Reduced Modeling of Strongly Stratified Shear Flows

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Themes

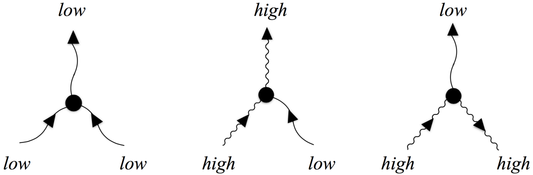
1. Stratified turbulence in absence of boundaries
2. Illustration of asymptotic derivation of QL system (temporal scale separation)
3. Extension of QL: slow spatial variation of “mean” field

   $\Rightarrow$ Generalized Quasi-Linear (GQL) approx. [Marston, Chini & Tobias, *PRL* 2016]
4. Increasing efficiency of QL: full exploitation of temporal scale separation

   $\Rightarrow$ New type of amplitude equation (which Guillaume will discuss. . .)

Quasi-Linear (QL) and Generalized Quasi-Linear (GQL) Formulations

- Decomposition into low and high modes in x and energy-conserving retention and removal of triadic interactions, w/cut-off Λ enabling homotopy b/w QL and DNS:

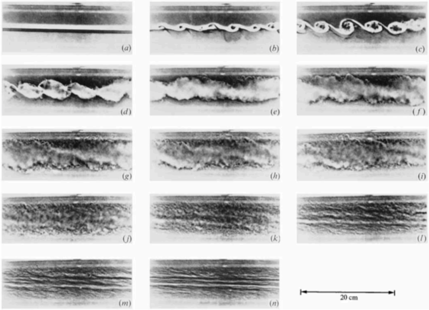


$$\textbf{Low modes:} \quad \bar{u}(x,z,t) = \sum_{|k|\le\Lambda} e^{ikx} u_k(z,t)$$

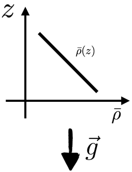
$$\textbf{High modes:} \quad u'(x,z,t) = \sum_{|k|>\Lambda} e^{ikx} u_k(z,t)$$

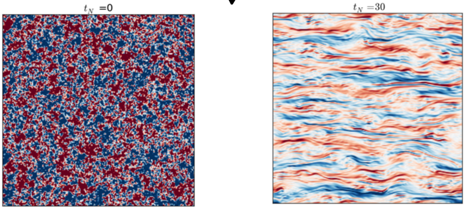
- **QL** obtained for $\Lambda = 0$, while **DNS ("NL")** obtained for $\Lambda \to \infty$
- **Accuracy** significantly improved for $0 < \Lambda \lesssim 5$
- **Efficiency** achieved by directly simulating *Statistical* Dynamical System (GCE2)?

Stratified Turbulence: Phenomenology & Significance



–Thorpe (1971)



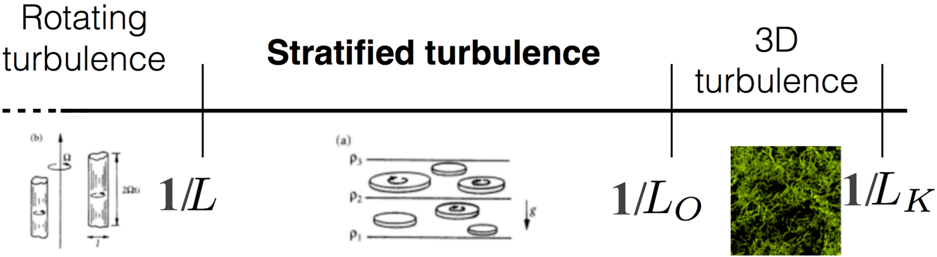


–1024x1024 DNS (C. Rocha)

- Anisotropic layers with $L \gg h$ coexist with small-scale isotropic features
- Mechanism for diabatic mixing (e.g. crucial for closing ocean circulation)
- SGS process in regional circulation and in computational climate models

