A CONTEMPORARY CHRONICLE OF OCEAN WAVE / SEA ICE INTERACTION RESEARCH: CONTEXT, MODELS, DELUSIONS AND IMPACTS

Vernon Squire

University of Otago, New Zealand

Kavli Institute
20th–24th May 2018
Santa Barbara, USA

Isaac Newton Institute for Mathematical Sciences

The talk

- Background for non sea ice folk.
- Background for non wave-ice interaction folk, including the marginal ice zone.
- Two modelling paradigms I and II, to be defined.
- My focus has primarily been paradigm I.
- But there is also a demand to operationalize paradigm II models, e.g. in global climate models and global-scale wave forecasting models such as WAVEWATCH III®.

The talk

- Background for non sea ice folk.
- Background for non wave-ice interaction folk, including the marginal ice zone.
- Two modelling paradigms I and II, to be defined.
- My focus has primarily been paradigm I.
- But there is also a demand to operationalize paradigm II models, e.g. in global climate models and global-scale wave forecasting models such as WAVEWATCH III®.

The talk

- Background for non sea ice folk.
- Background for non wave-ice interaction folk, including the marginal ice zone.
- Two modelling paradigms I and II, to be defined.
- My focus has primarily been paradigm I.
- But there is also a demand to operationalize paradigm II models, e.g. in global climate models and global-scale wave forecasting models such as WAVEWATCH III®.

The talk

- Background for non sea ice folk.
- Background for non wave-ice interaction folk, including the marginal ice zone.
- Two modelling paradigms I and II, to be defined.
- My focus has primarily been paradigm I.
- But there is also a demand to operationalize paradigm II models, e.g. in global climate models and global-scale wave forecasting models such as WAVEWATCH III®.

The talk

- Background for non sea ice folk.
- Background for non wave-ice interaction folk, including the marginal ice zone.
- Two modelling paradigms $\mathbb{I}$ and $\mathbb{II}$, to be defined.
- My focus has primarily been paradigm $\mathbb{I}$.
- But there is also a demand to operationalize paradigm $\mathbb{II}$ models, e.g. in global climate models and global-scale wave forecasting models such as WAVEWATCH III®.

The talk

- Background for non sea ice folk.
- Background for non wave-ice interaction folk, including the marginal ice zone.
- Two modelling paradigms $\mathbb{I}$ and $\mathbb{II}$, to be defined.
- My focus has primarily been paradigm $\mathbb{I}$.
- But there is also a demand to operationalize paradigm $\mathbb{II}$ models, e.g. in global climate models and global-scale wave forecasting models such as WAVEWATCH III®.

Sea ice 101 – a deterrent for aspiring modellers

Some convenient (or otherwise) over-simplified facts about sea ice ...

- It is cold (air temperature) at the top and warm ($\sim -1.8^\circ$C) at the bottom, *in situ*.
- Its properties depend on its growth history.
- It has a complex structure composed of ice crystals, air and brine (and other salts) plus, in some circumstances, natural organic matter.
- It is anisotropic and spatially and temporally heterogeneous.
- Its mechanical behaviour depends on strain rate and spatial scale.
- It can appear in Nature as various types of continuous sheets (fast ice, grease ice, nilas, etc.) or separate floes 1–2 m across (pancakes) to vast floes of $\gtrapprox$km size.
- Level sea ice thicknesses can range from O(mm) up to O(10 m) or even greater, but pressure ridges can be up to 50 m keel-to-sail.
- It may have snow on top and frazil crystals and/or platelets underneath.
- The region of loose ice floes between the open ocean and the interior quasi-continuous pack ice is often called the marginal ice zone (MIZ).
- Surface gravity waves, up/downwelling, jets, streamers and bands, and eddies are symptomatic of the MIZ.

Wave/ice interaction 101

We know that ocean waves in the range of periods $T \approx 4$–20 s ...

- Reduce in amplitude as they travel through sea ice fields due to
  - *dissipative* energy loss
  - *conservative* wave scattering, i.e. redistribution of wave energy.
- Break up ice floes, if waves are sufficiently energetic.
- Move floes around.

With climate warming, we expect to see ...

- Weakened and more compliant Arctic summer sea ice.
- Perennial Arctic sea ice becoming more seasonal.
- More mobile ice floes, with a greater incidence of leads and polynyas.
- A less compact ice canopy that resembles a MIZ.
- As confirmed by satellite radar altimeter data, higher and longer ocean waves generated globally (also locally because of larger aggregated fetches), which
  - penetrate further into the ice cover, as their amplitudes are larger
  - have greater destructive payload to pummel and break up the sea ice
  - promote further melting in summer or freezing in winter by helping winds and currents move ice floes around.

Wave/ice interaction 101

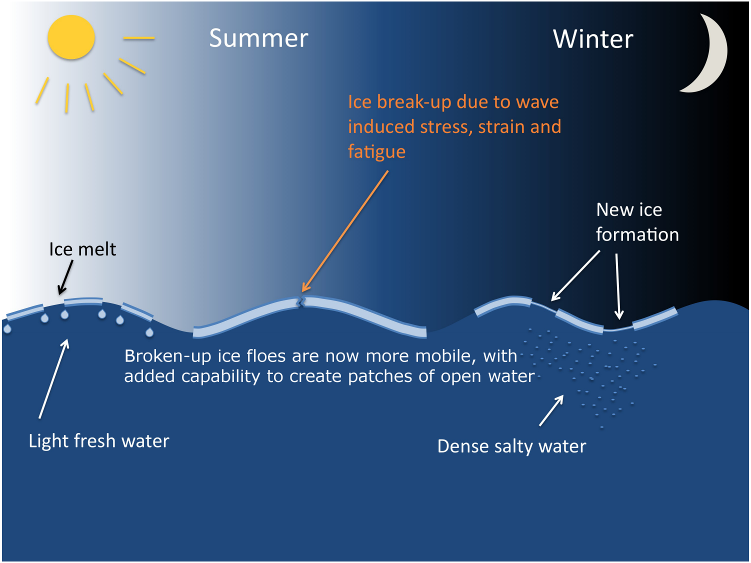
We know that ocean waves in the range of periods $T \approx 4$–$20\,\mathrm{s}$ ...

