Direct Statistical Simulation of Geophysical Flows

arXiv: 1412.0381

I. 2 Purposes:

1) Better understanding. Forest from trees. (Lorenz) Global and Local Models

Practical:

2) Possible gains in resolution SGS Local (SGS) models. (Not yet realized — "curse of dimensionality" must be addressed) Access to extreme events.

II. What statistics?

1) Single-point PDFs? ✓ Joint PDFs?

2) Low order moments (equal time) ✓

3) Autocorrelations (different times)

4) Extreme events?

✗ In general: Non-local, anisotropic + heterogeneous.

III. What averaging?

1) Time - difficult technically

2) Ensemble ✓

3) Spatial ✓

IV Models

1) Deterministic vs. Stochastic?

2) Quadratic non-linearities? $(\vec{v}\cdot\vec{\nabla})\vec{v}$ ✓
(N.B. Latent Heat)

$$\begin{cases} \partial_t q = F + L[q] + Q[q,q] + \eta(t) \\ \dot{q}_i = F_i + L_{ij}\, q_j + Q_{ijk}\, q_j\, q_k + \eta_i(t). \end{cases}$$

V. Hopf Generating Functional

$$\Psi[\vec{p},\vec{q}] = e^{i\vec{p}\cdot\vec{q}(t)}$$

$$\overline{\Psi}[\vec{p},t] = \overline{e^{i\vec{p}\cdot\vec{q}(t)}}$$

$$i\frac{\partial\Psi}{\partial t} = -\vec{p}\cdot\frac{\partial\vec{q}}{\partial t}\,\Psi$$

$$\frac{\partial\Psi}{\partial p_i} = i\,q_i\,\Psi \qquad \frac{\partial^2\Psi}{\partial p_i\,\partial p_j} = -q_i\,q_j\,\Psi$$

$$i \frac{\partial \Psi}{\partial \tau} = -p_i \left[F_i - i L_{ij} \frac{\partial}{\partial p_j} - Q_{ijk} \frac{\partial^2}{\partial p_j \partial p_k} \right] \Psi$$

Linear, so $\overline{\Psi}$ also a solution.

$$\overline{\Psi} = \overline{e^{i \vec{p} \cdot \vec{q}(\tau)}}$$ Generates equal-time moments.

$$\Psi(p=0, \tau) = 1 \qquad \Psi(p \to \infty, t) = 0$$

$$-i \frac{\partial \Psi}{\partial p_i}\bigg|_{p=0} = \overline{q_i}, \qquad -\frac{\partial^2 \Psi}{\partial p_i \partial p_j}\bigg|_{p=0} = \overline{q_i q_j}, \text{ etc.}$$

Fourier x-form to get PDF

$$\left(\frac{1}{2\pi}\right)^D \int d^D p \, e^{-i \vec{p} \cdot \vec{x}} \, \overline{e^{i \vec{p} \cdot \vec{q}(\tau)}} = \overline{\delta(\vec{x} - \vec{q}(\tau))} = P(\vec{x}, \tau)$$

Duality $\quad -i \frac{\partial}{\partial p_j} \leftrightarrow x_j \qquad p_k \leftrightarrow +i \frac{\partial}{\partial x_k} \qquad \Psi \Leftrightarrow P$

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x_i} \left[F_i + L_{ij} x_j + Q_{ijk} x_j x_k \right] P$$

$$= -\vec{\nabla} \cdot (\vec{v} P)$$

$$\frac{\partial P}{\partial t} = - \vec{\nabla} \cdot (\vec{V} P) + \Gamma \vec{\nabla}^2 P = \hat{L}_{FPE} P$$

stochastic noise.

$$\vec{V}(\vec{x}) = \vec{F} + \vec{L}(\vec{x}) + \vec{Q}[\vec{x}, \vec{x}]$$

Steady-state: $\hat{L}_{FPE} P = 0$.

(Show PDF's for Lorenz Attractor)

Cumulant Expansion

$$\Psi(\vec{p},\tau) = \overline{e^{i\vec{p}\cdot\vec{q}(\tau)}} = e^{iC_i p_i - \frac{1}{2}C_{ij}p_i p_j}$$

$$\left.\frac{\partial \Psi}{\partial p_i}\right|_{p=0} = i\overline{q_i} = iC_i \qquad C_i = \overline{q_i}$$

$$\left.\frac{\partial^2 \Psi}{\partial p_i \partial p_j}\right|_{p=0} = -\overline{q_i q_j} = -C_i C_j - C_{ij}$$

$$\rightarrow \quad C_{ij} = \overline{q_i q_j} - \overline{q_i}\,\overline{q_j}$$

$$= \overline{q_i' q_j'}$$

$q = \overline{q} + q'$ Reynolds decomposition.