Stratified Turbulence: Scales and Parameters

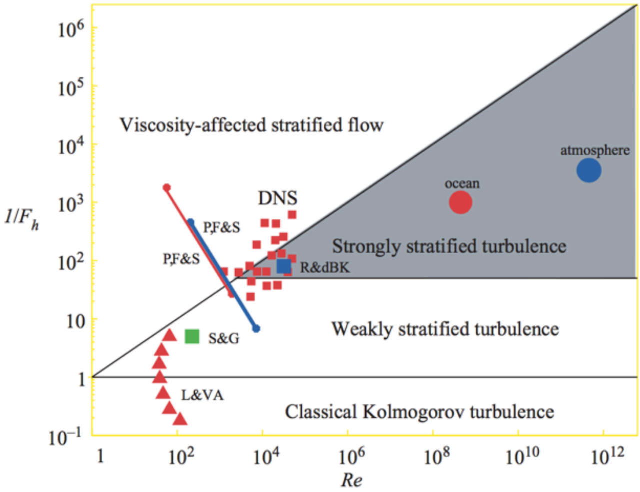
Buoyancy frequency $N \equiv \sqrt{-\frac{g}{\rho_0}\frac{d\bar{\rho}}{dz}}$



- (Horizontal) Froude number: $Fr \equiv \frac{U}{NL}$
- Reynolds number: $Re \equiv \frac{UL}{\nu}$

Simple scaling arguments give $L/L_O = O(Fr^{-3/2})$, where $L_O = (\epsilon/N^3)^{1/2}$

Stratified Turbulence: Computational Challenge



–Brethouwer *et al.* (2007)

Note: $F_h \equiv Fr$

Stratified Turbulence: Fundamental Questions

- What sets vertical scale(s)?
- **Mixing properties?**
- Is horizontal spectrum of horizontal KE independent of Fr as $Fr \to 0$ for $Re \gtrsim 10/Fr^2$?

Bartello & Tobias (2013) estimate that to demonstrate this independence even over one decade in Fr*, namely, over the parameter range* $0.01 \leq Fr \leq 0.1$*, would require the ratio of maximum to minimum resolvable scale to be in the* **millions** *(i.e. in a single spatial direction), yielding a formidable computational challenge.*

Stratified Turbulence: Governing Equations

Anisotropic Scaling

$\mathbf{x}_\perp : L \qquad z : h \qquad t : L/U \qquad \mathbf{u}_\perp : U \qquad w : Fr^2 UL/h \qquad p : \rho_0 U^2 \qquad b : U^2/h$

Non-Rotating Boussinesq Equations

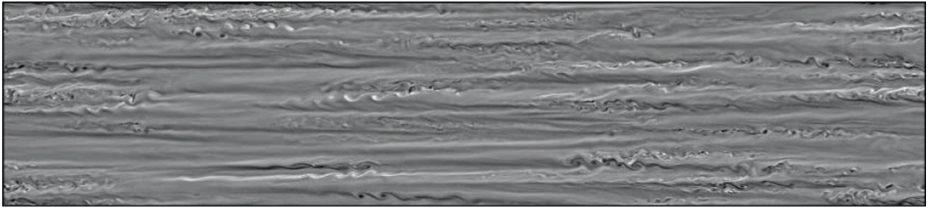
$$\begin{aligned}
\partial_t \mathbf{u}_\perp + (\mathbf{u}_\perp \cdot \nabla_\perp)\,\mathbf{u}_\perp + \frac{Fr^2}{\alpha^2} W \partial_z \mathbf{u}_\perp &= -\nabla_\perp p + \mathcal{D}(\mathbf{u}_\perp) + \mathbf{f}_\perp \\
Fr^2 \left[\partial_t W + (\mathbf{u}_\perp \cdot \nabla_\perp)\, W + \frac{Fr^2}{\alpha^2} W \partial_z W \right] &= -\partial_z p + b + Fr^2 \mathcal{D}(W) \\
\nabla_\perp \cdot \mathbf{u}_\perp + \frac{Fr^2}{\alpha^2} \partial_z W &= 0 \\
\partial_t b + (\mathbf{u}_\perp \cdot \nabla_\perp)\, b + \frac{Fr^2}{\alpha^2} W \partial_z b &= -W + Pr^{-1} \mathcal{D}(b)
\end{aligned}$$

where: $\mathcal{D} = \frac{1}{Re}\left[\nabla_\perp^2 + \frac{1}{\alpha^2}\partial_z^2\right]$ and $\alpha \equiv h/L$, $Fr \equiv U/(NL)$

Limit Equations

- $Fr/\alpha \to 0$ as $Fr \to 0$ (Lilly 1983): Layerwise **2D** flow
- $Fr/\alpha = O(1)$ as $Fr \to 0$ (Billant & Chomaz 2001): Anisotropic **3D** flow