- Reduce in amplitude as they travel through sea ice fields due to
  - *dissipative* energy loss
  - *conservative* wave scattering, i.e. redistribution of wave energy.
- Break up ice floes, if waves are sufficiently energetic.
- Move floes around.

With climate warming, we expect to see ...

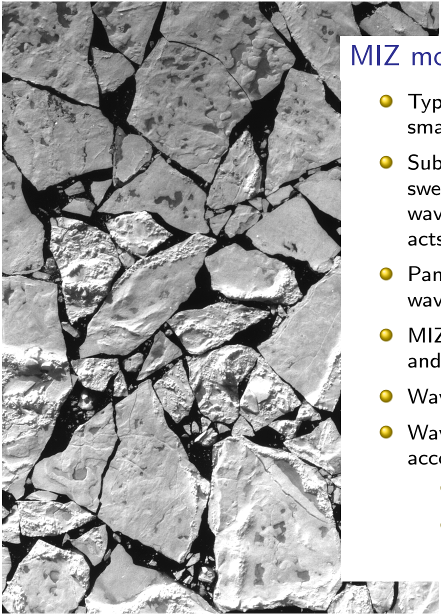
- Weakened and more compliant Arctic summer sea ice.
- Perennial Arctic sea ice becoming more seasonal.
- More mobile ice floes, with a greater incidence of leads and polynyas.
- A less compact ice canopy that resembles a MIZ.
- As confirmed by satellite radar altimeter data, higher and longer ocean waves generated globally (also locally because of larger aggregated fetches), which
  - penetrate further into the ice cover, as their amplitudes are larger
  - have greater destructive payload to pummel and break up the sea ice
  - promote further melting in summer or freezing in winter by helping winds and currents move ice floes around.

Seasonal effects of ocean waves entering ice fields[1]



[1] Courtesy Alison Kohout 2013, with addendum

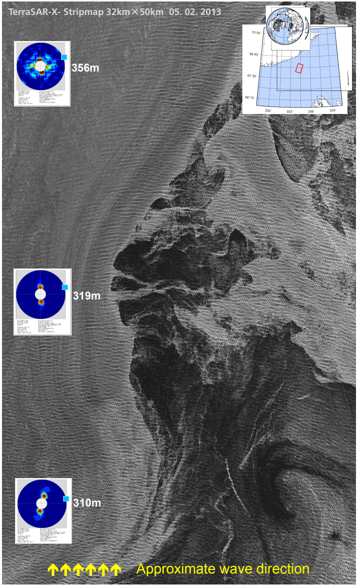
MIZ genesis and maintenance due to waves



MIZ morphology

- Typically composed of a random distribution of small floes with radius $r \sim O(10\text{–}100)$ m.
- Substantial wave activity typically exists in the swell regime $T = 4\text{–}20$ s, so that $kr =$ wave number $\times$ floe radius $= O(1)$ and scattering acts to redistribute wave energy.
- Pancake ice where $r \lessapprox 1$ m is unlikely to scatter waves.
- MIZ created primarily by wave-induced ice breaking and jostling.
- Wave energy dissipation is also important.
- Waves attenuate and radiate preferentially, according to their frequency, viz.
    - shorter waves reduce in amplitude more quickly than longer ones do (low pass filter);
    - shorter waves spread angularly more quickly than longer ones do.

Example. German TerraSAR-X image



⟸

32 km × 50 km, 1-m-resolution TerraSAR

X-band SAR image of wave propagation into the Greenland Sea MIZ.[2] A storm located south of Greenland causes substantial ocean waves to propagate into the sea ice, where they are attenuated with increasing distance from the ice edge with an apparent growth of aggregated wavelength and a possible broadening of directional spread. The small growth in wavelength is probably due to spectral evolution caused by the preferential attenuation of shorter period waves as opposed to dispersion.

[2] Lehner et al. (2013)

Models of MIZ wave/ice interactions

There are two preeminent modelling paradigms for wave/ice interaction . . .

- **Paradigm I.** Methodology that endeavours to represent the physics of each constituent process as faithfully as possible through state variables, acknowledging from the outset that approximations are inevitable.
- **Paradigm II.** A continuum ersatz, which leads naturally into parameterizations that can potentially easily be incorporated into global-scale wave forecasting models such as WAVEWATCH III® or global climate models.

  But do such parameterizations faithfully represent waves propagating in the MIZ, for example?

Moreover

- Mostly all current models of wave propagation through sea ice use $d_x A = -\alpha A$ to describe how the wave amplitude $A = A(x)$ reduces with distance x travelled, i.e. they fit an exponential curve to data.[3] Some observations suggest that a paradigm II model of the form $d_x A = -\alpha A^n$ might be worth considering, arising from nonlinear dissipation.

[3] $A(x)$ is a proxy wave amplitude normally derived from energy density spectra collected by buoys located at increasing distances from the ice edge

Models of MIZ wave/ice interactions

There are two preeminent modelling paradigms for wave/ice interaction ...

- **Paradigm I.** Methodology that endeavours to represent the physics of each constituent process as faithfully as possible through state variables, acknowledging from the outset that approximations are inevitable.
- **Paradigm II.** A continuum ersatz, which leads naturally into parameterizations that can potentially easily be incorporated into global-scale wave forecasting models such as WAVEWATCH III® or global climate models.

  But do such parameterizations faithfully represent waves propagating in the MIZ, for example?

Moreover

- Mostly all current models of wave propagation through sea ice use $d_x A = -\alpha A$ to describe how the wave amplitude $A = A(x)$ reduces with distance x travelled, i.e. they fit an exponential curve to data.[3] Some observations suggest that a paradigm II model of the form $d_x A = -\alpha A^n$ might be worth considering, arising from nonlinear dissipation.

