Selection mechanism for rotating patterns in weakly excitable media

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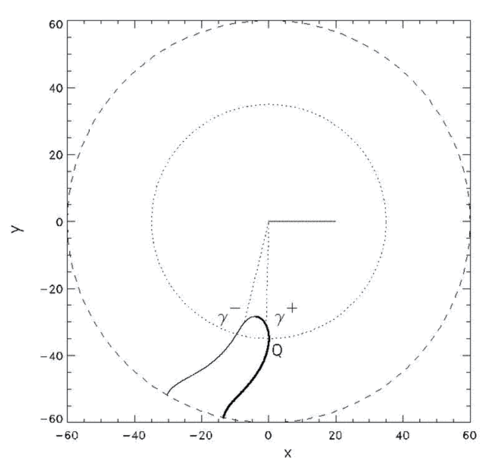
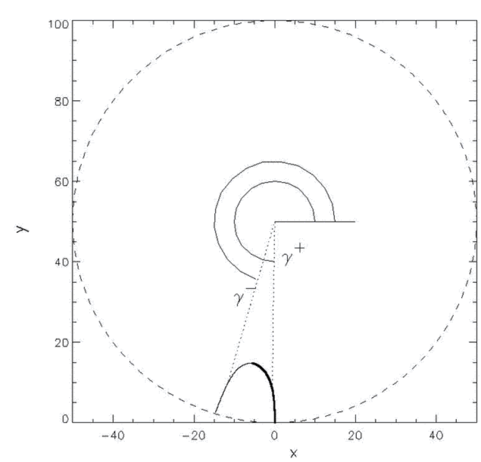
- Introduction
- Reaction-diffusion model
- Free-boundary description
- Spot pinned to the no-flux boundary of a disk
- Spiral wave within a disk
- Spiral wave in an unbounded medium
- Summary