Emergence of Multiple Vertical and Horizontal Scales



–Waite (2014)

- As $Fr \to 0$, vertical length scale self-adjusts so that $h = O(U/N) \;\Rightarrow\; \alpha = O(Fr)$ [Note: then $w : hU/L$]

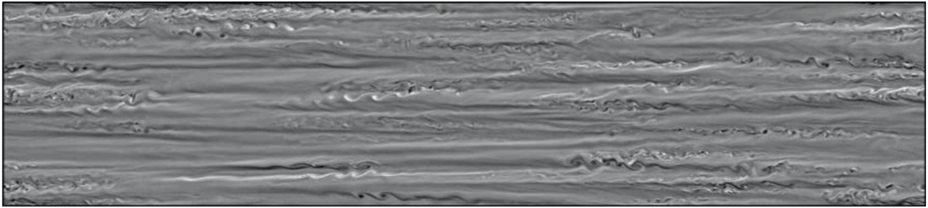
- Stratified turbulence regime *defined* by $Fr \to 0$ with:

  $\alpha = O(Fr)$ and **buoyancy Reynolds number** $\mathcal{R} \equiv ReFr^2 \gtrsim 10$

- Clear evidence of shear instabilities on horizontal scales $l \ll L$, and perhaps modulation on $O(L)$ vertical scales

. . . Opportunity for asymptotically-reduced multiscale modeling. . .

Multiple Scale Asymptotic Analysis



–Waite (2014)

1. Identify relevant **distinguished limit**: $\alpha = Fr$ and $\mathcal{R} \equiv ReFr^2 = O(1)$ as $Fr \to 0$
2. Introduce *fast* horizontal and temporal scales: $\chi_\perp = \mathbf{x}_\perp / Fr$ and $\tau = t/Fr$ so that

$$\nabla_\perp \to \nabla_\mathbf{x} + \frac{1}{Fr}\nabla_\chi \qquad \partial_t \to \partial_t + \frac{1}{Fr}\partial_\tau$$

3. Introduce fast averaging operation and mean/fluctuation decomposition:

$\phi(\mathbf{x}_\perp, z, t) \to \phi(\chi_\perp, \mathbf{x}_\perp, z, \tau, t) = \overline{\phi}(\mathbf{x}_\perp, z, t) + \phi'(\chi_\perp, \mathbf{x}_\perp, z, \tau, t)$, where $\overline{\phi'} \equiv 0$

Expansion *Ansatz*

- Introduce $\epsilon \equiv \sqrt{Fr}$ and posit following asymptotic expansions for various fields:

$$\begin{aligned} [\mathbf{u}_\perp, b, p] &\sim [\mathbf{u}_{0\perp}, b_0, p_0] + \epsilon[\mathbf{u}_{1\perp}, b_1, p_1] + \epsilon^2[\mathbf{u}_{2\perp}, b_2, p_2] + \dots \\ W &\sim \epsilon^{-1}W_{-1} + W_0 + \epsilon W_1 + \dots \end{aligned}$$

- Key prescription is that vertical velocity (normalized by αU) is $O(\epsilon^{-1})$ *on fine horizontal scales* (i.e. when scaled by U, vertical velocity $w \sim \epsilon w_1 + \epsilon^2 w_2 + \dots$)

- This (re-)scaling ensures that feedback of fluctuations upon mean fields through vertical Reynolds stress divergence $\partial_z \left[\overline{W' u'_\perp}\right]$ arises at proper order s.t. there is a dominant balance with tendency $\partial_t \bar{\mathbf{u}}_\perp$ and vertical diffusion $\mathcal{R}^{-1}\partial_z^2 \bar{\mathbf{u}}_\perp$

  Rescaling simultaneously ensures that fine-scale dynamics are **isotropic**

  Compatible with DNS/scaling analysis showing $< w^2 >= Fr\, U^2$ at **small scales** (Maffioli & Davidson, JFM 2016)

- Can then *deduce* that **fluctuating** horizontal velocity, buoyancy, and pressure fields arise at $O(\epsilon)$, a key simplification