[3] $A(x)$ is a proxy wave amplitude normally derived from energy density spectra collected by buoys located at increasing distances from the ice edge

Paradigm I examples

- Hydroelastic models of wave propagation into and within continuous sea ice,[4] including across cracks, changes of thickness or other physical properties, open and refrozen leads, pressure ridges, etc., and combinations of such features.
- Viscous[5] or viscoelastic[6] surface layer models of grease ice or pancake ice at high concentrations.
- Representations of ocean waves entering and propagating in MIZs.[7]

Paradigm I models are constructed from physically plausible assumptions that can be independently verified experimentally, e.g. using a Kirchhoff-Love plate or a viscoelastic layer with prescribed state variables such as the elastic shear modulus G and viscosity μ, floating on an inviscid ocean.

I am not saying such models are perfect or 100% accurate. I am arguing for the fidelity of the parameterization and the strength of its link to physics.

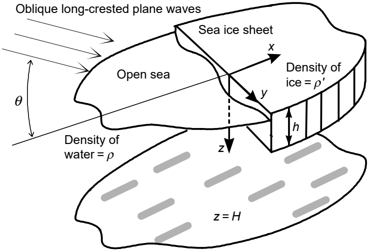
[4] e.g. Fox and Squire (1990, 1994), where the reflexion and transmission coefficients for surface gravity waves entering a fast ice sheet are calculated precisely for the first time

[5] Keller (1998), Carolis and Desiderio (2002)

[6] Wang and Shen (2010), Zhao and Shen (2015a,b), Zhao et al. (2015)

[7] e.g. Masson and Leblond (1989), Perrie and Hu (1996, 1997), Vaughan et al. (2009), Montiel et al. (2016), Montiel and Squire (2017), ...

Canonical problem – a uniform semi-infinite sea ice sheet[4]



- Many published solutions – most early ones assuming zero draught, with a multitude of methods (eigenfunctions, integral equation).
- Assume $\Phi(\boldsymbol{x}, t) = \mathrm{Re}\left[\phi(x, z)e^{i(ly-\omega t)}\right]$.
- Represent ice sheet by Kirchhoff-Love plate with flexural rigidity D. Characteristic length and time defined $L_c = \sqrt[4]{D/\rho g}$, $\tau_c = \sqrt{L_c/g}$.
- Nondimensionalize. $L = L_c(\omega\tau_c)^{-2/5} = \sqrt[5]{D/\rho\omega^2}$ and time with respect to $\tau = \tau_c \implies$ ice and wave properties embedded and ice dispersion relation described by $\varpi = \lambda - \mu = g/L\omega^2 - h\rho'/L\rho$.
- Define $\mathcal{L}(x, \partial_x) = D(x)(\partial_x^2 - l^2)^2 + \lambda - \mu m(x)$, such that $D(x)$ and $m(x)$ are 0 for water or 1 for ice.

$$(\nabla^2 - l^2)\phi(x, z) = 0$$

$$\mathcal{L}(x, \partial_x)\phi_z(x, 0) + \phi(x, 0) = 0$$

$$\phi_x(x^+, z) - \phi_x(x^-, z) = \phi(x^+, z) - \phi(x^-, z) = 0, \ \phi_z(x, H) = 0$$

$$\mathcal{L}_-(\partial_x)\phi_z(0^+, 0) = 0, \ \mathcal{L}_+(\partial_x)\phi_{zx}(0^+, 0) = 0 \text{ with } \mathcal{L}_\pm(\partial_x) = (\partial_x^2 - l^2) \mp (1-\nu)l^2.$$

Radiation conditions $\implies \phi(x, z) \sim \begin{cases} (e^{ikx} + Re^{-ikx})\varphi(z, k) & \text{as } x \to -\infty \\ Te^{i\alpha_0 x}\varphi_0(z, \alpha_0) & \text{as } x \to \infty. \end{cases}$

Wave numbers are $\kappa = (k^2 + l^2)^{1/2}$ and $\gamma_0 = (\alpha_0^2 + l^2)^{1/2}$.

$s|T|^2 = 1 - |R|^2$ links the reflection and transmission coefficients where s is the intrinsic admittance.

An algebraic solution[8] is found for $\mu = 0$, viz. $|R| = (k - \alpha_0)/(k + \alpha_0)$, $|T| = 2k\alpha_0/(k + \alpha_0)$, $s = \alpha_0/k\omega^2$.

[8] Tkacheva (2011)

Tweaking the canonical 1

Propagation across cracks

N cracks located at points $x = x_n$ in the closed finite interval $[0, d]$ is close to that just set out, with minor changes as follows

$$\phi(x,z) \sim \begin{cases} (e^{i\alpha_0 x} + Re^{-i\alpha_0 x})\varphi_0(z,\alpha_0) & \text{as } x \to -\infty \\ Te^{i\alpha_0 x}\varphi_0(z,\alpha_0) & \text{as } x \to \infty \end{cases}$$

$$\mathcal{L}(\partial_x) = (\partial_x^2 - l^2)^2 + \varpi, \; x \in (-\infty, \infty) \setminus \cup \left\{ (x_n^-, x_n^+) \right\}$$

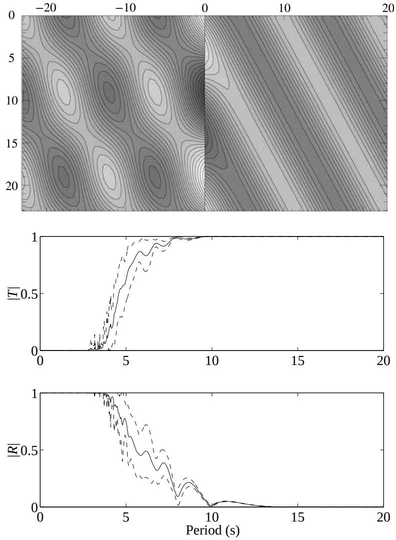
For each x_n, $\mathcal{L}_-(\partial_x)\phi_z(x_n^\pm, 0) = 0$

$$\mathcal{L}_+(\partial_x)\phi_{zx}(x_n^\pm, 0) = 0$$

$|T|^2 = 1 - |R|^2$, i.e. $s = 1$.

