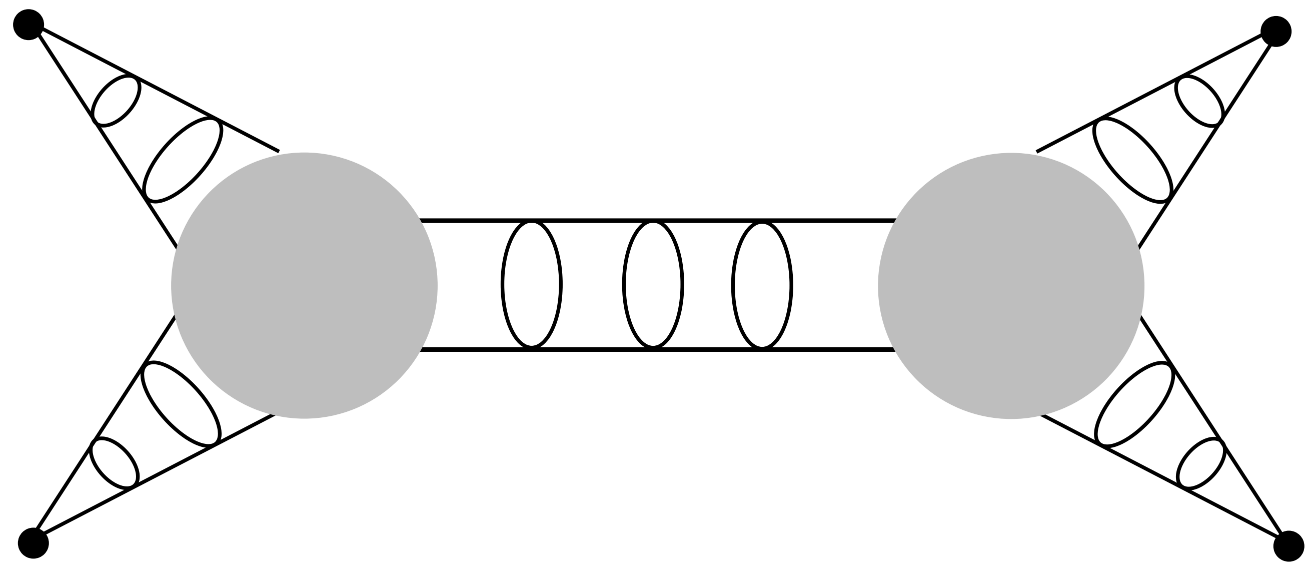
+ cross-channels
- no exchanged melons
+ 4-pt contact

Gross & V.R.

8-pt: glue 4-pt functions

(the lines on the higher-point functions are really dressed propagators)

One can derive a simple formula for the diagrams appearing in the higher point functions.