Multiscale Reduced PDEs

Mean Equations

$$
\begin{aligned}
\left[\partial_t + (\overline{\mathbf{u}}_{0\perp}\cdot\nabla_{\mathbf{x}}) + \overline{W}_0\partial_z\right]\overline{\mathbf{u}}_{0\perp} + \partial_z\left(\overline{W'_{-1}\mathbf{u}'_{1\perp}}\right) &= -\nabla_{\mathbf{x}}\overline{p}_0 + \frac{1}{\mathcal{R}}\partial_z^2\overline{\mathbf{u}}_{0\perp} + \overline{\mathbf{f}}_{0\perp} \\
0 &= -\partial_z\overline{p}_0 + \overline{b}_0 \\
\nabla_{\mathbf{x}}\cdot\overline{\mathbf{u}}_{0\perp} + \partial_z\overline{W}_0 &= 0 \\
\left[\partial_t + (\overline{\mathbf{u}}_{0\perp}\cdot\nabla_{\mathbf{x}}) + \overline{W}_0\partial_z\right]\overline{b}_0 + \partial_z\left(\overline{W'_{-1}b'_1}\right) &= -\overline{W}_0 + \frac{1}{Pr\mathcal{R}}\partial_z^2\overline{b}_0
\end{aligned}
$$

Multiscale Reduced PDEs

Mean Equations

$$\left[\partial_t + (\overline{\mathbf{u}}_{0\perp} \cdot \nabla_{\mathbf{x}}) + \overline{W}_0 \partial_z\right] \overline{\mathbf{u}}_{0\perp} + \partial_z \left(\overline{W'_{-1} \mathbf{u}'_{1\perp}}\right) = -\nabla_{\mathbf{x}} \overline{p}_0 + \frac{1}{\mathcal{R}} \partial_z^2 \overline{\mathbf{u}}_{0\perp} + \overline{\mathbf{f}}_{0\perp}$$

$$0 = -\partial_z \overline{p}_0 + \overline{b}_0$$

$$\nabla_{\mathbf{x}} \cdot \overline{\mathbf{u}}_{0\perp} + \partial_z \overline{W}_0 = 0$$

$$\left[\partial_t + (\overline{\mathbf{u}}_{0\perp} \cdot \nabla_{\mathbf{x}}) + \overline{W}_0 \partial_z\right] \overline{b}_0 + \partial_z \left(\overline{W'_{-1} b'_1}\right) = -\overline{W}_0 + \frac{1}{Pr\mathcal{R}} \partial_z^2 \overline{b}_0$$

Fluctuation Equations

$$(\partial_\tau + \overline{\mathbf{u}}_{0\perp} \cdot \nabla_\chi)\, \mathbf{u}'_{1\perp} + W'_{-1} \partial_z \overline{\mathbf{u}}_{0\perp} = -\nabla_\chi p'_1 + \frac{Fr}{\mathcal{R}} \left(\nabla_\chi^2 + \partial_z^2\right) \mathbf{u}'_{1\perp}$$

$$(\partial_\tau + \overline{\mathbf{u}}_{0\perp} \cdot \nabla_\chi)\, W'_{-1} = -\partial_z p'_1 + b'_1 + \frac{Fr}{\mathcal{R}} \left(\nabla_\chi^2 + \partial_z^2\right) b'_1$$

$$\nabla_\chi \cdot \mathbf{u}'_{1\perp} + \partial_z W'_{-1} = 0$$

$$(\partial_\tau + \overline{\mathbf{u}}_{0\perp} \cdot \nabla_\chi)\, b'_1 + W'_{-1} \partial_z \overline{\mathbf{b}}_0 = -W'_{-1} + \frac{Fr}{Pr\mathcal{R}} \left(\nabla_\chi^2 + \partial_z^2\right) b'_1$$

Attributes of Multiscale System

- In absence of Reynolds stress ("eddy-flux") divergences (RSDs), mean equations reduce to hydrostatic primitive equations (Billant & Chomaz 2001)
- Vertical RSDs provide crucial feedback of fluctuating fine-scale dynamics on evolution of mean fields, given here without need for *ad hoc* closure
- Fluctuation dynamics are **quasi-linear (QL)** about local mean fields:
  - Reverting to single set of $(\mathbf{x}_\perp, t)$ scales yields [and interpreting $\overline{(\cdot)}$ as strict horizontal mean] yields a **QL reduction** of full Boussinesq equations
  - Suggests **2nd-order cumulant expansion (CE2**) approaches used by Marston & Tobias, Farrell & Ioannou, and Young & Srinivasan can be **formally justified** for stratified shear turbulence in the limit $Fr \to 0$
- By retaining multiple horizontal and temporal scales, can rationally extend QL/CE2 schemes ⇒ **Physical space variant of spectral GQL**

