Algebraic Approach to Dualities

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KITP "Beyond Optical Lattices" - November 2010

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Unified Framework to Quantum and Classical Dualities

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State of Affairs

The notion has been around for a very long time, and it has proven to be of great importance in Statistical Mechanics, Quantum Field Theory, and many other fields. Still, the best description available was:

"Dualities are certain mathematical transformations"

To be more precise some authors would add:

"Transformations like the ones considered by Kramers and Wannier in their study of the Ising model"

It is not clear how to find or derive dualities and whether classical and quantum versions are related or not

State of Affairs

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So, What are Dualities?

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So, What are Dualities?

Unitary Maps of Bond algebras
Local to Local

Why Dualities?

- Phase transitions and diagrams: Self-dual $\leftrightarrow$ Critical Points
- Nature of (topological) excitations: Classification?
- Connect seemingly unrelated (dual) (lattice or field)theories:
  Orbital ordered and superconducting theories
  AdS/CFT: Conjectured
- Allow exact solutions in special cases:
  Kitaev Toric Code model at finite T
  Kitaev honeycomb model in topological sectors

Why Dualities?

- Dimensional reduction and TQO
- Find simpler classical actions
- Numerical Simulations:
  Stochastic, Hierarchical Mean-fields or Renormalization Group Methods
- Unravel the power of Quantum Computation?

Old friends from the Zoo of Dualities

Classical Dualities

Wisdom: "Symmetries" of equations of motion

Maxwell's equations in empty space (vacuum)

$$\nabla \times E = -\dot{B} \qquad \nabla \times B = \dot{E}$$

$$\nabla \cdot E = 0 \qquad \nabla \cdot B = 0$$

CLASSICAL ELECTROMAGNETIC DUALITY

$$E \to -B \qquad B \to E$$

Things that look different are equivalent and interchangeable

Classical Dualities
(Classical Stat Mech and Field Theory)

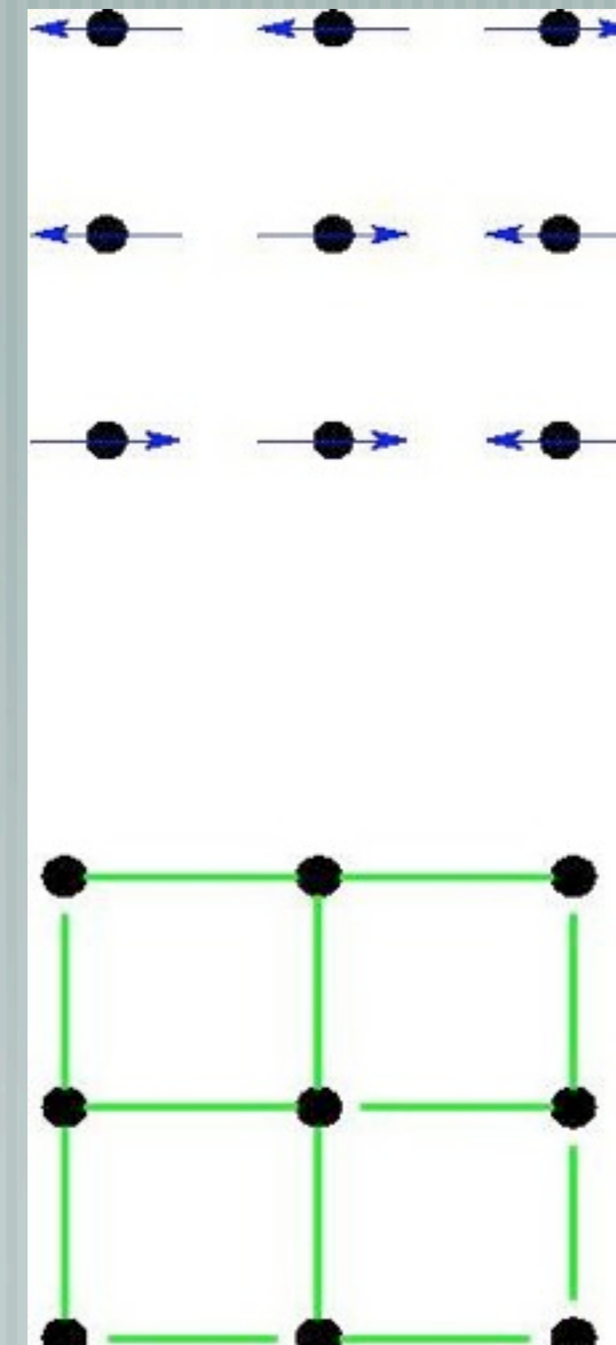
Wisdom: Low-temperature-to-High-temperature relations

Kramers-Wannier Self-Duality of the D=2 Ising Model

$$\mathcal{Z}(K) = \sum_{\sigma} \exp\Big[K \sum_{\langle i,j \rangle} \sigma_i \sigma_j\Big]$$

$$\sigma_i = 1 \text{ or } -1$$

$$K = \beta J \qquad \beta = 1/k_B T$$



has a remarkable property:

it is self-dual,

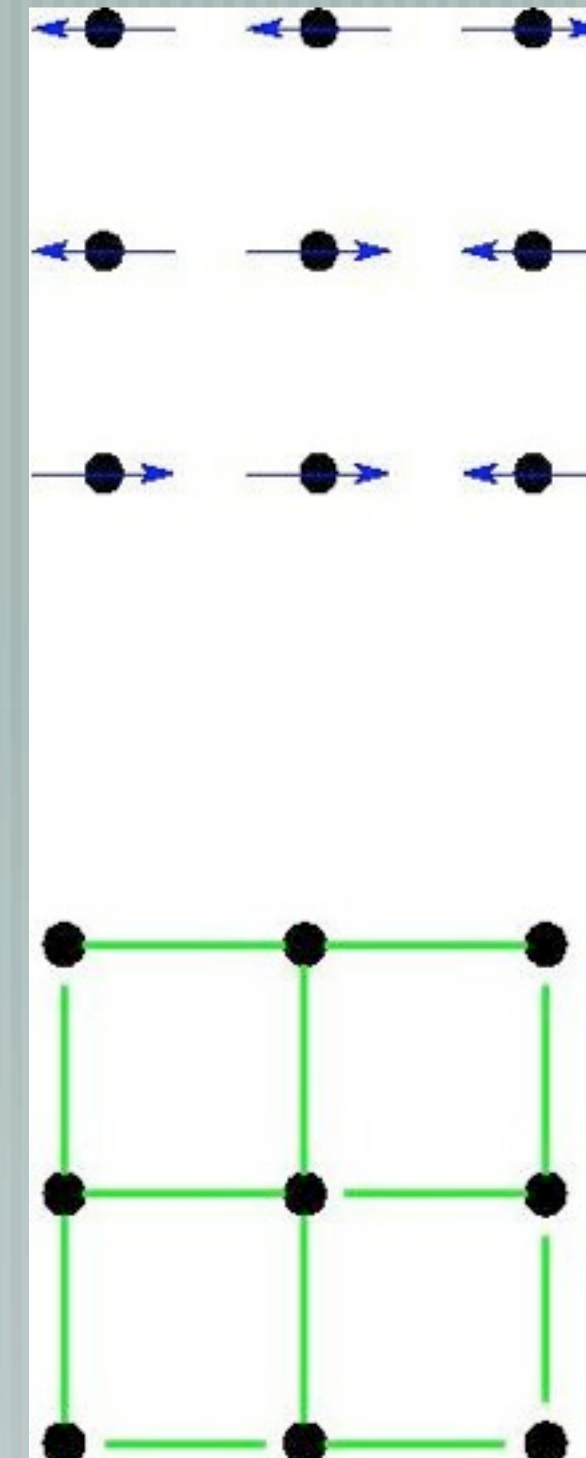
meaning...

KW SELF-DUALITY RELATION

$$\frac{\mathcal{Z}(K)}{\sinh(2K)^{N^2/2}} = \frac{\mathcal{Z}(K^*)}{\sinh(2K^*)^{N^2/2}}$$

whenever $\sinh(2K)\sinh(2K^*) = 1$

CONCEPT:
High-T ⟷ Low-T relation



The critical point is located at the self-dual point: $K = K^* = K_c$

$$K_c = \frac{1}{2}\ln(1+\sqrt{2})$$

Quantum Dualities

Wisdom: Strong-coupling-to-Weak-coupling relations

Quantum dualities

$$H_{\mathrm{IC}} = \sum_i j\sigma_i^z \sigma_{i+1}^z + h\sigma_i^x = \sum_i h\mu_i^z \mu_{i+1}^z + j\mu_{i+1}^x$$

$$\mu_i^x = \sigma_{i-1}^z \sigma_i^z \qquad \mu_i^z = \sigma_i^x \sigma_{i+1}^x \sigma_{i+2}^x \sigma_{i+3}^x \cdots$$

The new operators are spin-1/2 operators as well, thus it has to be that

$$E_{\mathrm{IC}}(j,h) = E_{\mathrm{IC}}(h,j)$$

CONCEPT:
Strong-coupling ⟷ Weak-coupling relation

"Hand-waving" approach to quantum dualities

This is the traditional approach to Quantum Self-Dualities and Dualities

Idea: if you suspect a connection, try to prove it by GUESSING an OPERATOR (VERY NON-LOCAL) MAPPING, and good luck in finding it!!!!