In 1d, h is dimension

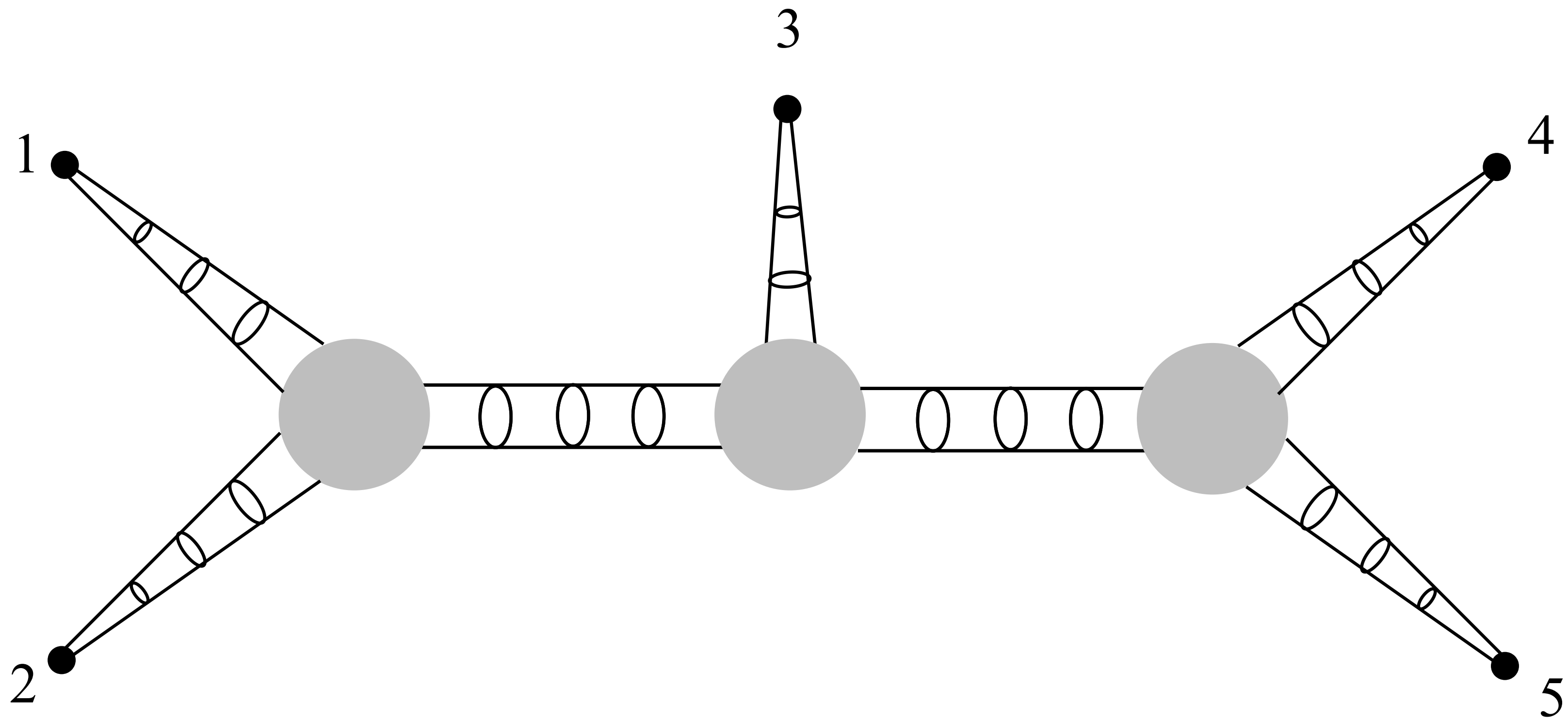
$$\int_{\mathcal{C}} \frac{dh}{2\pi i}\, \widetilde{\rho}(h)\, c_{h_1 h_2 h}\, c_{h_3 h_4 h}\, G_h^{h_1,h_2,h_3,h_4}(\tau_i)$$