Strict QL Model

- Re-interpreting $\overline{(\cdot)}$ as strict horizontal mean
- Single horizontal scale χ and revert to single time scale τ via $\partial_t \to (1/Fr)\partial_\tau$

Mean Equations

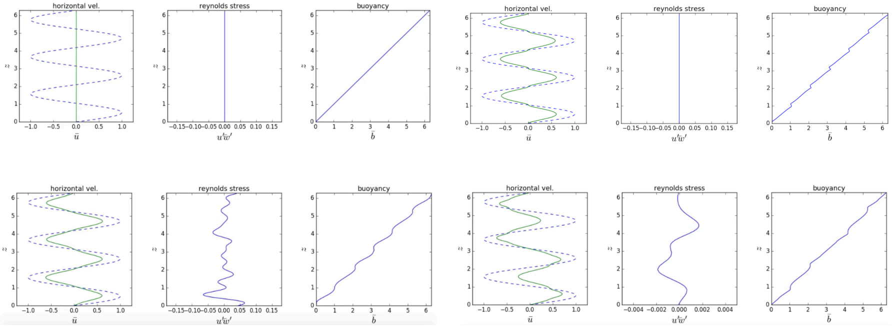
$$\begin{aligned}
\frac{1}{Fr}\partial_\tau \overline{\mathbf{u}}_{0\perp} + \partial_z \left(\overline{W'_{-1}\mathbf{u}'_{1\perp}}\right) &= \frac{1}{\mathcal{R}}\partial_z^2 \overline{\mathbf{u}}_{0\perp} + \overline{\mathbf{f}}_{0\perp} \\
0 &= -\partial_z \overline{p}_0 + \overline{b}_0 \\
\frac{1}{Fr}\partial_\tau \overline{b}_0 + \partial_z \left(\overline{W'_{-1} b'_1}\right) &= \frac{1}{Pr\mathcal{R}}\partial_z^2 \overline{b}_0
\end{aligned}$$

Fluctuation Equations

$$\begin{aligned}
\left(\partial_\tau + \overline{\mathbf{u}}_{0\perp} \cdot \nabla_\chi\right) \mathbf{u}'_{1\perp} + W'_{-1}\partial_z \overline{\mathbf{u}}_{0\perp} &= -\nabla_\chi p'_1 + \frac{Fr}{\mathcal{R}}\left(\nabla_\chi^2 + \partial_z^2\right) \mathbf{u}'_{1\perp} \\
\left(\partial_\tau + \overline{\mathbf{u}}_{0\perp} \cdot \nabla_\chi\right) W'_{-1} &= -\partial_z p'_1 + b'_1 + \frac{Fr}{\mathcal{R}}\left(\nabla_\chi^2 + \partial_z^2\right) b'_1 \\
\nabla_\chi \cdot \mathbf{u}'_{1\perp} + \partial_z W'_{-1} &= 0 \\
\left(\partial_\tau + \overline{\mathbf{u}}_{0\perp} \cdot \nabla_\chi\right) b'_1 + W'_{-1}\partial_z \overline{\mathbf{b}}_0 &= -W'_{-1} + \frac{Fr}{Pr\mathcal{R}}\left(\nabla_\chi^2 + \partial_z^2\right) b'_1
\end{aligned}$$

Evolution of Mean Fields for (2D) Strongly Stratified Kolmogorov Flow

- Body forcing $\overline{f}_{0x} = (m^2/\mathcal{R})\sin(mz)$ and imposed (initially) linear stratification
- Parameters: $m = 3$, $\mathcal{R} = 200$, $Fr = 0.02$



Extending QL Framework for Increased Accuracy and Efficiency

Some observations. . .

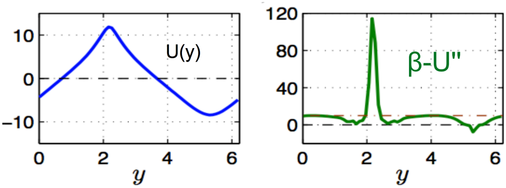
- QL simulations can be as expensive as DNS (!) owing to temporal stiffness and inclusion of many fluctuation (zonally-varying) modes, particularly in large domains
- Generalized QL (GQL) formulation has been shown to increase accuracy, but not necessarily efficiency
- Statistical formulations of (G)QL dynamics [(G)CE2] may offer certain efficiencies but increases spatial dimensionality. . .