Two types of rotating patterns



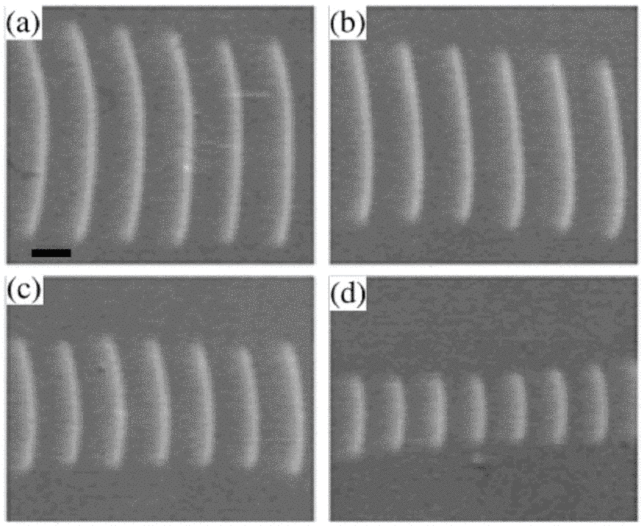
Spiral wave rotating within a disk



Excited sport pinned to the disk boundary

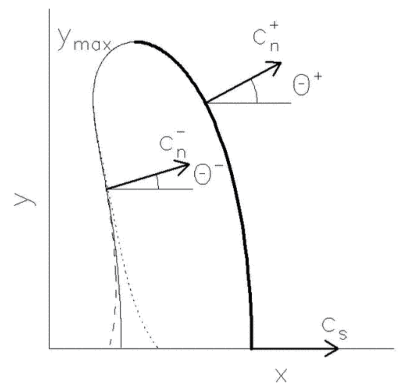
Stabilized wave segments

$\phi_0 = -0.0744\,mWcm^{-2}$ $\phi_0 = -0.0248\,mWcm^{-2}$



$\phi_0 = 0.0248\,mWcm^{-2}$ $\phi_0 = 0.0744\,mWcm^{-2}$

Mihaliuk, Sakurai, Chhirila, Showalter, PRE, 2002



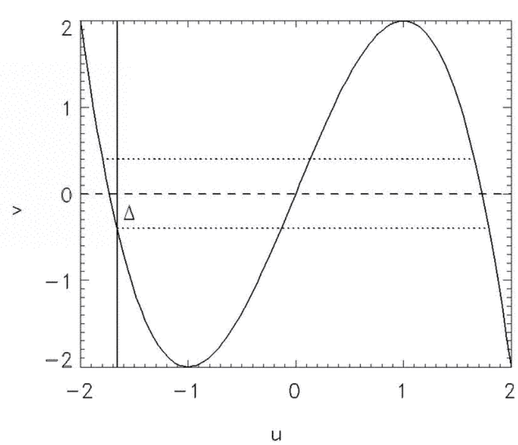
“Critical finger” Karma, PRL, 1991;
” Wave segments”
Zykov, Showalter, PRL, 2005

Excitable medium model

$$\frac{\partial u}{\partial t} = D\nabla^2 u + F(u,v),$$
$$\frac{\partial v}{\partial t} = \varepsilon G(u,v),$$

$$F(u,v) = 3u - u^3 - v,$$
$$G(u,v) = u - \delta$$

$$D = 1, \quad \delta = -1.63,$$
$$0.0068 < \varepsilon < 0.0111$$
$$\Delta = 0.56$$



Wave front velocity

$$\xi = x - c_0 t$$

$$u^3 - 3u + v_f$$

$$\frac{d^2u}{d\xi^2} = (u-u_1)(u-u_2)(u-u_3) - c_0\frac{du}{d\xi}$$

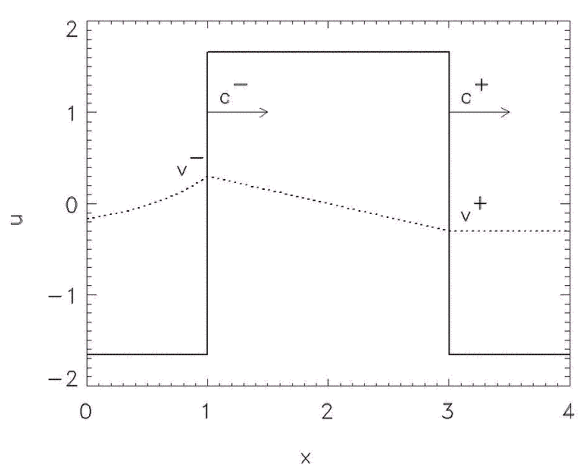
$$u \to u_3, \xi \to -\infty; \quad u \to u_1, \xi \to \infty.$$

$$\frac{du}{d\xi} = \eta(u-u_1)(u-u_3) \quad \Rightarrow \quad \eta = (2D)^{-1/2}, \quad c_0 = \sqrt{D/2}(u_1+u_3-2u_2)$$

$$v^* - v_f = \Delta << 1$$

$$u_1 \approx -\sqrt{3} + \frac{\Delta}{6}; \quad u_2 \approx -\frac{\Delta}{3}; \quad u_3 \approx \sqrt{3} + \frac{\Delta}{6} \quad \Rightarrow \quad c_0 = \sqrt{\frac{D}{2}}\Delta = \alpha(v^* - v_f)\sqrt{D}$$

Impulse in a 1-D medium



$$c^- = -c^+ \quad \Rightarrow \quad v^- = -v^+$$

$$v^- = v^+ + \frac{G^*\varepsilon}{c^+}(x^+ - x^-) \quad \Rightarrow \quad x^+ - x^- = -\frac{2v^+c^+}{G^*\varepsilon}$$

Velocity – curvature relationship

$$\frac{\partial u}{\partial t} = F(u) + D\frac{\partial^2 u}{\partial r^2} + \frac{D}{r}\frac{\partial u}{\partial r} \quad \Rightarrow \quad \frac{\partial u}{\partial t} = F(u) + D\frac{\partial^2 u}{\partial r^2} + \frac{D}{R}\frac{\partial u}{\partial r}$$

$$\xi = r - c(R)t$$

$$D\frac{d^2 u}{d\xi^2} = -F(u) - [c(R) + \frac{D}{R}]\frac{du}{d\xi}$$

$$c(R) + \frac{D}{R} = c_0 \quad \Rightarrow \quad c(R) = c_0 - \frac{D}{R} \quad \Rightarrow \quad c = c_0 - Dk$$