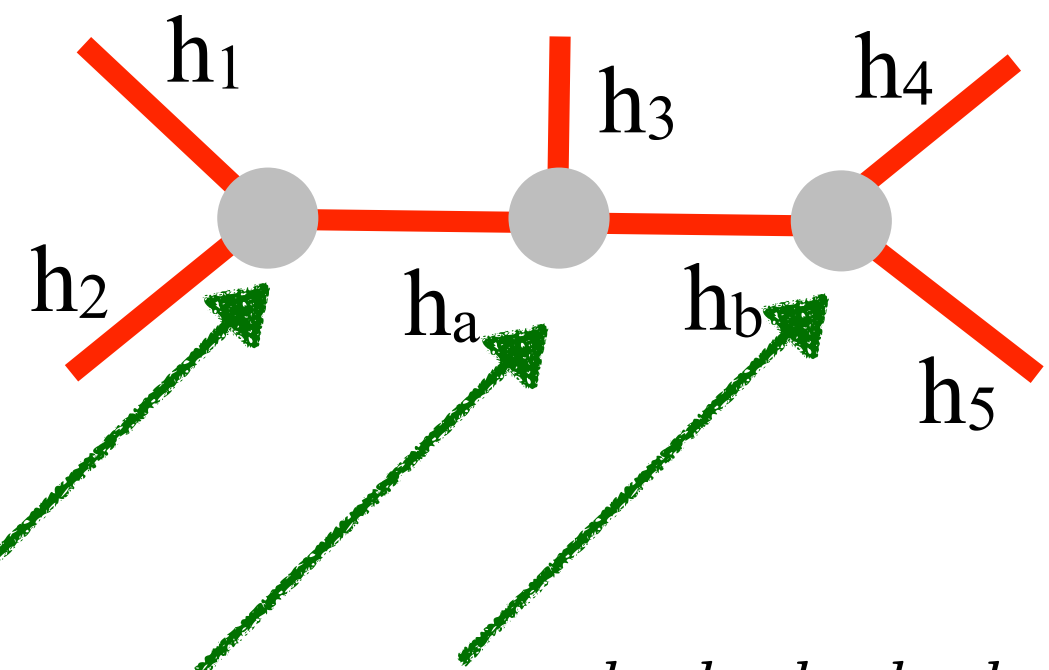
4-pt function fundamentals (sum of ladders)

3-pt function bilinears

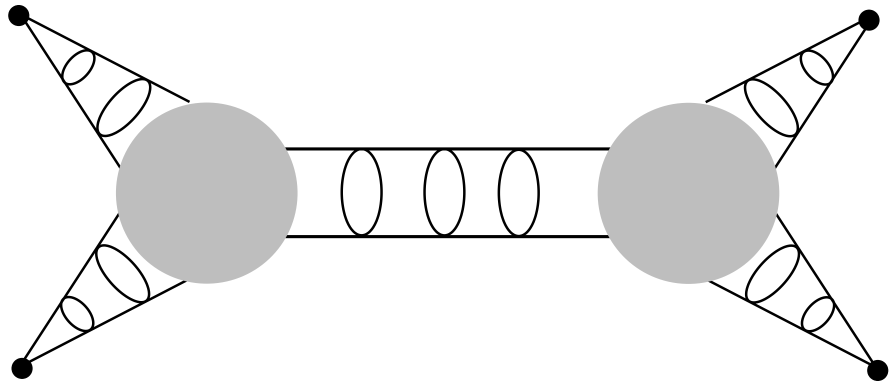
Conformal Block

Gross & V.R.





$$\int_{\mathcal{C}} \frac{dh_a}{2\pi i}\, \widetilde{\rho}(h_a) \int_{\mathcal{C}} \frac{dh_b}{2\pi i}\, \widetilde{\rho}(h_b)\; c_{h_1 h_2 h_a} c_{h_a h_3 h_b} c_{h_b h_4 h_5}\; G^{h_1,h_2,h_3,h_4,h_5}_{h_a,h_b}(\tau_i)$$