Alternative approach for systematically extending QL reduction and increasing computational efficiency relies on 3 ingredients:

1. Self-tuning of QL systems toward **"statistical marginal stability"**
2. Evolution of QL fluctuation energy spectrum toward **narrow-banded distribution**
3. Exploitation of **multiple scales** in t, x (and possibly z)

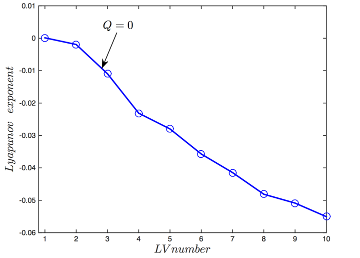
Statistical Marginal Stability of QL/CE2 Systems

DSS of Barotropic Jets



N. Constantinou. Formation of large-scale structures by turbulence in rotating planets. arXiv:1503.07644, 2015.

DSS of Plane Couette Flow

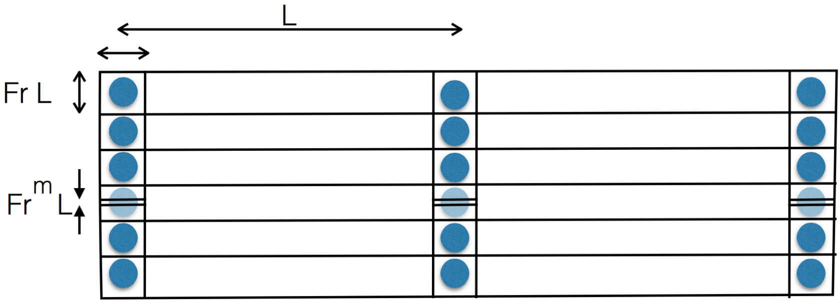


$$\partial_t \mathbf{U} = \mathbf{P}_L \left(-\mathbf{U} \cdot \nabla \mathbf{U} + \frac{1}{R} \Delta \mathbf{U} \right) + \mathcal{L}_k \mathbf{C}_k,$$

$$\partial_t \mathbf{C}_k = \mathbf{A}_k(\mathbf{U})\, \mathbf{C}_k + \mathbf{C}_k \left(\mathbf{A}_k(\mathbf{U}) \right)^\dagger + \mathbf{Q}_k.$$

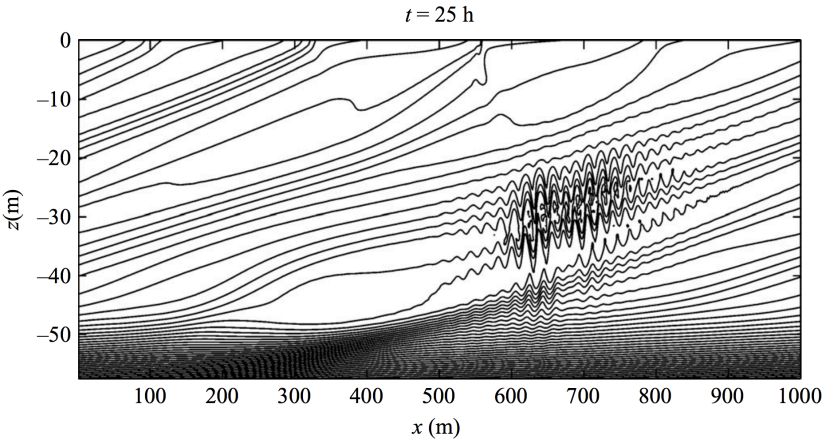
B. Farrell & P. Ioannou. Structure and mechanism in a second-order statistical state dynamics model of self-sustaining turbulence in plane Couette flow. arXiv:1607.05020, 2016.

Beyond QL Dynamics: Multiple Scales in t and x and/or z



- Desirable to extend model (i.e. beyond a QL representation) either to enhance accuracy or efficiency or both
- Develop multiscale numerical algorithm based on retention of multiple space and time scales

Other Applications: Secondary Shear Instability of Symmetrically-Unstable Fronts



J. Taylor & R. Ferrari. On the equilibration of a symmetrically unstable front via a secondary shear instability (JFM 2009).