Particle-Wave Duality

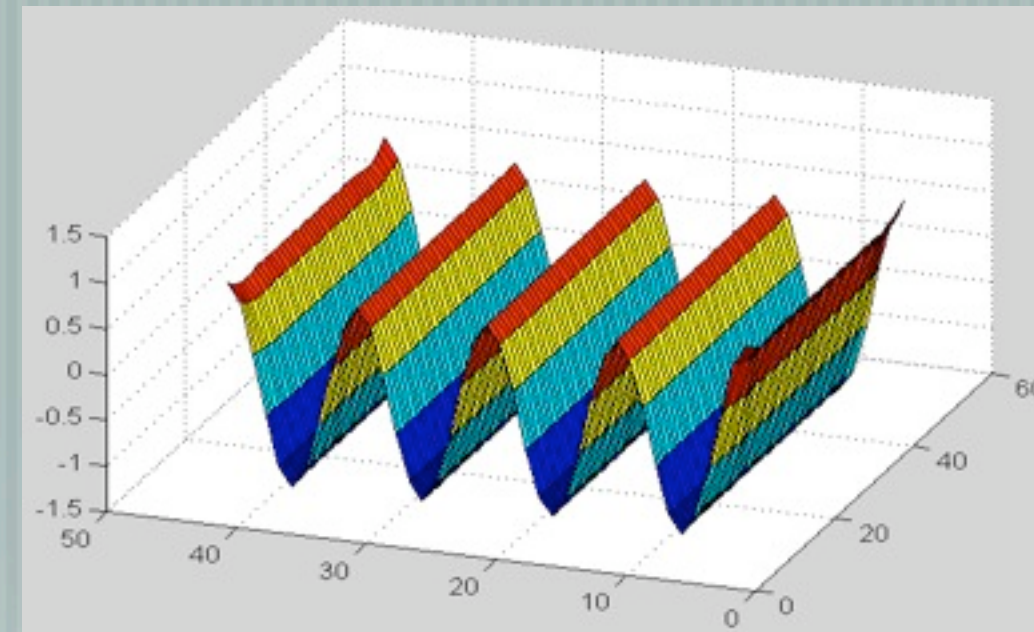
Unconventional view on Particle-wave duality

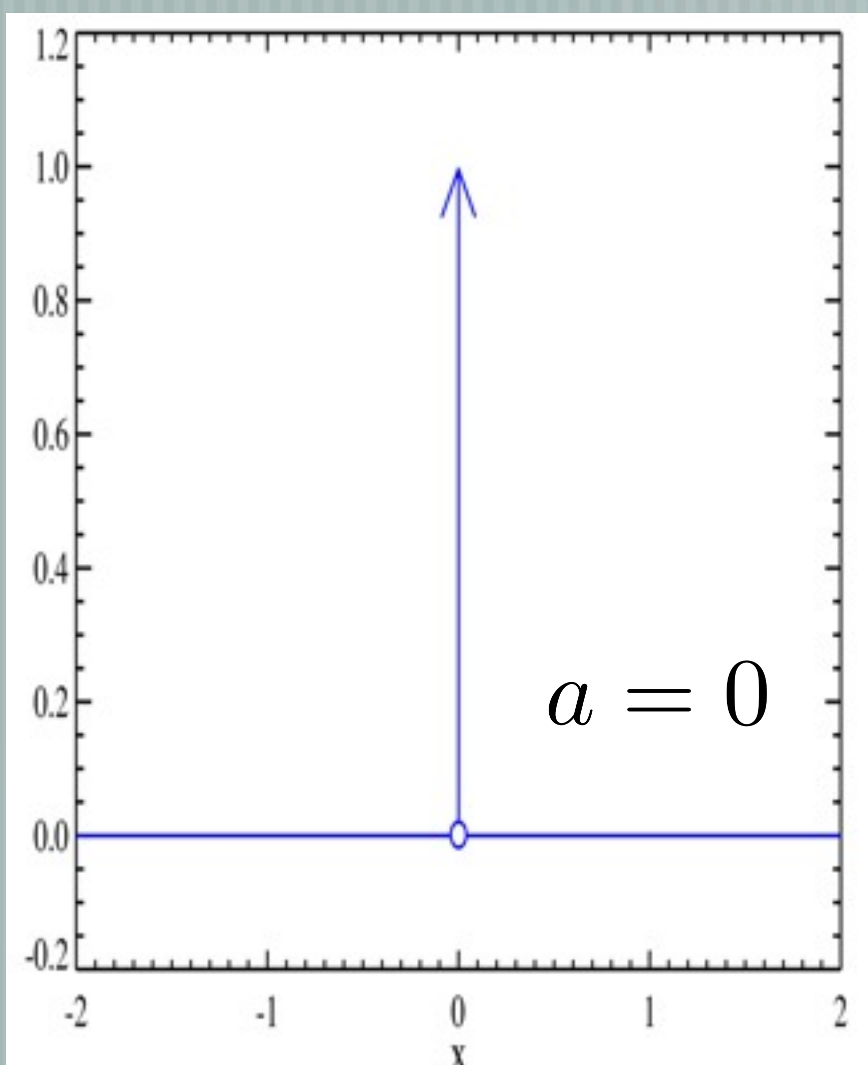
Our intuition about quantum motion has to deal with Heisenberg's Uncertainty Relation and its descendants and relatives

$$[x, p] = i$$

A particle with a definite momentum p is in a wavy state (wave)

$$\psi = \frac{1}{\sqrt{2\pi}} e^{ipx}$$





A particle with a definite position a is localized in space (particle)

$$\psi = \delta(x - a)$$

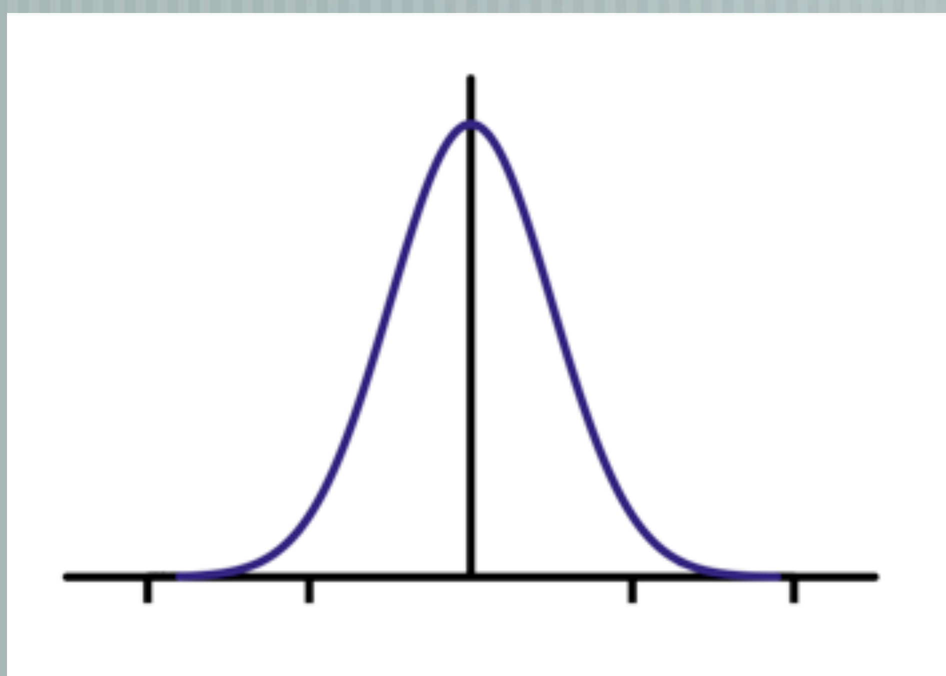
Both pictures are incompatible due to the Heisenberg uncertainty relation

Imagine one introduces new position and momentum operators:

$$x' = -p \qquad\qquad p' = x$$

$$\boxed{[x', p'] = i}$$

This **unitary** transformation has surprising consequences:



A quantum state which is localized in momentum can be thought of as being localized in position, and viceversa.

Things that look very different seem equivalent

With only this info, one cannot distinguish between position and momentum

How can one distinguish position from momentum?

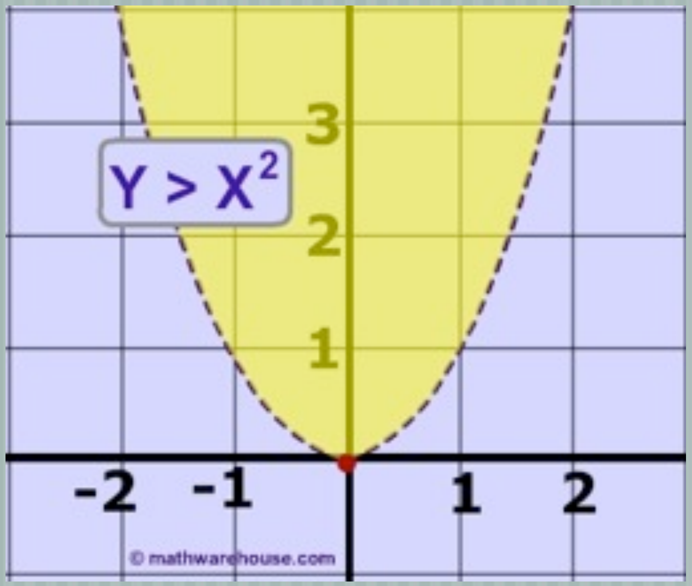
DYNAMICS BREAKS THE EQUIVALENCE

$$H = \frac{1}{2m}p^2 + V(x)$$

The Hamiltonian breaks the symmetry of the Heisenberg algebra

$$[x, p] = i$$

But for the Harmonic oscillator...



$$\frac{1}{2m}p^2 + \frac{1}{2}kx^2 \quad \leftrightarrow \quad \frac{1}{2}kp^2 + \frac{1}{2m}x^2$$

$$E_n = \sqrt{\frac{k}{m}}(n + \frac{1}{2}) \quad \leftrightarrow \quad E_n = \sqrt{\frac{k}{m}}(n + \frac{1}{2})$$

Elementary SELF-DUALITY RELATION

The Harmonic Oscillator shares the symmetry of the Heisenberg algebra, but with new consequences

What all these Dualities have in common?

All these Dualities are examples of Unitary equivalence

So... What are Dualities?

One would like to understand:

1)Their physical content and meaning in **classical** and **quantum** physics and their **connection if any**

2)A precise **mathematical characterization**

3)Methods to look for dualities systematically in **any space-time** dimension

4) New Applications

* **E. Cobanera, G. Ortiz, Z. Nussinov, "Unified approach to classical and quantum dualities", PRL 104, 020402 (2010), http://arxiv.org/abs/0907.0733**

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Is there any connection ?

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Our bond-algebraic approach

Bond algebras and their symmetries

Quantum Hamiltonians are built as a sum of quasi-local operators

We call these **BONDS**:

$$H = \sum_R J_R \mathcal{O}_R$$

A bond algebra for H is the set of all linear combinations of products of bonds

$$\mathcal{A}_H = \{1, \alpha\mathcal{O}_{\mathcal{R}}, \beta\mathcal{O}_R\mathcal{O}_{R'}, \mathcal{O}_R - \mathcal{O}_R\mathcal{O}_{R'}\mathcal{O}_{R''}, \cdots\}$$

It knows a lot about the Hamiltonian...

