Lessons from $^3\text{He}$

A. J. Leggett

UIUC

Normal state of $^3\text{He}$: Fermi liquid, $k_F \sim 1\text{ Å}^{-1}$

Intratomic potential:

$k_F r_0 \sim 2.5$

$\Rightarrow$ s-wave pairing

suppressed by hard core,

$V_{kk'} \sim f(k,k') \sum \xi_{k} \rho_{k}(\xi_{k'})$

slowly varying, can approximate $k = k' = \frac{1}{2}$

$V_0 > 0, V_1$ (and $V_2$?) $< 0$

$\Rightarrow$ pair form in $l = 1$ state (well verified

in expr., e.g. ultrasound abs.)

What does this mean?

Yang: $g_2(\mathbf{r},\mathbf{r}',\mathbf{r}'',\mathbf{r}''') = \sum \xi_k \chi_0(\mathbf{r},\mathbf{r}'') \chi_0(\mathbf{r}',\mathbf{r}''')$

if $N_0 \sim N$, $\chi_0(\mathbf{r},\mathbf{r}') = \chi_0(\mathbf{0},\mathbf{0})$

pair $\leftrightarrow \sim Y_{1m}(\xi)$
ABM phase ($^3\text{He}-\Lambda$):

\[ \chi_0(\mathbf{p}, \mathbf{q}; \sigma_0, \sigma_2) = (\sin \theta - \text{c.c.}) \cos \omega t \]

The $\not c + \not K$ question: what is $\langle L \rangle$?

\[ \theta: \text{not "obvious" that it is } N\hbar/2 \]

[cf: $^3\text{He}-\Lambda$, all pairs form in $\uparrow \downarrow$ state, but $\langle S \rangle \neq N$!]

"Polar rot." describes these particles which are condensed, not liquid or a chain!

In literature, find at least:

1. $\langle L \rangle = N\hbar/2$
2. $\langle L \rangle \approx N\hbar/2 \times (\Delta/\epsilon_F)$
3. $\langle L \rangle \approx N\hbar/2 \times (\Delta/\epsilon_F)^3$

Complication: boundary and $\not c' + \not f'$ on $\not f'$!
"Naive" calculation (ignore boundary conditions):

(a) BCS:

\[ \Psi_{\text{BCS}} = \prod_b (u_b + v_b a_b^+ a_{-b}^+) \ket{\text{vac}} \]

\[ u_b = |u_b|, \quad v_b = |v_b| \exp \frac{i}{\hbar} \phi_b \]

\[ \langle N \rangle = \sum_b |v_b|^2 \Rightarrow \text{infinite deg. from } N/2 \text{ pairs} \]

\[ w_\alpha, m_m \Rightarrow \langle N \rangle = N \hbar/2 \]

Formally, take N-particle project of \( \Psi_{\text{BCS}} \):

\[ \Psi_N = \langle \text{const.} \rangle \hat{\Psi}^{N/2} \ket{\text{vac}} \]

\[ \hat{\Psi} = \sum_b c_b a_{-b}^+ a_b^+ \]

\[ c_b = \frac{v_b}{u_b} \exp \frac{i}{\hbar} \phi_b \]

\[ [\hat{L}_z, \hat{\Psi}] = -\imath \hbar \sum_b \left( \frac{\partial c_b}{\partial \phi_b} \right) a_{-b}^+ a_{b}^+ = \hbar \hat{N} \]

\[ \Rightarrow \quad \hat{L}_z \Psi_N = \frac{N \hbar}{2} \Psi_N \]

Note: results in 5th of \( \Delta \)!

\[ \Rightarrow \text{macroscopic discontinuity in } \langle L \rangle \]

at \( \Delta = 0 \).

(? limits \( \Delta \to 0, N \to \infty \)
\[ \text{do not commute?} \]?)
(6) alternative approach: consider 

\[ \Psi_n' = (\hat{\Psi}^+)^n (\hat{\Psi}^-)^n \quad |FS\rangle \]

with appropriate spin indices

\[ \hat{\Psi}^+ = \sum_{k>0} c_k a_k^+ a_k \]
\[ \hat{\Psi}^- = \sum_{k<0} d_k a_{-k}^+ a_k \]
\[ \begin{align*}
N_+ &= \sum_{k>0} \frac{16k^2}{1+16k^2} \\
N_- &= \sum_{k<0} \frac{16k^2}{1+16k^2}
\end{align*} \]

This gives some value of K.E. as \( \Psi_n \)

It does not as it should give some value of P.E.

But, it does not allow scattering of a pair of particles into two states and vice versa. To remedy this, set for the moment \( N_+ = N_- = 1 \) and consider

\[ \Psi_n'' = \sum_{f_f} f_f (\hat{\Psi}^+)^f (\hat{\Psi}^-)^f |FS\rangle \quad \sum_f f_f^2 = 1 \]

with \( f_f \) being varying. Irrespective of form of \( u_k \), this gives some value of both K.E. and P.E. as \( \Psi_{0,2} \).

In fact, in some case it is simply a trivial renormalization of \( \Psi_n \) (\( \sim \Psi_{0,2} \)). But...
WHAT IS VALUE OF \( \langle L \rangle \) FOR "ALTERNATIVE" W.F. IN \( L \neq 0 \) CASE?

Ans.: \( \langle L \rangle = \frac{\alpha}{2} (\langle N_+ \rangle - \langle N_- \rangle) \)

and so in limit \( \langle N_+ \rangle = \langle N_- \rangle \) (\( \Delta \to 0 \))
\( \langle L \rangle = 0 \)

no discontinuity at \( \Delta = 0 \! \)!

Conjecture: for noninfinitesimal \( \Delta \),
\[ N_+ - N_- = N_{\nu} - N_{\bar{\nu}} \]

\((N_{\nu} \equiv \text{no. of states within Fermi sea with a kp given by } \frac{2\pi k}{\hbar} = \nu \text{. If so, then for small } \Delta \)
\[ \langle L \rangle \sim \frac{N_{\nu}}{2} (\Delta/\nu \pi)^2 \quad (\text{since } \mu(\Delta) - \epsilon_\nu \sim (\Delta/\nu \pi)^2) \]

but for \( \mu < 0 \) ("BEC" side of crossover)
\[ \langle L \rangle = N_{\nu}/2. \]

as expected.