Spot pinned to the boundary of a disk

Kinematical equations

$$\frac{dc_n}{ds} = \omega - kc_\tau$$

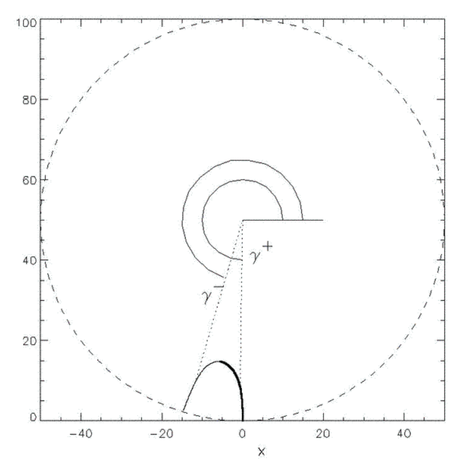
$$\frac{dc_\tau}{ds} = kc_n$$

Scaling

$$C = c/c_0, \quad S = sc_0/D$$

$$K = kD/c_0, \quad \Omega = \omega D/c_0^2$$

$$R = \frac{r_d c_0}{D}, \quad B = \frac{G^* \varepsilon}{\alpha^2 \Delta^3}$$



Wave front

$$\frac{dC_n^+}{dS^+} = \Omega - K^+ C_\tau^+$$

$$\frac{dC_\tau^+}{dS^+} = K^+ C_n^+$$

$$C_n^+ = 1 - K^+$$

$$K^+(S_{DR}^+) = K_{DR}, \quad C_\tau^+(S_{DR}^+) = 0$$

$$\Omega = (1 - K_{DR})/R$$

Wave back

$$\frac{dC_n^-}{dS^-} = \Omega - K^- C_\tau^-$$

$$\frac{dC_\tau^-}{dS^-} = K^- C_n^-$$

$$C_n^- = 1 - K^- - \frac{B}{\Omega}(\gamma^+ - \gamma^-)$$

$$C_n^-(0) = 0, \quad C_\tau^-(0) = C_t$$

$$C_\tau^-(S_R^-) = 0$$

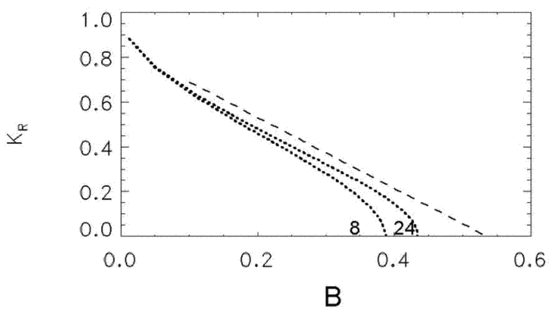
Selection mechanism

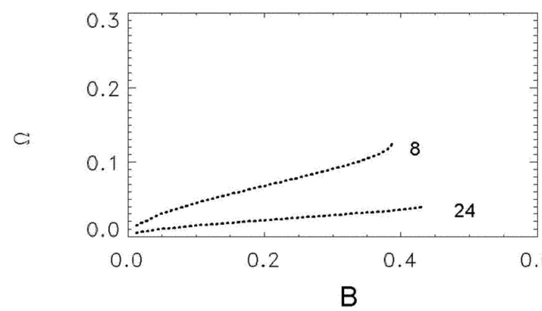
$$C_t = C_t(K_{DR}, R)$$

$$B = B_{RK}^*(C_t, R)$$

$$B = B_{RK}(K_{DR}, R)$$

Selected values





$$R \to \infty \Rightarrow K_R = (B_c - B)/\beta$$

$$\Omega(B) = (1 - K_R)/R$$

Spiral wave within a disk

Outer part of the front

$$\frac{dC_n^+}{dS^+} = \Omega - K^+ C_\tau^+$$

$$\frac{dC_\tau^+}{dS^+} = K^+ C_n^+$$

$$C_n^+ = 1 - K^+$$

$$K^+(S_Q^+) = K_Q, \quad C_\tau^+(S_Q^+) = 0$$

$$C_\tau^+(S_R^+) = 0 \Rightarrow \Omega = \Omega(K_Q, R)$$

Inner part of the front

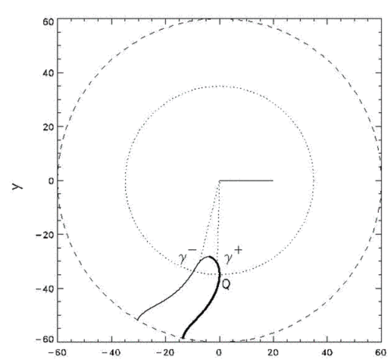
$$\frac{dC_n^+}{dS^+} = \Omega - K^+ C_\tau^+$$