Exposing Quantum Dualities

When are two Hamiltonias Dual?

H_1 **and** H_2 **are dual if there is an homomorphism between their bond algebras**

DUALITIES **are one to one, onto mappings between bond algebras that preserve every algebraic relation between bonds:**

$$\mathcal{O}^1_{R_1} \leftrightarrow \mathcal{O}^2_{R_2}$$

Self-Dualities are automorphims of bond algebras

that preserve the form of the Hamiltonian

In other words:

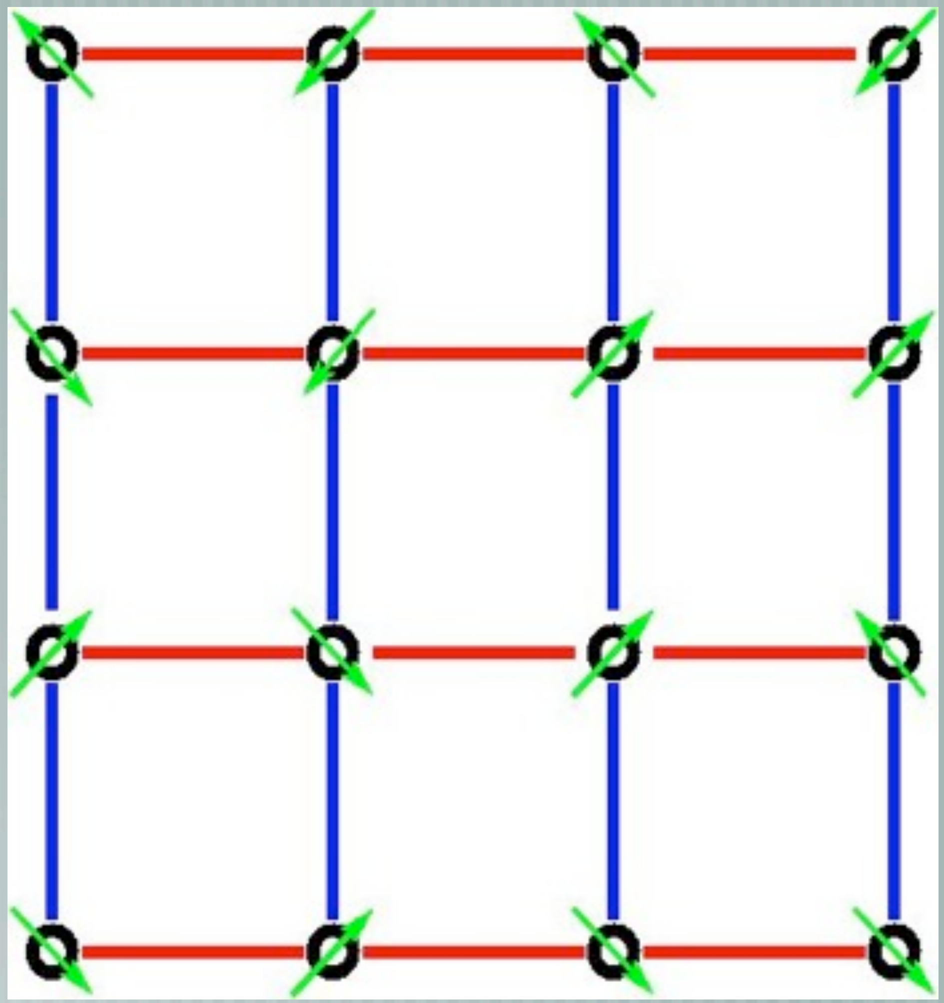
A Self-Duality is a symmetry of the bond algebra

that preserves the form of the Hamiltonian

Quantum Mechanics requires these mappings to be

UNITARILY IMPLEMENTABLE

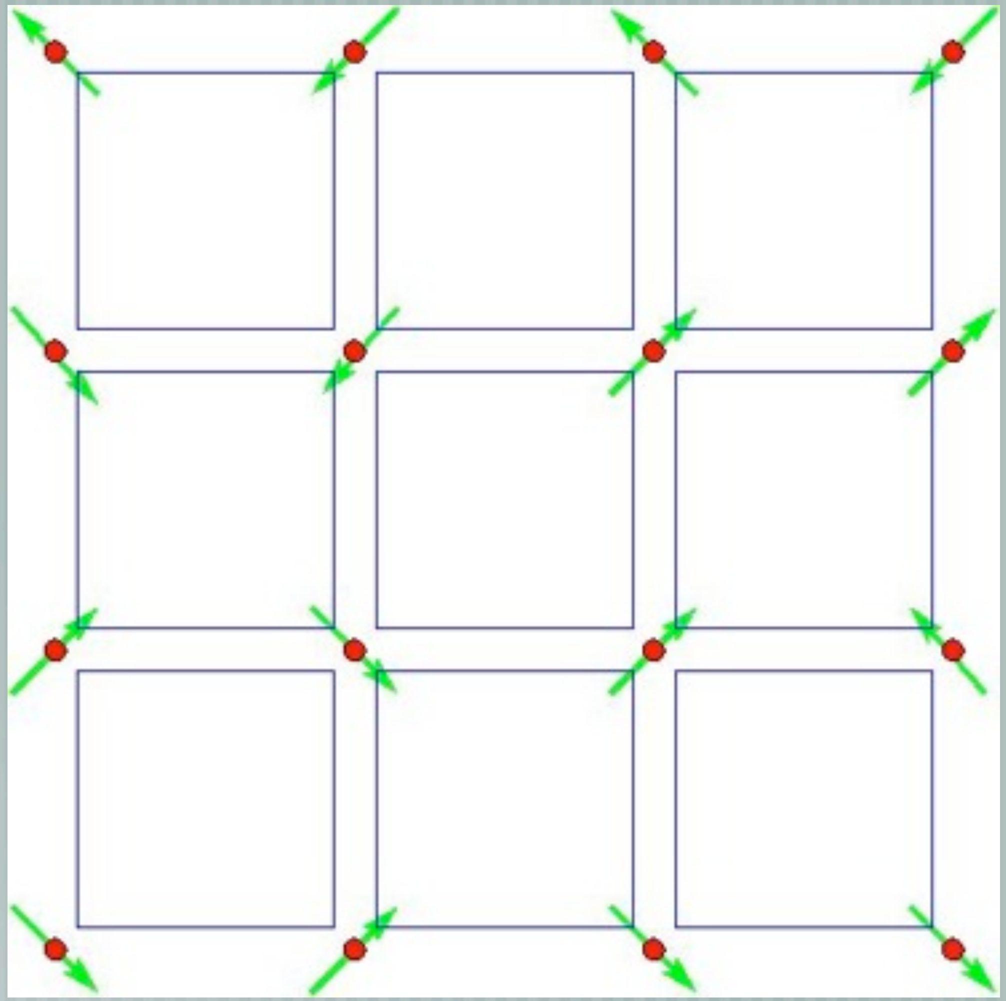
Example of Duality: Planar Orbital Compass (POC) and Xu-Moore Models (XM)



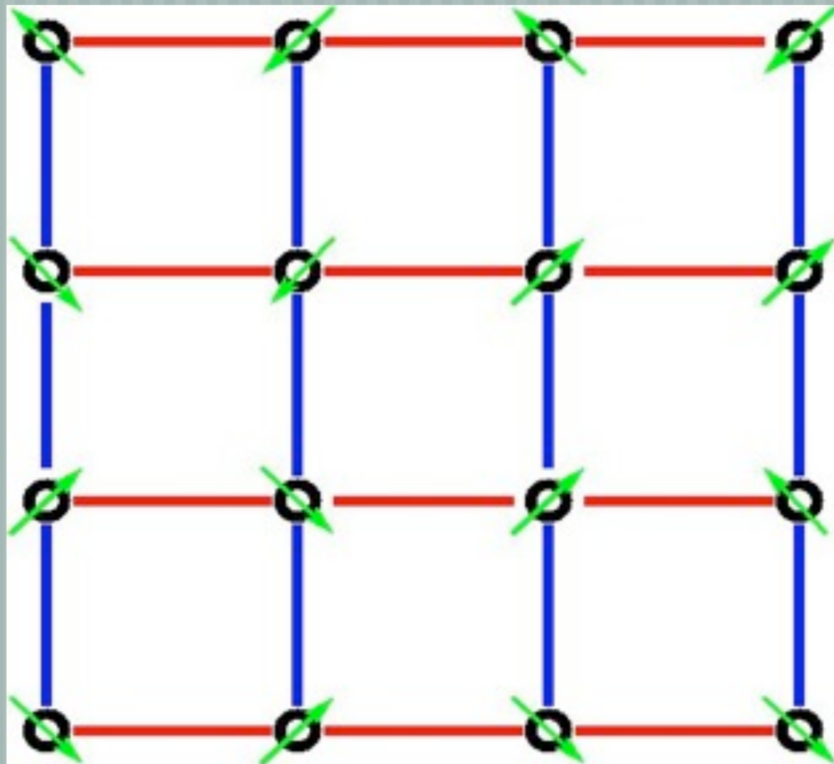
The POC model provides a simplified scenario to study orbital ordering in transition metal compounds