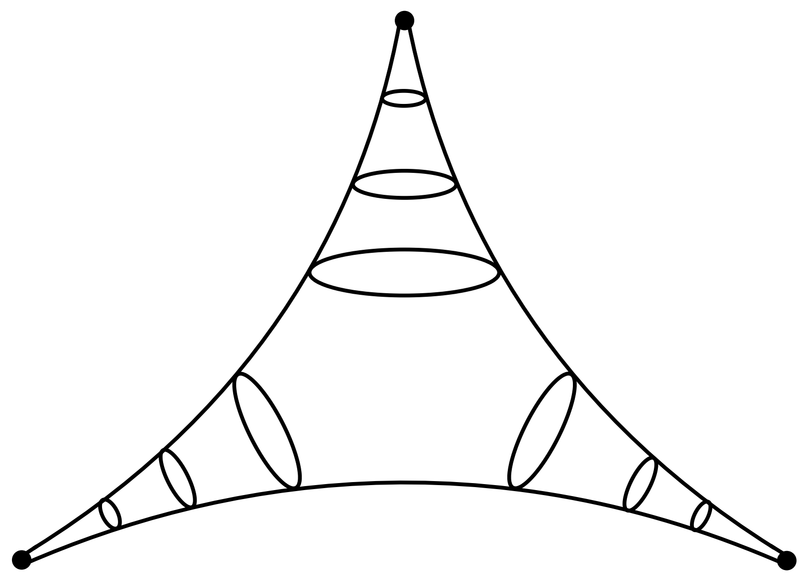


$$\int_{\mathcal{C}} \frac{dh}{2\pi i}\, \widetilde{\rho}(h)\, c_{h_1 h_2 h}\, c_{h_3 h_4 h}\, G_h^{h_1,h_2,h_3,h_4}(\tau_i)$$

- These are simple rules for summing an infinite number of diagrams. It doesn't matter that the four-point function is made up of ladders. These apply to any four-point functions.

- This is not just an OPE expansion. The $c_{h_1 h_2 h_3}$ are the analytically extended OPE coefficients of the single-trace operators. The four-point function is a sum of conformal blocks of single-trace operators and double-trace operators. This emerges upon closing the contour.

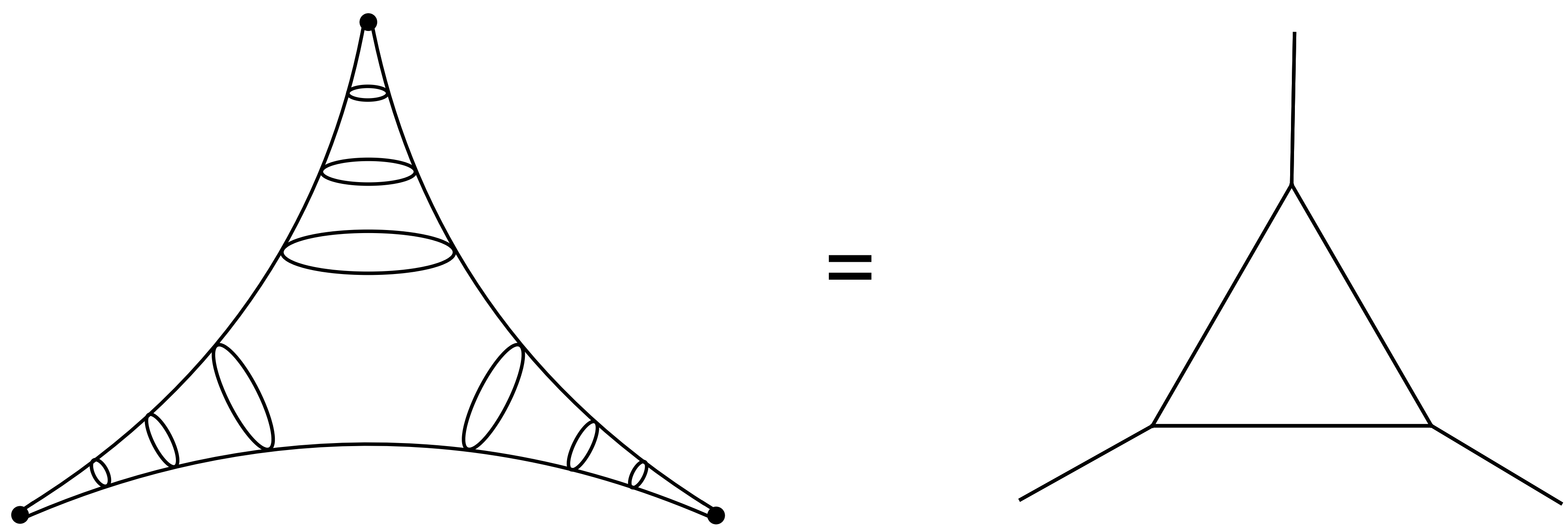
Three-point function of bilinears in SYK