Plots to the right show solutions found using Green's functions[9]

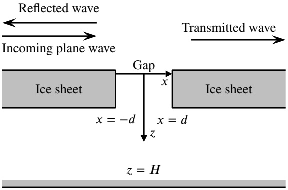
Upper. A snapshot of the surface displacement on either side of a crack for an obliquely incident wave ($\theta = 30^\circ$) from the left such that the wavelength/thickness ratio is 100 and the wavelength/depth ratio is 2.5.
Lower. Magnitude of the reflection and transmission coefficients when $\theta = 0$ due to 5 randomly-spaced open cracks located within 100 m of ice of thickness 1.0 m. One standard deviation about the curve is also plotted in each case.



[9] Squire and Dixon (2001), Williams and Squire (2002), Evans and Porter (2003)

Tweaking the canonical 2

Propagation across leads



Two semi-infinite ice sheets are separated by a gap of width $2d$, which may be open or refrozen.[10]

$$(D(x), m(x)) = (0,0) \text{ for } |x| < d$$
$$= (1,1) \text{ for } |x| > d$$

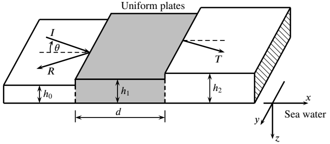
For $x = \pm d$, $\mathcal{L}_-(\partial_x)\phi_z(\pm d^\pm, 0) = 0$,

$$\mathcal{L}_+(\partial_x)\phi_{zx}(\pm d^\pm, 0) = 0$$

$|T|^2 = 1 - |R|^2$, i.e. $s = 1$.

Localized changes of thickness[10]

Thickness of any of the three plates may be set to zero.



Modified canonical system is

$$\phi(x,z) \sim \begin{cases} (e^{i\alpha_0 x} + Re^{-i\alpha_0 x})\varphi_0(z,\alpha_0) & \text{as } x \to -\infty \\ Te^{i\alpha_2 x}\varphi_2(z,\alpha_2) & \text{as } x \to \infty \end{cases}$$

$$(D(x), m(x)) = \begin{cases} (D_0, m_0) & \text{for } x < 0 \\ (D_1, m_1) & \text{for } 0 < x < d \\ (D_2, m_2) & \text{for } x > d \end{cases}$$

For free edges $x_e = 0$ and $x_e = d$

$$\mathcal{L}_-(\partial_x)\phi_z(x_e^\pm, 0) = 0 \, \mathcal{L}_+(\partial_x)\phi_{zx}(x_e^\pm, 0) = 0$$

For frozen edges $x_e = 0$ and $x_e = d$

$$\phi_z(x_e^+, 0) = \phi_z(x_e^-, 0), \; \phi_{zx}(x_e^+, 0) = \phi_{zx}(x_e^-, 0)$$
$$D(x_e^+)\mathcal{L}_-(\partial_x)\phi_z(x_e^+, 0) = D(x_e^-)\mathcal{L}_-(\partial_x)\phi_z(x_e^-, 0)$$
$$D(x_e^+)\mathcal{L}_+(\partial_x)\phi_{zx}(x_e^+, 0) = D(x_e^-)\mathcal{L}_+(\partial_x)\phi_{zx}(x_e^-, 0).$$

[10] Squire and Dixon (2001), Chung and Linton (2005), Williams and Squire (2006)

Tweaking the canonical 3

Realistic sea-ice terrain[11]

A region of variable properties welded to the surrounding sheets surrounded by uniform ice sheets. The operator is redefined as follows

$$\mathcal{L}(x,\partial_x)=(\partial_x^2-l^2)\left[D(x)(\partial_x^2-l^2)\right] + (1-\nu)l^2D''(x)+\lambda-\mu m(x)$$

where

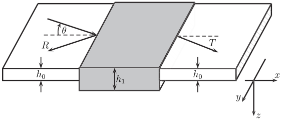
$$(D(x),m(x))=\begin{cases}(D(x),m(x)) & \text{for } x\in(0,d),\\ (1,1) & \text{for } x\notin(0,d)\end{cases}$$

$$\phi_z(x_c^+,0)=\phi_z(x_c^-,0),\ \phi_{zx}(x_c^+,0)=\phi_{zx}(x_c^-,0)$$

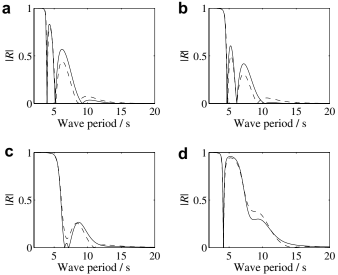
$$D(x_c^+)\mathcal{L}_-(\partial_x)\phi_z(x_c^+,0)= D(x_c^-)\mathcal{L}_-(\partial_x)\phi_z(x_c^-,0)$$

$$\left(D(x_c^+)\mathcal{L}_+(\partial_x)\partial_x+D_x(x_c^+)\mathcal{L}_-(\partial_x)\right)\phi_z(x_c^+,0)= \left(D(x_c^-)\mathcal{L}_+(\partial_x)\partial_x+D_x(x_c^-)\mathcal{L}_-(\partial_x)\right)\phi_z(x_c^-,0)$$

The inclusion of draft[12]



Scattering of normally incident waves by a 2-m-thick strip of width (a) 10 m, (b) 20 m, (c) 50 m and (d) 100 m with free edges floating on infinitely deep water between two semi-infinite 1-m-thick sheets. Solid curves are for realistic draught, dashed curves are for zero submergence.