$\overline{\overline{q}} = \overline{q}$, $\overline{q'} = 0$, $\overline{\overline{q}q} = \overline{q}\,\overline{q}$.

$C_{ijk} = \overline{q_i' q_j' q_k'}$ Centered moments.

$C_{ijkl} = \overline{q_i' q_j' q_k' q_l'} - C_{ij}C_{kl} - C_{ik}C_{jl} - C_{il}C_{jk}$.

$$\dot{C}_i = \overline{\dot{q}_i} = F_i + L_{ij}\overline{q_j} + Q_{ijk}\overline{q_j q_k}$$

$$= F_i + L_{ij}C_j + Q_{ijk}\left(C_j C_k + C_{jk}\right)$$

$$\dot{C}_{ij} = 2\left\{\overline{\dot{q}_i' \, q_j'}\right\} \qquad \{C_{ij}\} \doteq \tfrac{1}{2}(C_{ij} + C_{ji})$$

$$= 2\left\{\overline{(\dot{q}_i - \bar{\dot{q}}_i)\, q_j'}\right\}$$

$$\frac{\partial}{\partial r}\left(\overline{\bar{q}\, q'}\right) = 0$$

$$= 2\left\{\overline{\dot{q}_i \, q_j'}\right\}$$

$$= 2\left\{\overline{L_{ik}\, q_k\, q_j' + Q_{ike}\, q_k\, q_e\, q_j'}\right\} + 2\Gamma_{ij}$$

CE2

$$= \left\{2 L_{ik}\, C_{kj} + Q_{ike}\,(4 C_k C_{ej} + 2 \underbrace{\cancel{C_{kej}}}_{\text{3rd cumulant.}})\right\} + 2\Gamma_{ij}$$

$$\overline{q_m' \, q_n' \, q_j' \, q_k'} = C_{mn} C_{jk} + C_{mj} C_{nk} + C_{mk} C_{jn} + \cancelto{0}{C_{mnjk}}$$

CE3

Demonstrate Lorenz Attractor.

Stochastically-driven barotropic jets. (See Laura Cope's poster)

$\omega = \hat{r} \cdot \vec{\nabla} \times \vec{v}$

$$\dot{\omega} = J[\omega + f, \psi] + L[\omega] + \eta(t)$$

$$L[\omega] = -[\kappa - \nu_3 \nabla^6]\omega \qquad f(\phi) = 2\Omega \sin\phi$$

$$J[A, B] = \hat{r} \cdot (\vec{\nabla} A \times \vec{\nabla} B)$$

$$\omega = \nabla^2 \psi$$

More abstractly:

$$\dot{q} = \mathcal{L}[q] + \mathcal{Q}[q, q] + \eta$$

Reynolds decompose (could be mean-flow / eddy)

$$q = \bar{q} + q' \qquad \overline{q'} = 0, \quad \bar{\bar{q}} = \bar{q},$$

$$\overline{\bar{q}\,q} = \bar{q}\,\bar{q}$$

$$\overline{\dot{q}} = \dot{\bar{q}} = \overline{\mathcal{L}[q]} + \underbrace{\overline{\mathcal{Q}[q, q]}}_{} + 0$$

$$\overline{\mathcal{L}[q]} = \mathcal{L}[\bar{q}]$$

involves $\overline{q'(\vec{r}_1)\, q'(\vec{r}_2)} + \bar{q}(\vec{r}_1)\, \bar{q}(\vec{r}_2)$

If $\overline{}$ = zonal average = $\frac{1}{2\pi}\int_0^{2\pi} d\lambda$ Then

* Symmetry reduces dimensionality:

$$C_2(\phi_1, \phi_2, \lambda_1 - \lambda_2) = \overline{q'(\phi_1, \lambda_1)\, q'(\phi_2, \lambda_2)}$$

(Make no further assumptions of isotropy or homogeneity)

How to determine? (Closure problem)

$$\frac{\partial}{\partial t}\overline{q(\vec{r}_1)\, q(\vec{r}_2)} = \overline{q(\vec{r}_1)\, L[q(\vec{r}_2)]} + \overline{q(\vec{r}_1)\, Q[q(\vec{r}_2), q(\vec{r}_2)]} + (1 \leftrightarrow 2)$$

Connection to Structure Functions

Now need 3rd cumulant! Decouple by ignoring contribution of $\overline{q'q'q'}$ to tendency of 2nd cumulant. Farrell + Ioannou

<u>CE2</u> <u>S3T</u>

What does this look like for our fluid system?