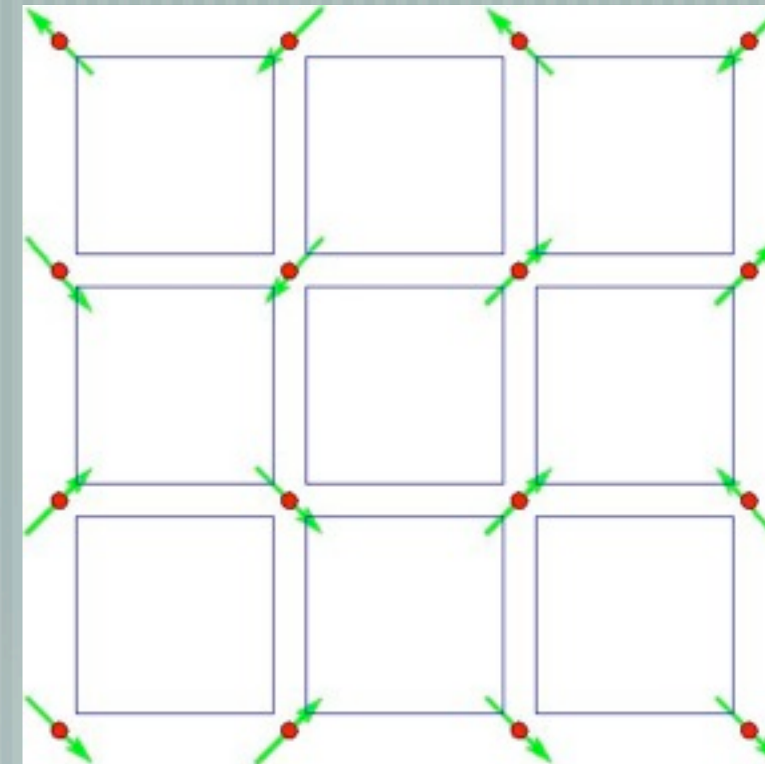
$$H_{POC} = \sum_{\vec{i}} j_x \sigma^x_{\vec{i}} \sigma^x_{\vec{i}+e_1} + j_y \sigma^y_{\vec{i}} \sigma^y_{\vec{i}+e_2}$$

The XM Hamiltonian was introduced as a simplified model of phase transitions in p+ip superconducting arrays



$$H_{XM} = \sum_{\vec{i}} j\sigma^z_{\vec{i}}\sigma^z_{\vec{i}+e_1}\sigma^z_{\vec{i}+e_1+e_2}\sigma^z_{\vec{i}+e_2} + h\sigma^x_{\vec{i}}$$





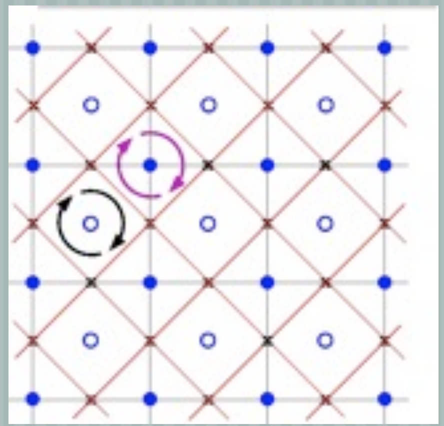
The two models are DUAL

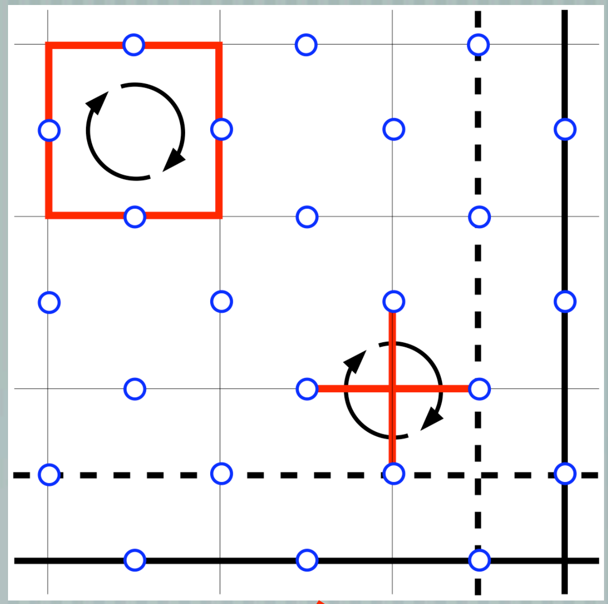
$$\sigma^x_{\vec{i}}\sigma^x_{\vec{i}+\vec{e}_1} \mapsto {\sigma^z}_{\vec{i}}{\sigma^z}_{\vec{i}+e_1}{\sigma^z}_{\vec{i}+e_1+e_2}{\sigma^z}_{\vec{i}+e_2}$$

$$\sigma^y_{\vec{i}}\sigma^y_{\vec{i}+\vec{e}_2} \mapsto \sigma^x_{\vec{i}+\vec{e}_2}$$

THE TWO HAMILTONIANS ARE UNITARILY EQUIVALENT

Kitaev's toric code model:





$$H_K = -\sum_s A_s - \sum_p B_p$$

$$A_s = \prod_{ij \in star(s)} \sigma^x_{ij}$$

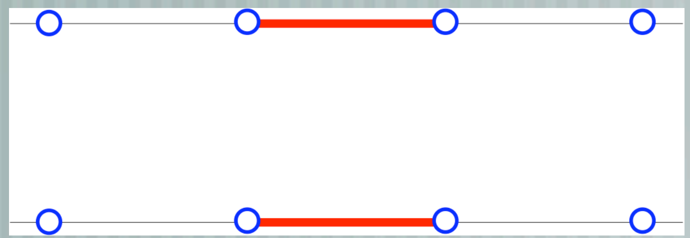
$$B_p = \prod_{ij \in boundary(p)} \sigma^z_{ij}$$

(Identical spectra)

Duality mappings: Non-local

(Dimensional reduction)

2 Ising chains:

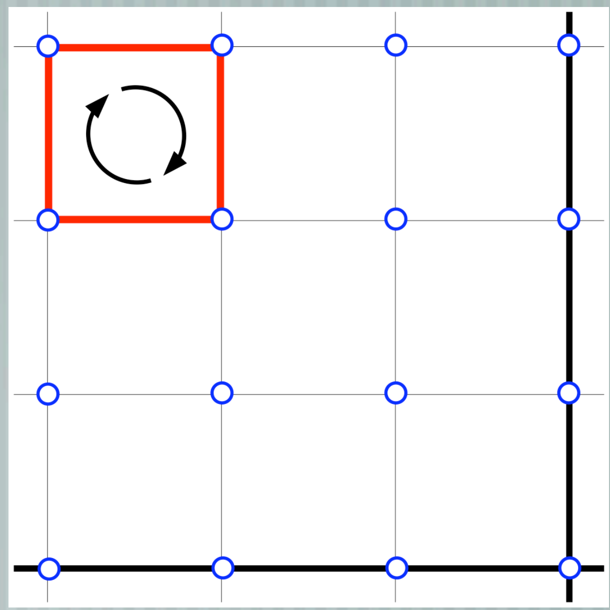


$$H_I = -\sum_s \sigma^z_s \sigma^z_{s+1} - \sum_p \sigma^z_p \sigma^z_{p+1}$$

(Nussinov-Ortiz 2006)

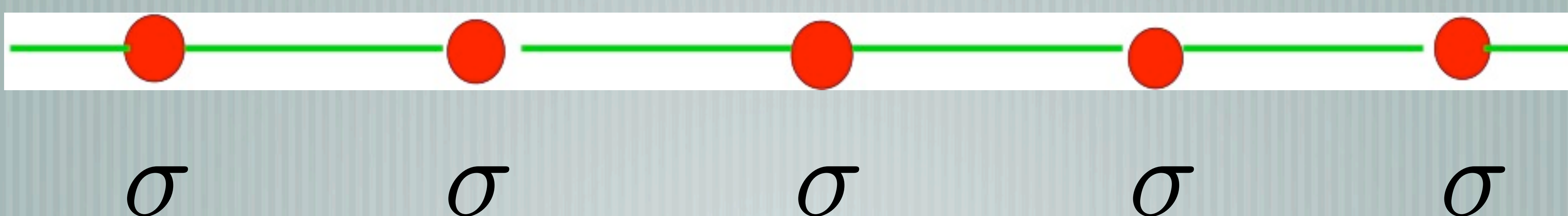
Wen's plaquette model:

$$H_W = -\sum_i \sigma^x_i \sigma^y_{i+\hat{e}_x} \sigma^x_{i+\hat{e}_x+\hat{e}_y} \sigma^y_{i+\hat{e}_y}$$



Example of Self-Duality:
Ising chain in a transverse field

$$H[j,h] = \sum_i j\sigma_i^z \sigma_{i+1}^z + h\sigma_i^x$$



BOND ALGEBRA

$\sigma^z_{i-1}\sigma^z_i$ $\sigma^z_i\sigma^z_{i+1}$

σ^x_{i-1} σ^x_i σ^x_{i+1}

Every bond $\sigma^z\sigma^z$ anti-commutes with two bonds σ^x

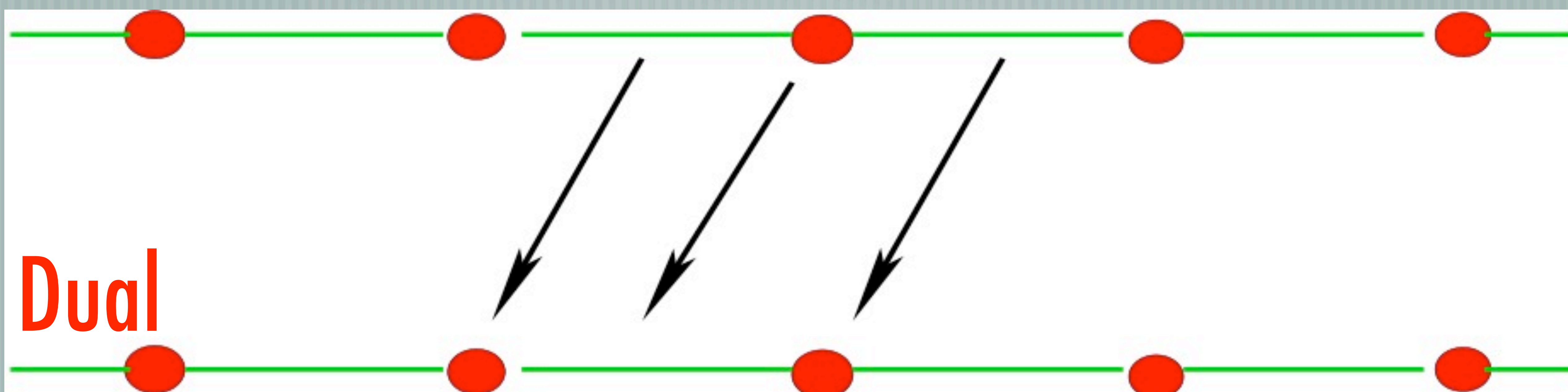
Every bond σ^x anti-commutes with two bonds $\sigma^z\sigma^z$