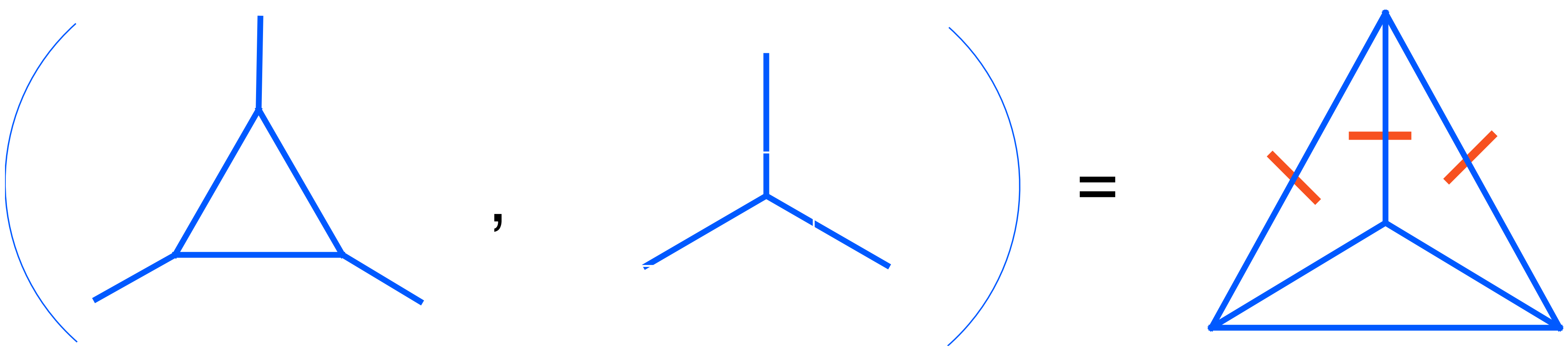
The planar diagram contribution

The melons are not important here. What is important is that we are gluing three 3-point functions.