Hopf.

$$\begin{cases} c_1(\vec{r}_1) \equiv \overline{\omega(\vec{r}_1)} & p_1(\vec{r}_1) \equiv \overline{\psi(\vec{r}_1)} \\ c_2(\vec{r}_1, \vec{r}_2) \equiv \overline{\omega'(\vec{r}_1)\,\omega'(\vec{r}_2)} & p_2(\vec{r}_1, \vec{r}_2) \equiv \overline{\omega'(r_1)\,\psi'(r_2)} \end{cases}$$

Also, $\psi(\vec{r}_1) = \int \delta(\vec{r}_1 - \vec{r}_2)\,\psi(\vec{r}_2)\, d^2 r_2$

So $\overline{J_1[\omega(\vec{r}_1), \psi(\vec{r}_1)]} = J_1[\overline{\omega(\vec{r}_1)\,\psi(\vec{r}_2)}, \delta(\vec{r}_1 - \vec{r}_2)]\, d^2 r_2$

$$\frac{\partial c(\vec{r}_1)}{\partial \tau} = L_1[c(\vec{r}_1)] + \overbrace{\int J_1[p(\vec{r}_1, \vec{r}_2), \delta(\vec{r}_1 - \vec{r}_2)] \, d^2 r_2}^{\text{Reynolds forcing.}}$$

Since $J_1[c(\vec{r}_1) + f(\vec{r}_1), p(\vec{r}_1)] = 0$, as the 1st cumulants don't vary with longitude.

$$C_2(\vec{r}_1, \vec{r}_2) = \nabla_2^2 \, p(\vec{r}_1, \vec{r}_2).$$

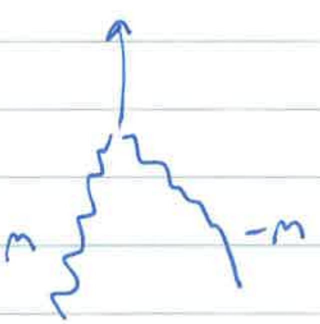
↓ advection of eddies by mean flow

$$\frac{\partial c(\vec{r}_1, \vec{r}_2)}{\partial \tau} = \{2 L_1[c(\vec{r}_1, \vec{r}_2)] + J_1[c(\vec{r}_1) + f(\vec{r}_1), p_2(\vec{r}_1, \vec{r}_2)] + J_1[c(\vec{r}_1, \vec{r}_2), p(\vec{r}_1)]\} + \Gamma(\vec{r}_1, \vec{r}_2).$$

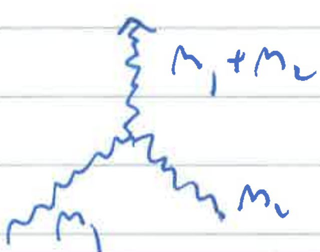
$$\langle \eta(\vec{r}_1, \tau_1)\, \eta(\vec{r}_2, \tau_2) \rangle = 2\Gamma(\vec{r}_1, \vec{r}_2)\, \delta(\tau_1 - \tau_2)$$

short time average. Analytically!

$$\{C(\vec{r}_1, \vec{r}_2)\} \equiv \frac{1}{2}[C(\vec{r}_1, \vec{r}_2) + C(\vec{r}_2, \vec{r}_1)]$$



condensate.



CE2 exact for quasi-linear approximation, hence realizable + conservative.

Speed-ups: (1) Average over stochastic forcing analytically.

(2) Fixed-pt or slowly varying statistics. Semi-implicit schemes.

(3) But: Need dimensional reduction. POD / PCA / ML ?

Topologically-protected waves. (Antoine Venaille).

Boundary Layers – SGS.

Beyond CE2: (1) CE3 / CE2.5

(2) GQL / GCE2.

(3) Ensemble averaging.

Other Non-linearities: Latent Heat.

APS GPC

Planetary Boundary Layers Conference @ KITP
May 21–25.