SELF-DUALITY AUTOMORPHISM

Homomorphism Φ_D:

$$\sigma^z_i \sigma^z_{i+1} \mapsto \sigma^x_i \qquad\qquad \sigma^x_i \mapsto \sigma^z_{i-1}\sigma^z_i$$

$\sigma^z_{i-1}\sigma^z_i \quad \sigma^x_i \quad \sigma^z_i\sigma^z_{i+1}$



$\sigma^x_{i-1} \quad \sigma^z_{i-1}\sigma^z_i \quad \sigma^x_i \quad \sigma^z_i\sigma^z_{i+1}$

Mapping is Unitarily implementable

$$\mathcal{U}_D \sigma_i^z \sigma_{i+1}^z \mathcal{U}_D^\dagger = \sigma_i^x \qquad \mathcal{U}_D \sigma_i^x \mathcal{U}_D^\dagger = \sigma_{i-1}^z \sigma_i^z$$

Ising chain in a transverse field is self-dual, meaning:

$$\mathcal{U}_D H[j,h] \mathcal{U}_D^\dagger = H[h,j]$$

$$j \leftrightarrow h$$

Advantages:

- Better suited for systematic (ALGORITHMIC) search of (self-)dualities

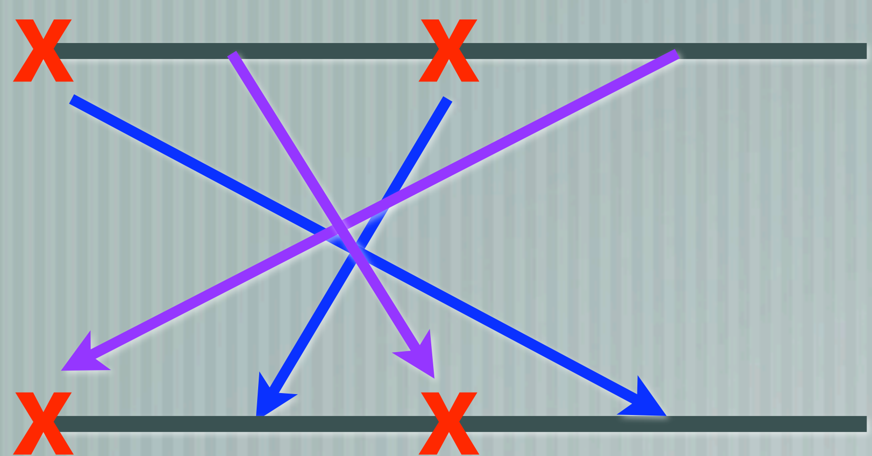
- Allows us to derive the (in general) non-local dual operator variables - the ones that had to be guessed in the past

Dualities in finite systems

(Role of boundary terms)

$$H = j(\sigma_1^z\sigma_2^z + \sigma_2^z\sigma_3^z) + h(\sigma_1^x + \sigma_2^x)$$

—— : $\sigma^z\sigma^z$ X: σ^x ○: σ^z



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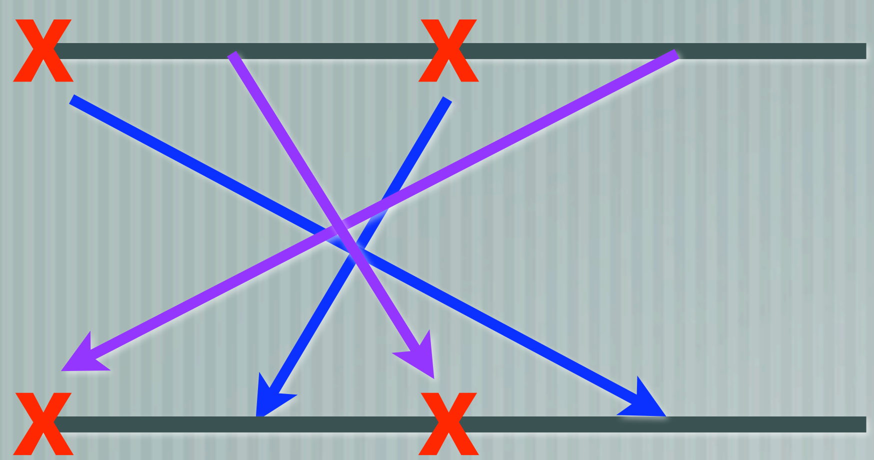
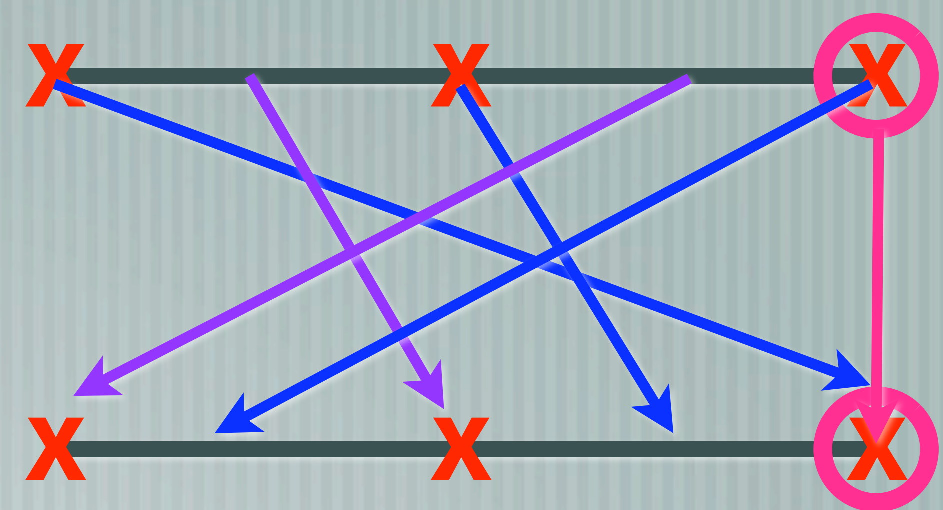


$j \leftrightarrow h$



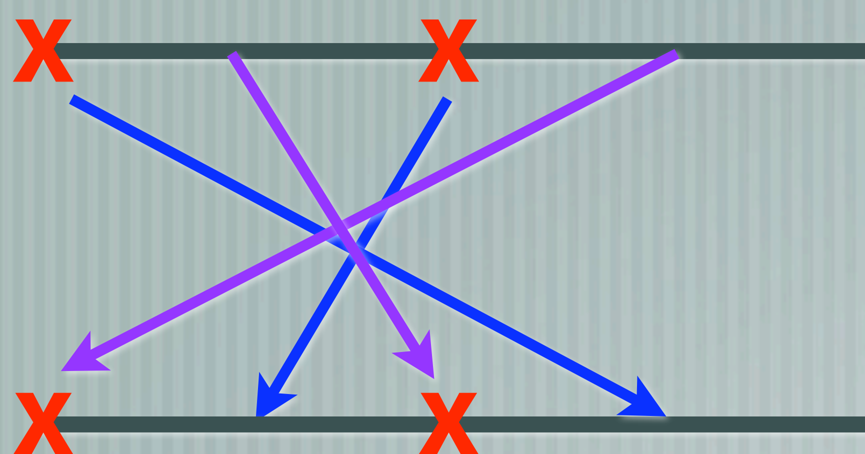
$$H = j(\sigma_1^z\sigma_2^z + \sigma_2^z\sigma_3^z + \sigma_3^z) + h(\sigma_1^x + \sigma_2^x + \sigma_3^x)$$

Dualities in finite systems

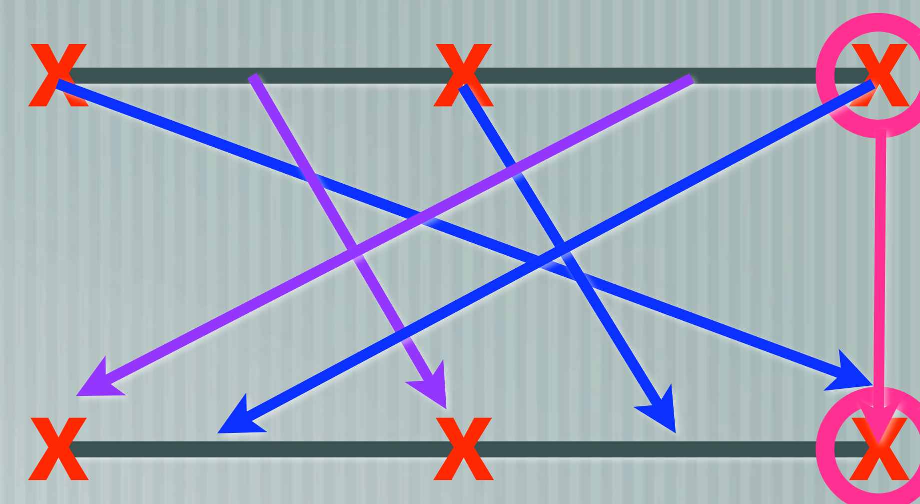
(Role of boundary terms)

$$H = j(\sigma_1^z\sigma_2^z + \sigma_2^z\sigma_3^z) + h(\sigma_1^x + \sigma_2^x)$$

—— : $\sigma^z\sigma^z$ x: σ^x ○: σ^z

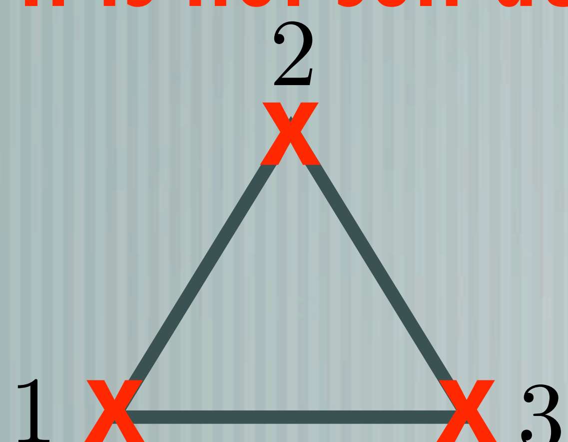


$j \leftrightarrow h$



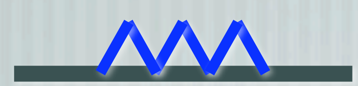
$$H = j(\sigma_1^z\sigma_2^z + \sigma_2^z\sigma_3^z + \sigma_3^z) + h(\sigma_1^x + \sigma_2^x + \sigma_3^x)$$