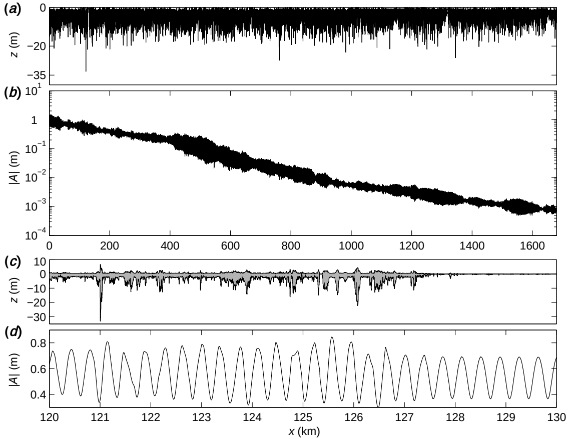


[11] Williams and Squire (2004)

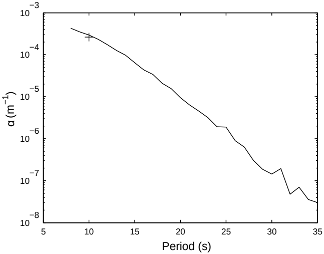
[12] Bennetts et al. (2007), Squire and Williams (2008), Williams and Squire (2008)

Flexural waves in continuous variable sea ice example[13]

(a) A 1670-km-long Arctic sea ice profile obtained in 1994 from submarine using upward looking sonar; (b) the magnitude of the vertical displacement $|A|$ of a 22 s ocean wave train as it progresses through ice terrain for the entire 1670 km transect; (c) 10 km example of the thickness profile used in the model found using isostasy, showing ridge sails and keels; and (d) 10 km detail of $|A|$. All are plotted against the horizontal coordinate x km.



Amplitude attenuation coefficients α appearing in $A = A_0 e^{-\alpha x}$, as a function of period T. The cross plotted just below the curve at $T = 10$ s shows α for no viscosity when the decay is all due to scattering. The functional shape is $\alpha(T) = 2 \times 10^{-2} \exp(-0.386\,T)$, roughly, which was unanticipated but the wave periods are rather long.

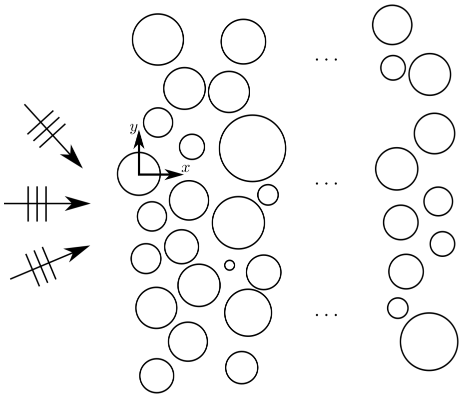


[13] Vaughan et al. (2009), Squire et al. (2009)

Paradigm I in the MIZ, a 2(+1)D model of wave scattering by a random MIZ[14]

Our model

Phase-resolving, linear wave scattering by random arrays of $O(10^3$–$10^5)$ circular floes.

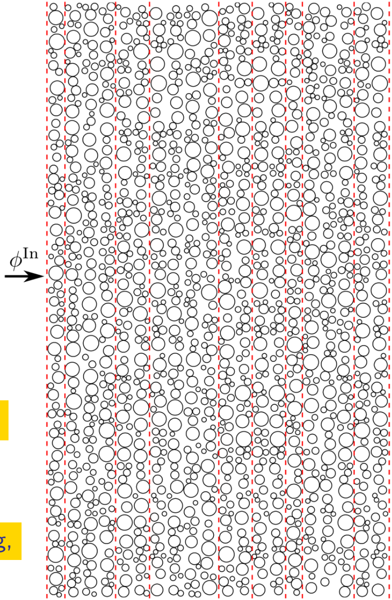


[14] Montiel et al. (2016), Montiel and Squire (2017)

The model

Physics and assumptions

- Linear periodic water wave theory
- Finite number of circular floes with prescribed randomized radius and thickness
- Each floe is represented as a thin elastic or viscoelastic plate
- Directional spectra
- Multiple scattering in deterministic framework
- Dissipative effects are essentially neglected (for now), e.g. floe collisions including ridge-building and rafting, overwash, viscous damping, sea ice inelasticity, turbulence, vortex shedding, etc.



Step 1: scattering by a single floe

Match the pressure and velocity fields at the interface between the ice-covered and ice-free domains, using a local coordinate system.

Exterior eigenfunction expansions

$$\text{Incident Field} \quad \rightarrow \quad \phi_p^{\mathrm{i}}(r_p, \theta_p, z) \approx \sum_{m=0}^{M} \xi_m(z) \sum_{n=-N}^{N} a_{m,n}^{(p)} \mathrm{J}_n(k_m r_p)\, \mathrm{e}^{\mathrm{i}n\theta_p}$$

$$\text{Scattered Field} \quad \rightarrow \quad \phi_p^{\mathrm{s}}(r_p, \theta_p, z) \approx \sum_{m=0}^{M} \xi_m(z) \sum_{n=-N}^{N} b_{m,n}^{(p)} \mathrm{H}_n(k_m r_p)\, \mathrm{e}^{\mathrm{i}n\theta_p}$$

with $\xi_m(z) = \cosh k_m(z+h)/\cosh k_m h$ and k_m the solutions of the open water dispersion relation $k \tanh kh = \alpha$.