In the notation from before, with each vertex denoting a three-point function,

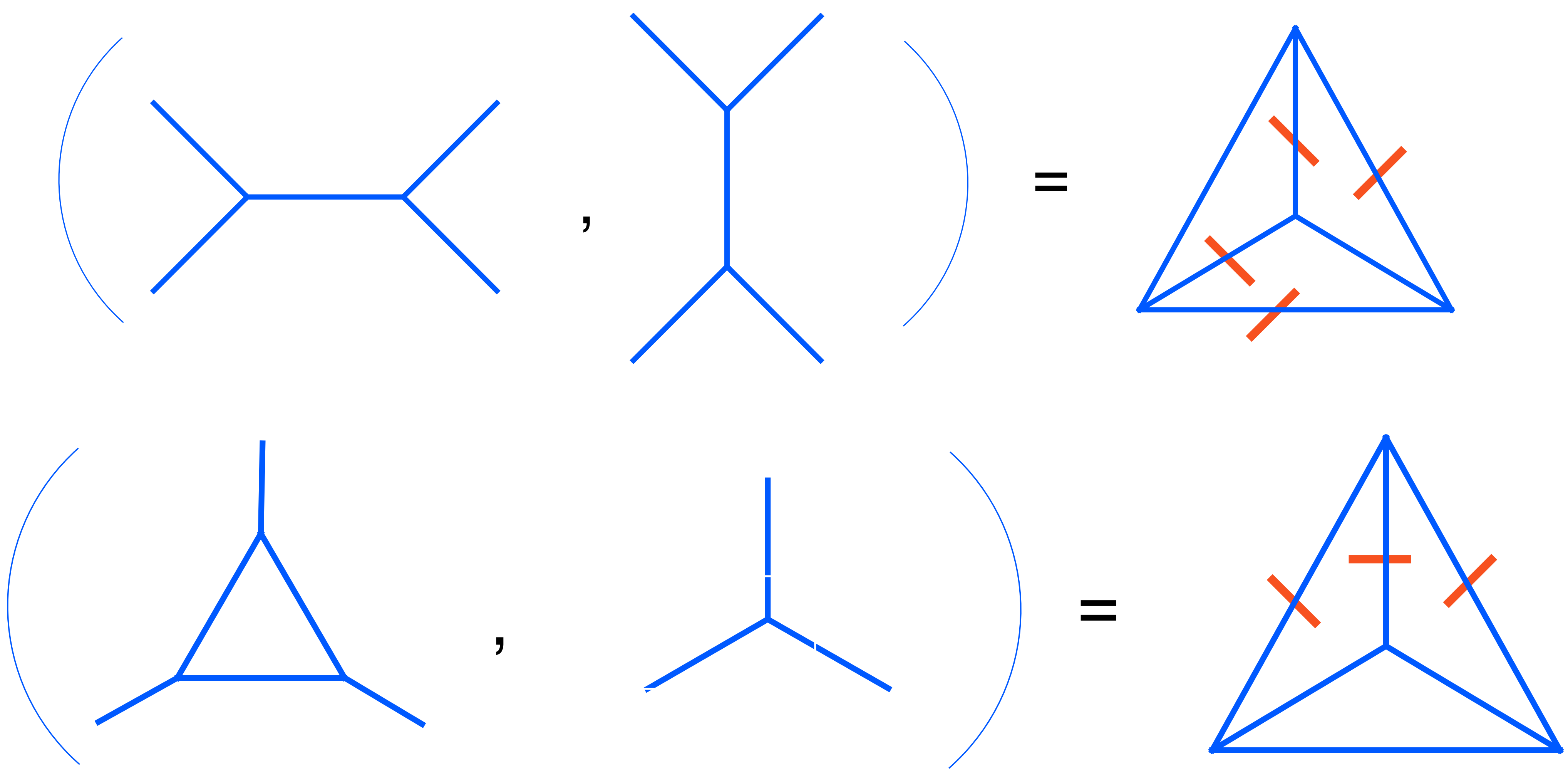


The functional form of a three-point function is fixed by conformal symmetry. We can extract the coefficient by contracting with a (bare) three-point function of shadow operators,



This is a tetrahedron: a 6j symbol

The overlap of two partial waves - a group theoretic quantity- and the planar Feynman diagrams in an SYK correlation function - a dynamical quantity- are just two different ways of splitting a tetrahedron



Summary

SYK is a solvable, strongly coupled, large N CFT.

It serves as a model of AdS/CFT, of black holes, and as a model of strange metals, among other applications. This is something we have not discussed, but has been discussed in many of the other talks.

SYK is solvable because i) as a CFT, all n-point correlation functions are expressed in terms of n-point conformal blocks. ii) The Feynman diagrams for n-point functions are built out of 4-point functions glued together.

Every solvable model is simple when viewed in a particular way.

Our formula for the n-point functions encodes the simplicity of SYK.

More generally, our formula applies to any conformal theory where the Feynman diagrams for n-point functions are built by gluing together 4-point functions.

The new CFT techniques entering the solution are in themselves interesting, and should have a number of applications.

Future

A mystery of AdS/CFT that remains is a clear understanding of how the bulk emerges from the CFT.

$\mathcal{N}$=4 Super Yang-Mills is integrable, so it would seem one should be able to answer the question in that context. Indeed, significant progress has been made. But the fact that it is not easily solvable, or in a direct way (by summing Feynman diagrams) seems to be a limitation.

SYK is easily solvable, as we saw in this talk. However, the solution doesn't seem to tell us what the bulk theory is, or if there even is one.

It would be useful to have a solvable model that is harder than SYK, but easier than $\mathcal{N}$=4 Super Yang-Mills , and preferably related to string theory.