It is not self-dual:

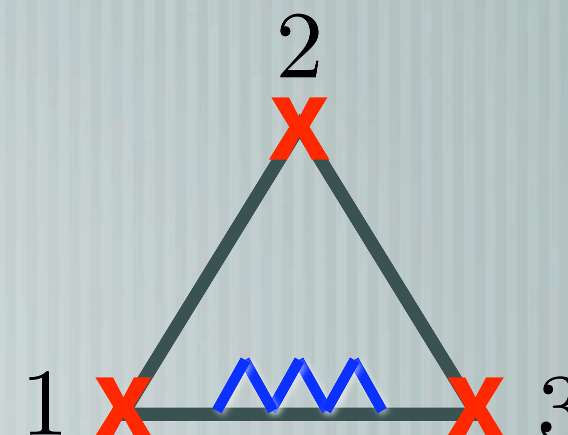


$\sigma_3^z\sigma_1^z$

It is self-dual:



String
$(\sigma_1^x\sigma_2^x\sigma_3^x)$



$\sigma_3^z\sigma_1^z(\sigma_1^x\sigma_2^x\sigma_3^x)$

Parameter-Dep bond algebras

Bond algebra: $A_i(j,h) = j\sigma_i^x\sigma_{i+1}^x + \ h\sigma_i^z$

$$B_i(j^{-1},h^{-1}) = j^{-1}\sigma_i^y\sigma_{i+1}^y + \ h^{-1}\sigma_{i+1}^z$$

Automorphism:

$$A_i(j,h) \to A_i(h,j), \qquad B_i(j^{-1},h^{-1}) \to B_i(h^{-1},j^{-1})$$

Self-dual Hamiltonian: $j \leftrightarrow h \qquad (m \text{ and } m' \text{ fixed })$

$$H = m\sigma_1^y + m'\sigma_N^x + \sum_{i=1}^{N-1} [(j\sigma_i^x\sigma_{i+1}^x + h\sigma_i^z) + (j^{-1}\sigma_i^y\sigma_{i+1}^y + h^{-1}\sigma_{i+1}^z)]$$

boundary terms

Parameter-Dep bond algebras

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boundary terms

Dual variables also depend on parameters

Abelian versus non-Abelian

Is the character of a duality (Abelian vs non-Abelian) determined by the group of symmetries of the Hamiltonian?

$$H = -\frac{j}{4}\sum_i (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z)$$ Heisenberg chain

Homomorphism Φ_D: (same as quantum Ising (Abelian))

$$\sigma_i^y \sigma_{i+1}^y = \sigma_i^x \sigma_i^z \sigma_{i+1}^z \sigma_{i+1}^x \xrightarrow{\Phi_D} -\sigma_{i-1}^z \sigma_i^x \sigma_{i+1}^z$$

$$H \xrightarrow{\Phi_D} -\frac{j}{4}\sum_i \left(\sigma_{i-1}^z \sigma_{i+1}^z - \sigma_{i-1}^z \sigma_i^x \sigma_{i+1}^z + \sigma_i^x\right)$$

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$$H \xrightarrow{\Phi_D} -\frac{j}{4}\sum_i (\sigma_{i-1}^z \sigma_{i+1}^z - \sigma_{i-1}^z \sigma_i^x \sigma_{i+1}^z + \sigma_i^x)$$

It is not illuminating to associate the character of a duality to the group of symmetries of the Hamiltonian

Dual operators, Disordered variables and Topological excitations

It is easy now to
COMPUTE DUAL OPERATOR VARIABLES:

Observation:

— Bond-algebraic mapping is local

— Mapping of microscopic degrees of freedom is non-local

Use bond-algebraic mapping to derive the dual variables

From bonds to dual variables: An example

$$\left.\begin{aligned} \mu_i^x &= \sigma_{i-1}^z \sigma_i^z \\ \mu_i^z \mu_{i+1}^z &= \sigma_i^x \end{aligned}\right\}$$ Bond algebra map

$$\mu_i^z \mu_{i+1}^z \mu_{i+1}^z \mu_{i+2}^z \cdots \mu_{i+4}^z \cdots = \sigma_i^x \sigma_{i+1}^x \sigma_{i+2}^x \sigma_{i+3}^x \cdots$$

Dual variables

$$\mu_i^z = \sigma_i^x \sigma_{i+1}^x \sigma_{i+2}^x \sigma_{i+3}^x \cdots$$

Generators of KINKS

From bonds to dual variables: An example

$$\mu_i^x = \sigma_{i-1}^z \sigma_i^z$$

$$\mu_i^z \mu_{i+1}^z = \sigma_i^x$$

Bond algebra map

$$\mu_i^z \mu_{i+1}^z \mu_{i+1}^z \mu_{i+2}^z \cdots \mu_{i+4}^z \cdots = \sigma_i^x \sigma_{i+1}^x \sigma_{i+2}^x \sigma_{i+3}^x \cdots$$

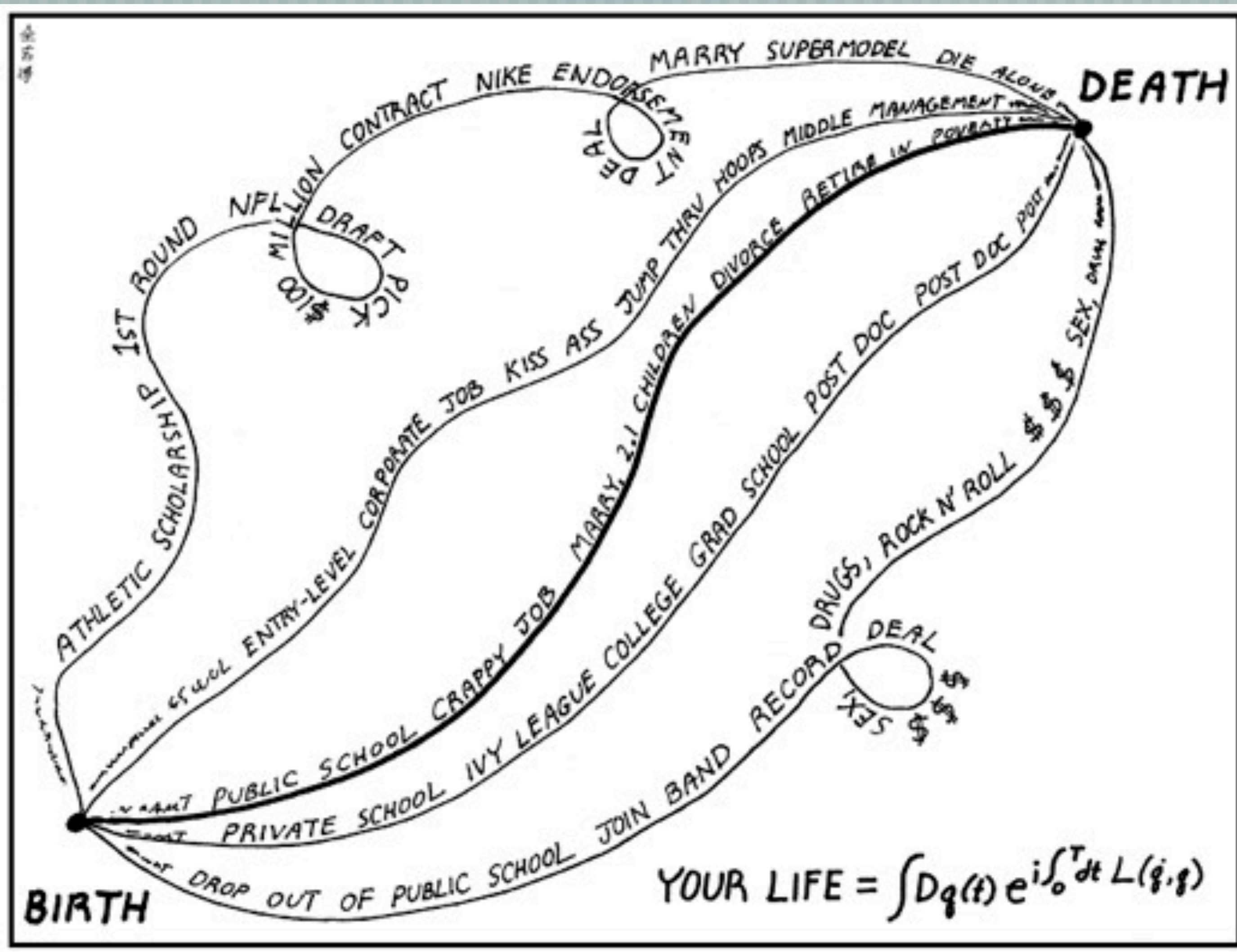
Dual variables

$$\mu_i^z = \sigma_i^x \sigma_{i+1}^x \sigma_{i+2}^x \sigma_{i+3}^x \cdots$$

Generators of KINKS

Connecting Classical and Quantum Dualities

1) Classical System in $D+1$ ⟷ Quantum system in D



The Path Integral Formulation of Your Life

Feynman Path Integral
alias
Transfer matrix
alias
Suzuki-Trotter decomposition
alias
World Line Monte Carlo
...

Thus we propose that

2) **Classical Dualities ⟷ Quantum Dualities**

3) **Characterize Quantum Dualities as Hamiltonian dependent bond-algebraic equivalences**

And hope for Quantum Dualities to be easier to deal with than classical ones.
After all, the classical problem comes from exponentiating a quantum one and taking a trace...