Floe interior eigenfunction expansion

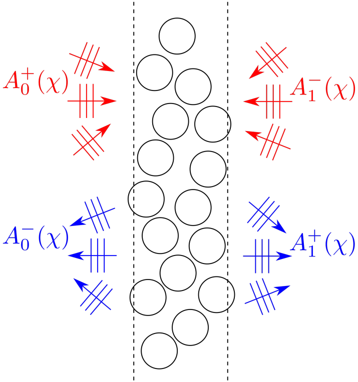
$$\text{Scattered Field} \quad \rightarrow \quad \phi_p^{\mathrm{f}}(r_p, \theta_p, z) \approx \sum_{m=-2}^{M} \psi_m(z) \sum_{n=-N}^{N} c_{m,n}^{(p)} \mathrm{J}_n(\kappa_m r_p)\, \mathrm{e}^{\mathrm{i}n\theta_p}$$

with $\psi_m(z) = \cosh \kappa_m(z+h)/\cosh \kappa_m(h-d)$ and κ_m the solutions of the ice-covered dispersion relation $(\beta\kappa^4 + 1 - \alpha d)\kappa \tanh \kappa(h-d) = \alpha$.

Step 2: multiple scattering in a single strip

Single strip scattering matrix

Invoking Graf's addition formula, we apply self-consistent interaction theory[15], for cylindrical waves which expresses the wave forcing on each ice floe as the coherent sum of the ambient incident wave and the scattered wave fields originating from all the other ice floes. The resulting solution is expressed in terms of cylindrical wave forms radiating from all floes.



- Mapping between incident and scattered waves
- Plane waves → cylindrical harmonics → plane waves
- Discretization of directional range:

$$\begin{pmatrix} \mathbf{A}_0^- \\ \mathbf{A}_1^+ \end{pmatrix} = \mathbf{S}_1 \begin{pmatrix} \mathbf{A}_0^+ \\ \mathbf{A}_1^- \end{pmatrix}$$

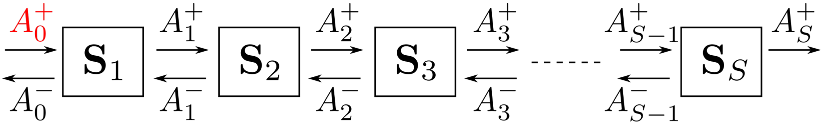
where $\mathbf{S}_1$ is the scattering matrix.

[15] Kagemoto & Yue (1986)

Step 3: strip-clustering method

We combine the strips by solving a multiple reflection/transmission problem, as each strip partially reflects and transmits the evolving directional wave spectrum radiating from preceding adjacent strips.

Multiple strip interactions become a 1D problem

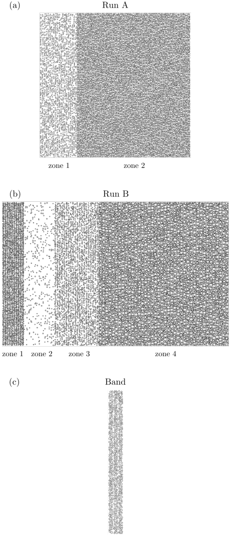


- Solution $\mathbf{A}_j^{\pm}$ is obtained iteratively (S-matrix method) in terms incident amplitudes $\mathbf{A}_0^+$ and $\mathbf{A}_J^-$.
- The interior field below each floe ϕ_p^{F} can then be calculated and the floe deflection

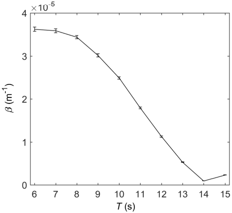
$$\eta_p = (1/\alpha)\partial_z \phi_p^{\mathrm{F}}, \quad z = -d \quad \text{if required.}$$ [14]

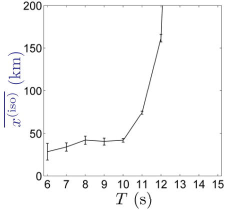
MIZ examples

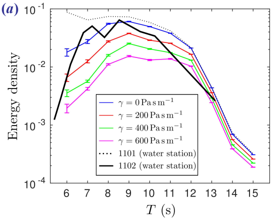
Models of MIZs showing zones and a band.

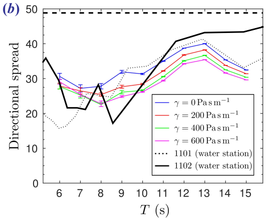


Upper plots show energy attenuation coefficients β appearing in $E = E_0 e^{-\beta x}$ (left) and distance to isotropy (right), for $T = 6$–15 s. Lower plots[16] are comparisons of predicted (colour) and observed (black) (*a*) energy spectra and (*b*) directional spread of the transmitted field for the band experiment in the same period range.









16 Squire and Montiel (2016)

Paradigm II examples

Unprecedented demand to embed ocean waves in global climate models or sea ice into a wave forecasting model such as WAVEWATCH III®, but ...

- Mass-loading and viscous models **used to represent discontinuous sea ice**, e.g. an MIZ.[17]
- Continuous ice solutions, whether originally intended to be applicable to its solid form[4] or to other continuous forms[6], **but used as so called effective media to represent composites of separate ice floes and cakes as found in the MIZ.** (Attenuation is introduced either by defining a complex shear modulus[18] or by introducing velocity-dependent damping.)

Unfortunately ...

- Mathematically, asymptotic behaviour should be checked and bounds sought on the *effective* MIZ's behaviour using homogenization theory but the latter is difficult.
- To calibrate the model with data, attenuation is measured in situ and the model is tuned by finding the best fit G and μ from the data set, i.e. the model is calibrated with the very thing we wish to predict.
- This causes variability and the possible occurrence of aberrant G_μ values, often larger than those for solid ice, justified on the basis that G_μ now encapsulates all the dissipative processes that affect waves. (A shear modulus G comparable with that for titanium!)

[17] e.g. Lamb (1916), Wadhams (1973), Weber (1987), Keller (1998), Newyear and Martin (1999), Liu and Mollo-Christensen (1988), Liu et al. (1991, 1992), Squire (1993), Carolis and Desiderio (2002), ...

[18] $G_\mu = G - i\omega\rho\mu$, where ρ is ice density

Current position[19]

Paradigm II in the MIZ

When used to represent the MIZ, the two favoured continuous ice sheet models[4,6] are invoked as effective media. Recall that they . . .