$$\frac{dC_\tau^+}{dS^+} = K^+ C_n^+$$

$$C_n^+ = 1 - K^+$$

$$K^+(S_Q^+) = K_Q, \quad C_\tau^+(S_Q^+) = 0$$

$$C_n^+(0) = 0 \Rightarrow C_t = C_t(K_Q, \Omega)$$



Wave back

$$\frac{dC_n^-}{dS^-} = \Omega - K^- C_\tau^-$$

$$\frac{dC_\tau^-}{dS^-} = K^- C_n^-$$

$$C_n^- = 1 - K^- - \frac{B}{\Omega}(\gamma^+ - \gamma^-)$$

$$C_\tau^+(0) = C_t$$

$$C_n^+(0) = 0$$

$$C_\tau^+(S_R^-) = 0$$

$$B = B_{RS}^*(C_t, \Omega, R)$$

Selection mechanism

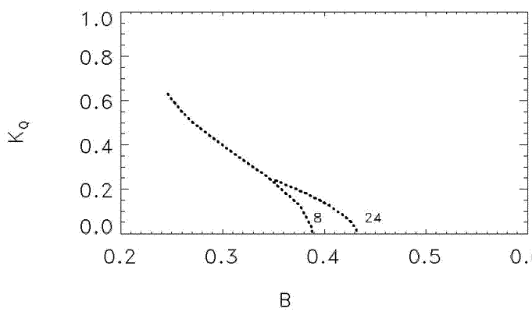
$$\Omega = \Omega(K_Q, R)$$

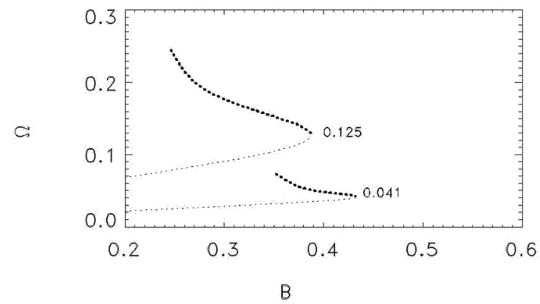
$$C_t = C_t(K_Q, R)$$

$$B = B_{RS}^*(C_t, \Omega, R)$$

$$B = B_{RS}(K_Q, R)$$

Selected values





$$\Omega = (1 - K_{DR}) / R_D$$

Spiral wave in an unbounded medium

Outer part of the front

$$\frac{dC_n^+}{dS^+} = \Omega - K^+ C_\tau^+$$

$$\frac{dC_\tau^+}{dS^+} = K^+ C_n^+$$

$$C_n^+ = 1 - K^+$$

$$K^+(S_Q^+) = K_Q, \quad C_\tau^+(S_Q^+) = 0$$

$$C_n^+(\infty) = 1 \Rightarrow \Omega = \Omega(K_Q)$$

Inner part of the front

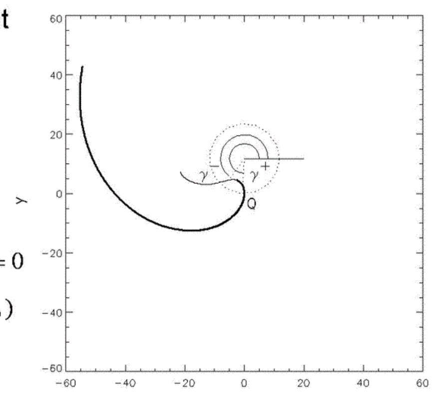
$$\frac{dC_n^+}{dS^+} = \Omega - K^+ C_\tau^+$$