Exposing Classical Dualities

Quantum-to-Classical: Suzuki-Trotter decomposition of the Ising chain

$$H[j,h] = -\sum_{i=1}^{N} j\sigma_i^z\sigma_{i+1}^z + h\sigma_i^x$$

Quantum Ising Chain in a Transverse Field

$$\mathrm{Tr}\; e^{-\Delta\tau H[j,h]} \approx \left(\frac{1}{2}\sinh 2h_M\right)^{\frac{MN}{2}} \times$$

$$\sum_{\{\sigma\}} e^{\left[-\frac{1}{2}\ln\tanh(h_M)\sum \sigma_{m,n}\sigma_{m,n+1} + j_M \sum \sigma_{m,n}\sigma_{m+1,n}\right]}$$

$$h_M = \frac{\Delta\tau\ h}{M} \qquad j_M = \frac{\Delta\tau\ j}{M}$$

The larger M, the better it gets

We can fine-tune the couplings of the Quantum Model to get an Isotropic classical Ising magnet

$$\text{Tr}\ e^{-\Delta\tau H[j,h]} \approx \left(\frac{1}{2}\sinh 2h_M\right)^{\frac{MN}{2}} \mathcal{Z}(K)$$

$$K \equiv -\frac{1}{2}\ln\tanh h_M = j_M$$

$$h_M = \frac{\Delta\tau\ h}{M} \qquad j_M = \frac{\Delta\tau\ j}{M}$$

On the other hand, exchanging couplings j and h gives

$$\text{Tr}\; e^{-\Delta\tau H[h,j]} \approx \left(\frac{1}{2}\sinh 2j_M\right)^{\frac{MN}{2}} \mathcal{Z}(\hat{K})$$

$$\boxed{\hat{K} = -\frac{1}{2}\ln\tanh(j_M) = h_M}$$

Any connection between the two?

YES!!!!!!

The QUANTUM Self-Duality guarantees that

$$\text{Tr}\; e^{-\Delta\tau H[h,j]} = \text{Tr}\; e^{-\Delta\tau H[j,h]}$$

OR BETTER, IN TERMS OF
CLASSICAL PARTITION FUNCTIONS

$$\left(\frac{1}{2}\sinh 2\hat{K}\right)^{\frac{MN}{2}} \mathcal{Z}(K) = \left(\frac{1}{2}\sinh 2K\right)^{\frac{MN}{2}} \mathcal{Z}(\hat{K})$$

MOREOVER, FROM THE EXPLICIT FORMULAS FOR THE CLASSICAL COUPLINGS IN TERMS OF THE QUANTUM ONES, THIS RELATION FOLLOWS:

$$\sinh(2K)\sinh(2\hat{K}) = 1$$

These altogether are nothing but the classical self-duality relation of Kramers and Wannier!!!!

Contrast: Quantum vs Classical

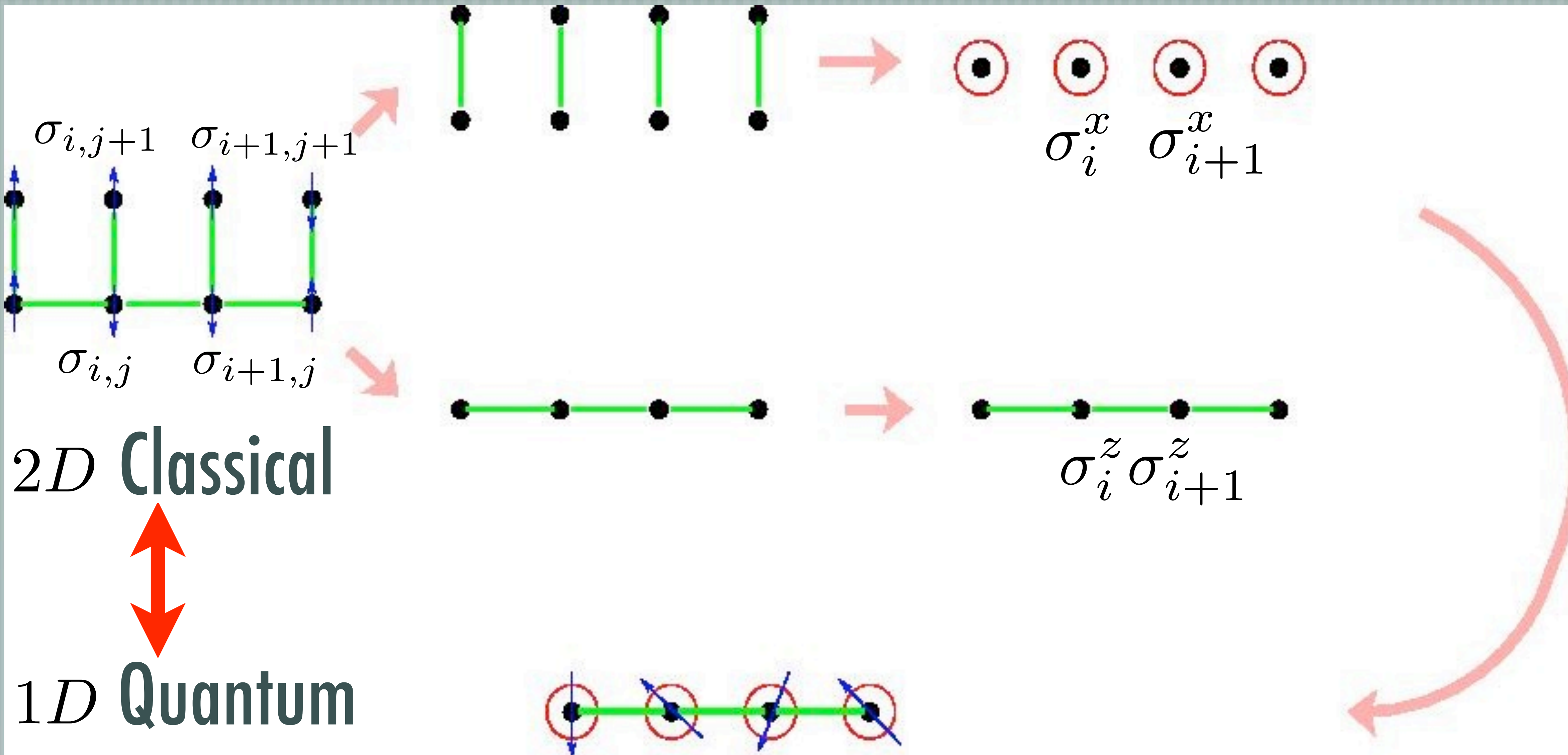
Quantum Self-duality relation

$$h = j$$

Classical Self-duality relation

$$\sinh(2K)\,\sinh(2\hat{K}) = 1$$

Classical-to-Quantum: From the transfer matrix to a quantum Hamiltonian



$$\langle \sigma' | \exp(\lambda \sigma^x) | \sigma \rangle = \sqrt{\frac{1}{2}\sinh(2\lambda)} \exp\left[-\frac{1}{2} \ln \tanh(\lambda) \sigma' \sigma \right]$$

Classical and Quantum
(Self-)Dualities
are equivalent and in correspondence:

We have managed to UNIFY them.

Dualities and New Symmetries

A self-duality does not preserve the form of the Hamiltonian but preserves its spectrum

$$\mathcal{U}_D H[j,h] \mathcal{U}_D^\dagger = H[h,j]$$

A self-duality is not a symmetry in general, but

$$\mathcal{U}_D^2 H[j,h] \mathcal{U}_D^{\dagger 2} = H[j,h]$$

Self-duality → $\sqrt{\text{Quantum Symmetry}}$

A self-duality is an emergent symmetry at the self-dual point

New (Self-)Dualities
Enlarging the Zoo of Dualities

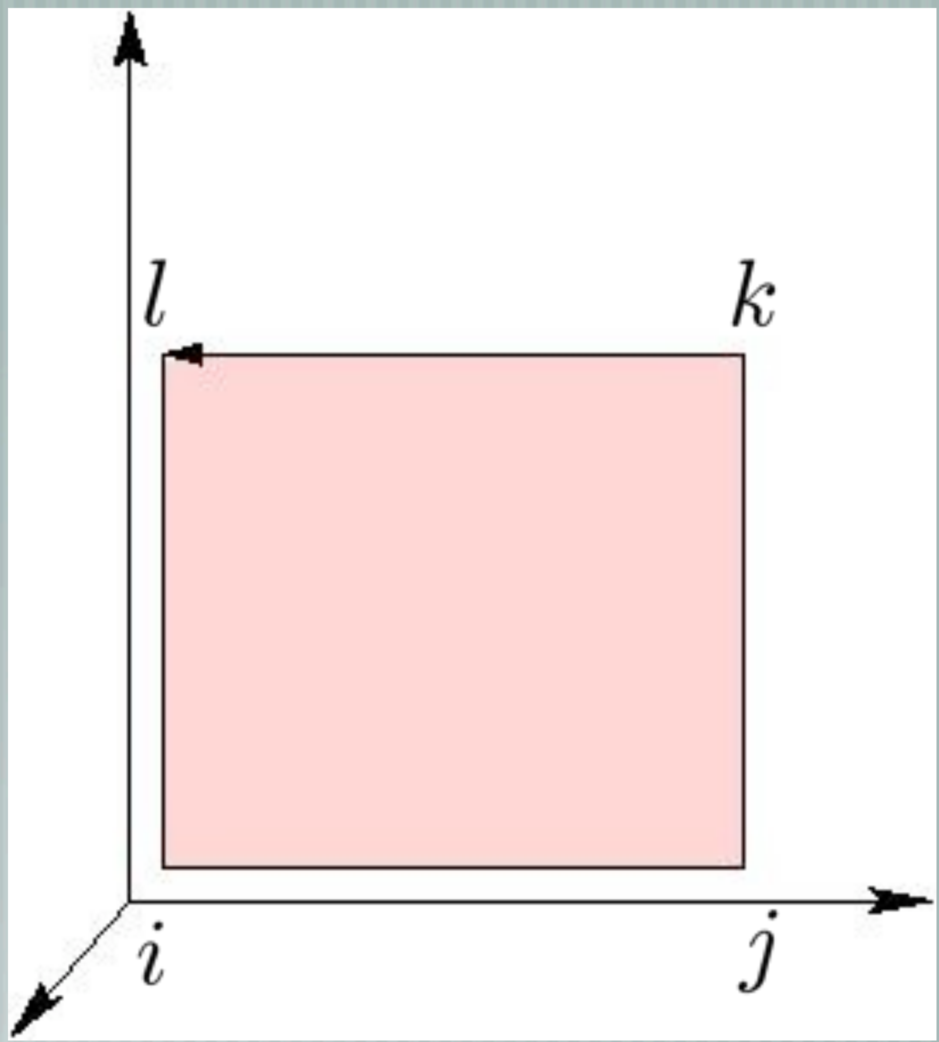
Gauge Field Theories

Four-Dimensional (D=4) Euclidean Lattice

‘t Hooft idea: the most important degrees of freedom in a confinement-deconfinement phase transition should be the field configurations taking values on $\mathbb{Z}_N$, the center of SU(N)