- **are built for continuous ice, not an MIZ composed of >10,000s of separate ice floes**.
- include a **shear modulus G and a viscosity μ that have no physical meaning whatsoever** and, in fact, can be physically nonsense as there is no way to relate the wave and ice conditions to G_μ in a general sense because the model is simply best-fitted to an observational outcome that only exists for the duration of the experiment.

But note,

- Each includes commensurate propagating wave modes but the viscoelastic layer model[6] also permits a rich set of legitimate roots that include flexural, longitudinal and surface waves, which range far and wide in parameter space and can be very close together in the complex plane making unambiguous root detection very challenging.
- Both use ice viscosity μ to represent overall dissipation, yet its effect is reduced as concentration decreases which is counterintuitive as more, not less, dissipation is likely to occur as concentration decreases.
- Indeed, should we even allow for $G \neq 0$, as nobody has ever observed anything other than open water dispersion in the MIZ? Why not just assume a viscous (layer) model? What physical justification can there be for a multitude of separated ice floes being elastic?

[19] Mosig et al. (2015)

A nonlinear ansatz

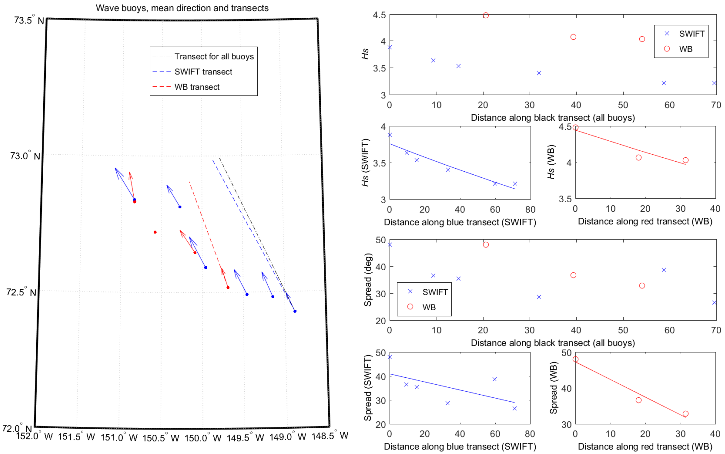
Exploring an alternative to $A = A_0 e^{-\alpha x}$ for paradigm II MIZ parameterizations

- Instead of the usual ode for exponential decay with distance travelled, x, i.e. $d_x A = -\alpha A$, where $\alpha = \mathrm{Im}(k) > 0$ denotes the spatial attenuation coefficient, we propose $d_x A = -\alpha A^n$.
- This has solution $A^{(1-n)} = A_0^{(1-n)} - (1-n)\alpha x$, where n and α are unknown parameters to be found, and A_0 is a constant representing the amplitude A at $x = 0$.
- When $n = 0, A = A_0 - \alpha x$, i.e. a linear law

  When $n = 1, A = A_0 e^{-\alpha x}$, i.e. exponential decay.
- In principle, n can take on other values of course.
- While, obviously, this gives an extra degree of freedom for data fitting, we foreshadow that n can be guesstimated by synthesizing results of *in situ* field data and, potentially SAR, relating to spectral form and the properties of $\alpha(\omega)$ or the energy attenuation coefficient $\beta(\omega)$.

Some observations. A snapshot from experiment WA3, collected in the Beaufort Sea in October 2015

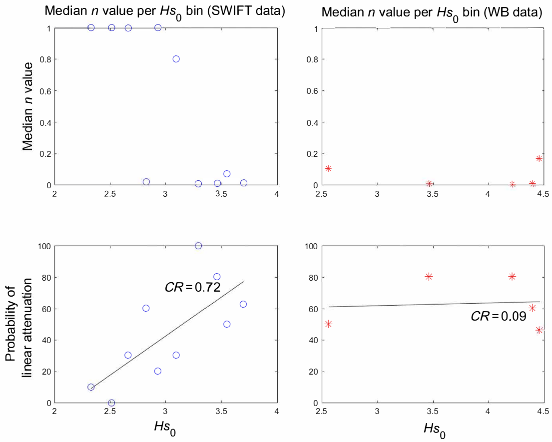
The figure shows the significant wave height attenuation profiles and directional spreading profiles for both the SWIFT and wave buoys. The sea ice is predominantly pancake ice.



Correlation analysis of n against Hs_0

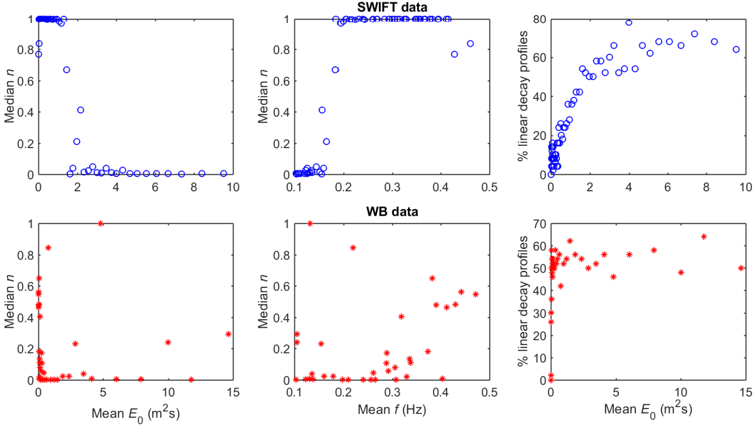
Median of n for 10 binned Hs values, also showing the percentage of linear decaying profiles, i.e. for $n<0.1$ ...

- For $Hs \lessapprox 3\,\text{m}$, $d_x Hs = -\alpha Hs$, with $\alpha = (4.49 \pm 0.45) \times 10^{-6}\,\text{m}^{-1}$
- For $Hs \gtrapprox 3\,\text{m}$, $d_x Hs = -\alpha$, with $\alpha = (5.73 \pm 0.54) \times 10^{-6}$.