$$\frac{dC_\tau^+}{dS^+} = K^+ C_n^+$$

$$C_n^+ = 1 - K^+$$

$$K^+(S_Q^+) = K_Q, \quad C_\tau^+(S_Q^+) = 0$$

$$C_n^+(0) = 0 \Rightarrow C_t = C_t(K_Q)$$



Wave back

$$\frac{dC_n^-}{dS^-} = \Omega - K^- C_\tau^-$$

$$\frac{dC_\tau^-}{dS^-} = K^- C_n^-$$

$$C_n^- = 1 - K^- - \frac{B}{\Omega}(\gamma^+ - \gamma^-)$$

$$C_\tau^+(0) = C_t$$

$$C_n^+(0) = 0$$

$$C_n^+(\infty) = 1$$

$$B = B_s^*(C_t, \Omega)$$

Selection mechanism

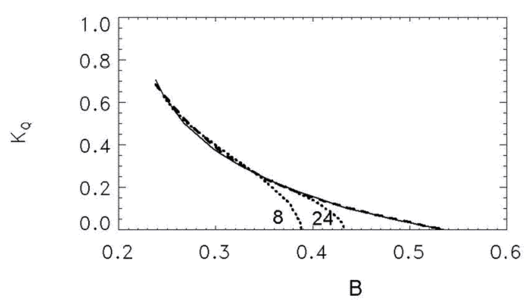
$$\Omega = \Omega(K_Q)$$

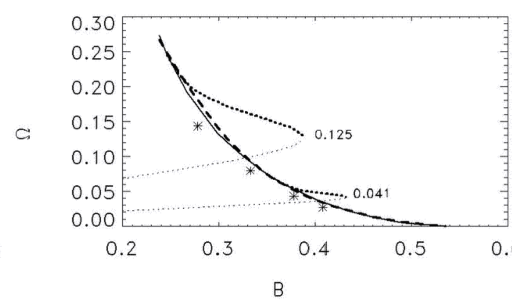
$$C_t = C_t(K_Q, \Omega)$$

$$B = B_s^*(C_t, \Omega)$$

$$B = B_S(K_Q)$$

Selected values

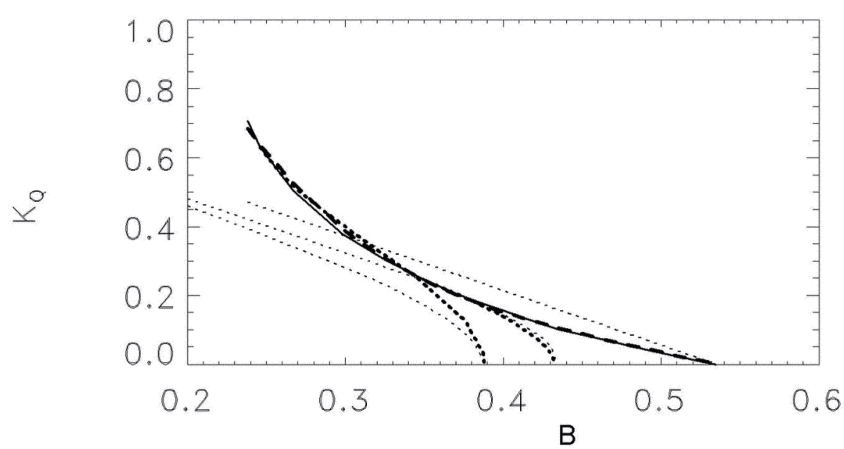




$$K_Q = (B_c - B) + 50(B_c - B)^4$$

$$\Omega(K_Q) = 0.685K_Q^{3/2} - 0.06K_Q^2 - 0.293K_Q^3$$

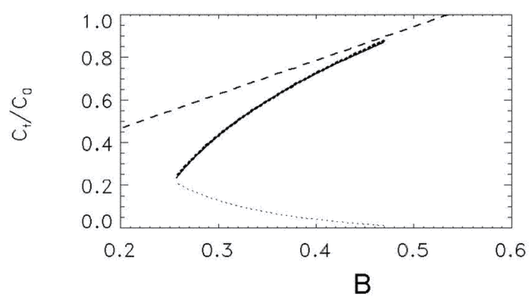
Selected values

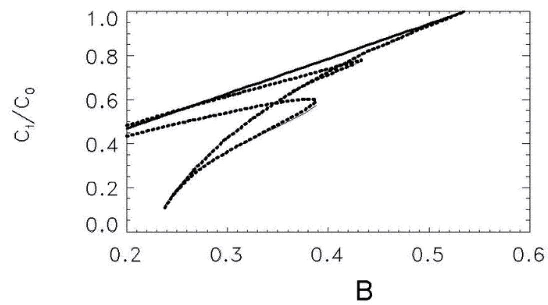


$K(S_{DR}^{+}) > 0 \Rightarrow$ spot

$K(S_{DR}^{+}) < 0 \Rightarrow$ spiral

Velocity of the spiral wave tip



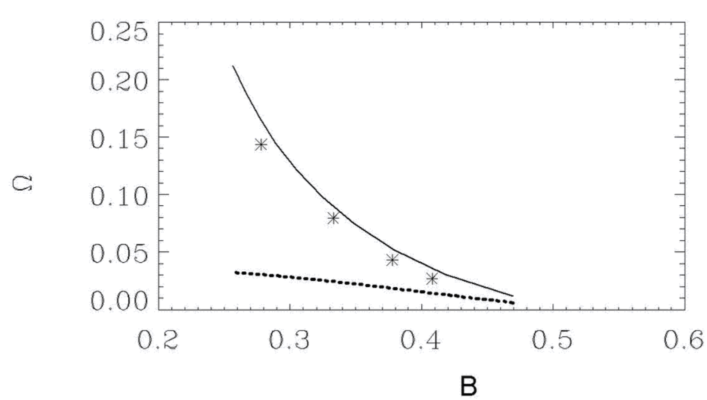


Spirals $\frac{c_t}{c_0} = 1-(B_c - B_s)/\beta - 1.5\Omega$

Spots $\frac{c_t}{c_0} = 1-(B_c - B_s)/\beta - 1.5\Omega + \Omega(1/R - \Omega)\chi(R),$

$\chi(R) \approx 2.5R$

Rotation frequency of an unbounded spiral wave



$$\frac{c_t}{c_0} = 1-(B_c - B)/\beta$$

$$R_0 = \frac{D}{c_0}\left[\frac{b\beta}{(B_c - B)}\right]^{3/2}$$

$$\Omega = \left(\frac{B_c - B}{b\beta}\right)^{3/2}$$

$$(b\beta)^{-3/2} = 0.395$$

Hakim, Karma, 1999

Summary

-Free-boundary model reveals the selection principle which determines the shape and the rotation frequency of excited spots and spiral waves rotating within a disk vs. the medium excitability

-It is shown that excited spots and spiral waves rotating within a disk represent two coexisting solutions, bifurcating from a critical point

-Spiral waves in an unbounded medium and stabilized wave segments represent limiting cases of these two coexisting solutions

-The proposed theoretical description is universal since it containes only the general characteristics of a two-component R-D model