With ‘t Hooft ideas in mind, several authors attempted rigorous studies of Wilson’s action for Lattice Gauge Field Theories

$$S = \frac{1}{g^2} \sum_{\square} \text{Re Tr } U_{ij} U_{jk} U^{\dagger}_{kl} U^{\dagger}_{li}$$



restricting however the fields to taking values on a unitary representation of $\mathbb{Z}_N$, that is, on Nth roots of unity

From Euclidean to Quantum Hamiltonian Formulation

The reverse of the Suzuki-Trotter decomposition gives a quantum problem in 3 dimensions

$$H_{LG} = \sum_{n} \sum_{i=1}^{3} V_n^i + \lambda\left(\frac{1}{g^2}\right)\Theta_n^i + h.c.$$

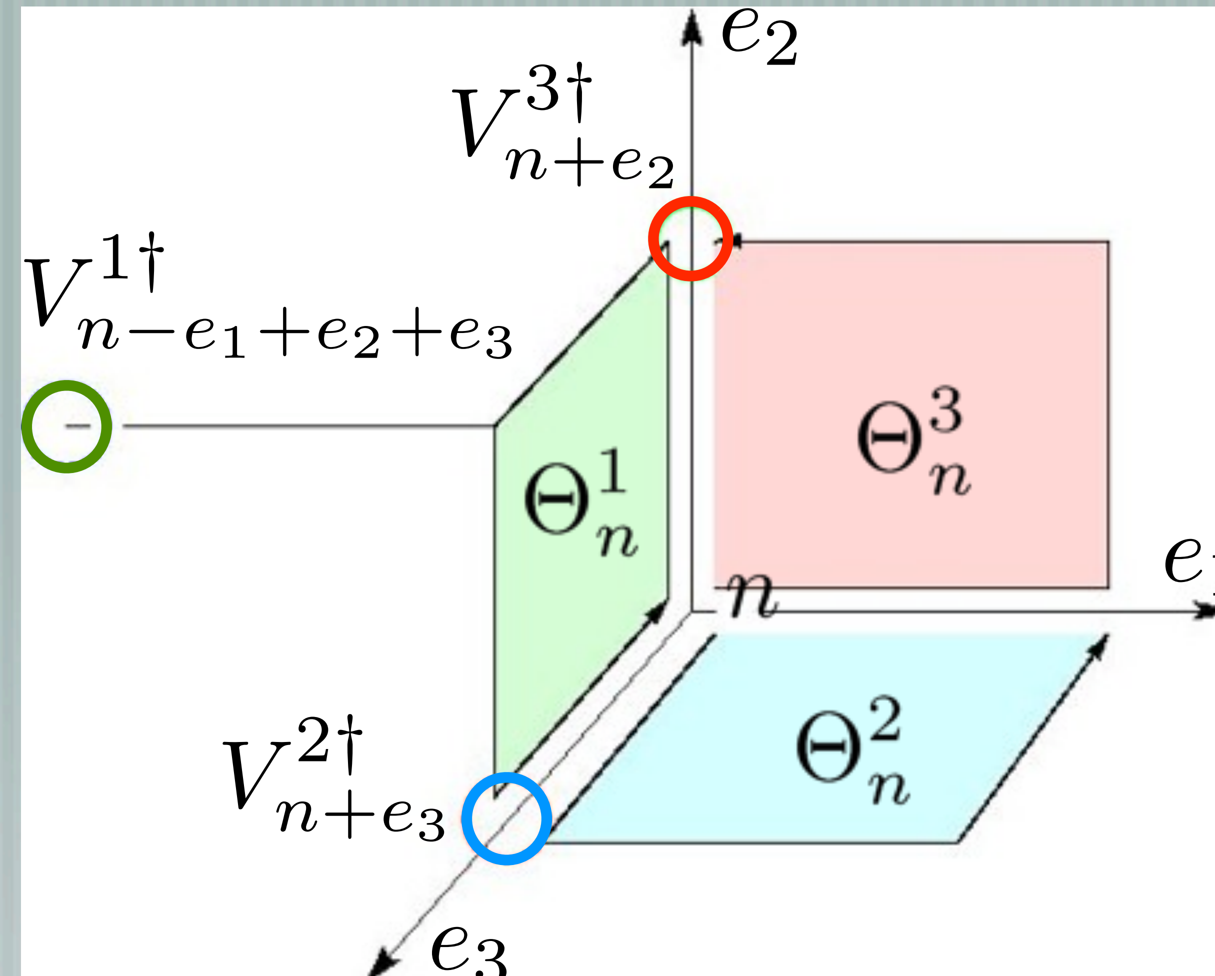
$$\Theta_n^1 = U_n^2 U_{n+e_2}^3 U_{n+e_3}^{2\dagger} U_n^{3\dagger}$$

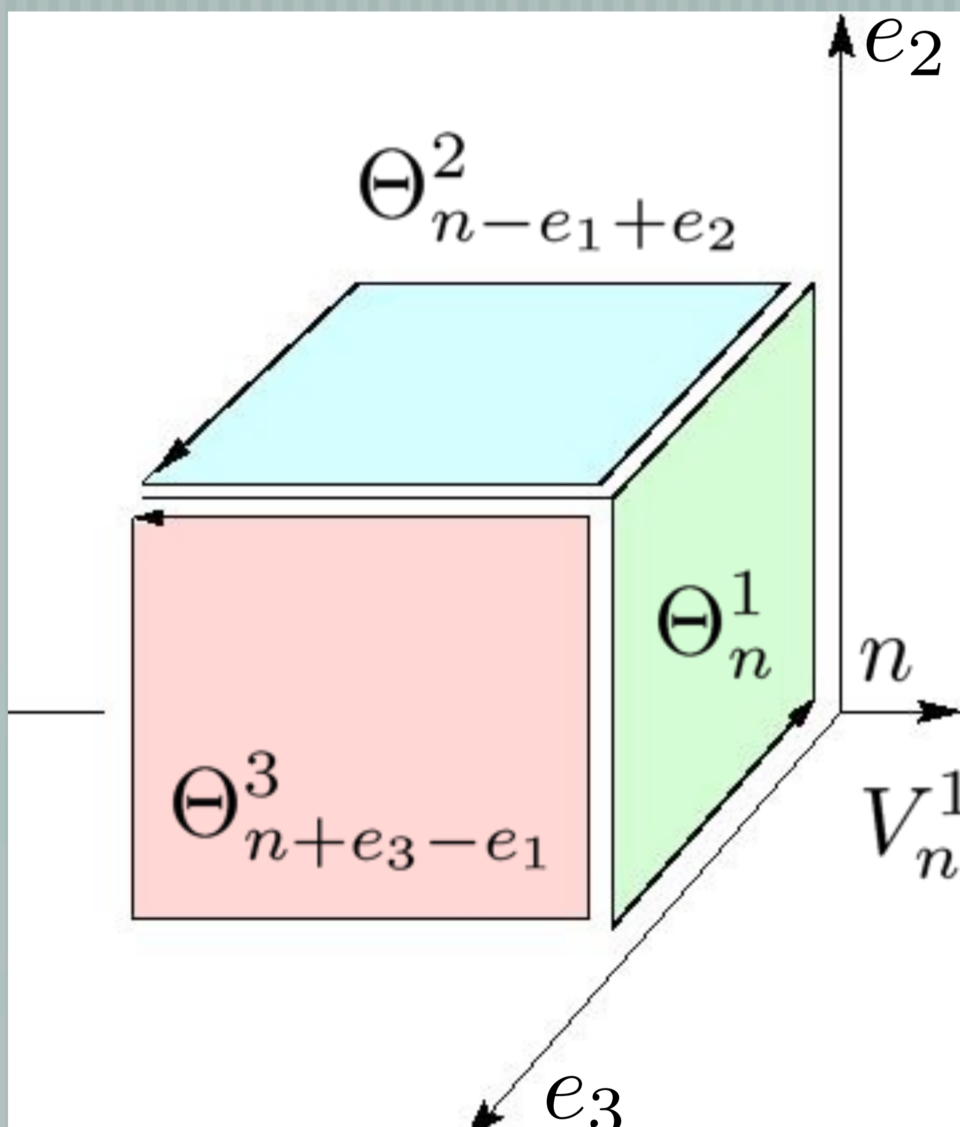
and cyclic permutations

$$VU = \omega UV, \qquad \omega = \exp i\frac{2\pi}{N}$$

The Weyl Algebra

The Self-Duality Mapping





$V^1_n, \; V^2_n, \; V^3_n$

$$\Phi_D^2 = \mathcal{C}$$

Discrete symmetry:Charge conjugation

Some features of the Self-Duality

- The critical couplings have to distribute symmetrically relative to the self dual point
- The self-duality unitary has period four, thus it reveals a new discrete symmetry of these theories
- An explicit analytic formula to compute the self-dual coupling can be obtained
- Prove connection between these GFTs and the vector Potts model
- We do not use Villain's trick. It is not necessary!

Summary of Main Results

- Quantum (self-)dualities can now be looked for systematically as bond algebra (unitary) mappings
- An algebraic approach to quantum (self-)dualities explains classical dualities as well, in any space dimension d
- Dual Variables can be computed and carry information on the topological excitations of the system
- New (self-)dualities can be discovered with this new algebraic approach. We showed the case of Abelian GFTs with a confinement-deconfinement phase transition

Summary of Main Results

- Dualities may emerge is certain sectors (emergent dualities)
- Self-dualities are square roots of symmetries
- Bond-algebra mappings allow exact solution of several many-body models in high space dimensions
- Same technique can be applied to QFTs
- Other Self-dualities: Potts, p-clock, etc. models
- Other Dualities: Extended Kitaev, Blume-Emery-Griffiths, etc. models

Big Questions:

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- How about non-Abelian dualities?
- Is Fourier transform on finite groups the end of the story?
- Can we classify topological excitations?