Energy density and power n

Left panels show n versus E_0, created from clusters of E_0 that each contain about 100 estimated values. Each point on the plot gives the mean of this sample of E_0 values versus the median of the corresponding sample of n values. The transition between n=1 and n=0 occurs at $E_0 \sim 1.5\,\mathrm{m^2s}$. The median of n versus the mean of the frequency calculated as the mean of all the frequency samples corresponding to the E_0 samples in a bin is shown in the middle panels. There is a dependence on frequency which follows closely the dependence on energy because the two are strongly correlated. The right panels show the percentage of linearly decaying profiles in each E_0 bin.



But, is there any physical justification for increasing the degrees of freedom by using $d_x A = -\alpha A^n$?

Yes – a granular floe jostling model

- A back-of-the-envelope analysis of pancake ice subjected to waves produces the same ode, i.e. $d_x A = -\alpha A^n$, by invoking collisions using granular flow theory.[20]
- This ode actually arises from the power-law fluid

$$\sigma_{ij}^{(d)} = 2\mu(\cdot)\dot{\epsilon}_{ij} = 2\left(\tfrac{1}{2}M^{-1}\tau^{(1-n)}\right)\dot{\epsilon}_{ij} = 2\left(\tfrac{1}{2}M^{-\frac{1}{n}}\dot{\epsilon}^{\frac{1-n}{n}}\right)\dot{\epsilon}_{ij},$$

  where M is constant, $\dot{\epsilon}_{ij}$ is the strain rate tensor and $\dot{\epsilon} = \sqrt{\frac{1}{2}\dot{\epsilon}_{ij}\dot{\epsilon}_{ij}}$ is effective strain rate.
- It implies that viscosity is not constant, as for Newtonian flow. Power-law fluids have a strain-rate-dependent apparent viscosity $\mu(\cdot) \propto \dot{\epsilon}^{\frac{1-n}{n}}$.
- The value that the index n takes on determines the way the material deforms . . .
  - when $n > 1$, the viscosity $\mu(\cdot)$ decreases as strain rate increases and the material is described as pseudoplastic;
  - when $0 \leq n < 1$, $\mu(\cdot)$ increases with increasing strain rate and it is said to be dilatant.

[20] Shen and Squire (1998)

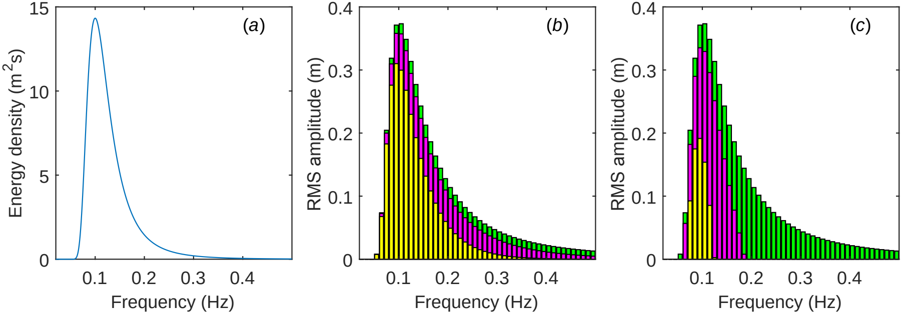
Is there a clever way of guessing n for waves in sea ice?

- Shorter period waves reduce in amplitude more rapidly than longer ones in sea ice, i.e. attenuation increases as frequency and strain rate increase $\Rightarrow$ the process exhibits apparent dilatancy *when taken as a whole.*
- Field experiments suggest $\alpha(\omega) \propto \omega^{2-3}$, with $\alpha(\omega) \propto \omega^2$ (inverse proportionality to wavelength) common.[21]
- Yet models being trialed in WAVEWATCH III® display $\alpha(\omega) \propto \omega^{11}, \omega^{7}, \omega^{7/2}, \omega^{3}$ asymptotic proportionality.
- In the experiment we also found that the actual energy density $E_0 \propto \omega^{-4}$ over the mid- to high-frequency bands where $d_x A = -\alpha A^n$ is postulated to hold, so A_0 and A will both reduce following ω^{-2}.
- We can argue that, for $\alpha(\omega) \propto \omega^2$ to hold under this constraint, we must have $n=2$ with the result that $\alpha = (A^{-1} - A_0^{-1})/x$.

[21] Wadhams (1975), Meylan et al. (2014), herein

Choosing n without a dispersion relation

(a) Pierson-Moskowitz spectrum $E_0(f)$. *(b)* The partially obscured green bar graph is an RMS amplitude spectrum $A_0(f)$ created from $E_0(f)$ by integrating across frequency bands of width 0.01 Hz with the central frequency at the mid-point. $A_0(f)$ is plotted as a bar graph to emphasize that each amplitude is valid over a frequency band, e.g. from 0.2–0.21 Hz with a central frequency 0.205 Hz, rather than at a single frequency. The magenta and yellow spectral amplitudes show how A_0 evolves exponentially as x increases. *(c)* Spectral amplitudes constructed in the same manner as for *(b)*, but for linear decay. Identical values for the constant of proportionality in $\alpha \propto f^2$ and the distances from the ice edge are used for *(b)* and *(c)*, chosen to emphasize the disparity between the two types of attenuation. It is the relativity between the same colours in plots *(b)* and *(c)* that is important, rather than the absolute values.



Thanks for listening. In conclusion ...

- Paradigm I models, based on state variables, are well advanced and have been tested positively in situ.
- While not doubting that they are needed, paradigm II parameterizations are at an earlier stage of development with disputable (or absent) physics.
- Because dissipation in the MIZ is highly nonlinear, $d_x A = -\alpha A$ may not be the best point to start a parameterization, unless we are absolutely sure that the wave slopes are modest, as it constrains data fitting to a decaying exponential shape.
- The added generality provided by the extra degree of freedom arising from $d_x A = -\alpha A^n$ potentially gives advantages to parameterization in global climate models or global-scale wave forecasting models, especially if the parameters α (or β) and n can be constrained.

Questions