The Fluid dual to Vacuum Einstein gravity

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Based on 1103.3022 and 1201.2678
with P. McFadden, K. Skenderis & M. Taylor.

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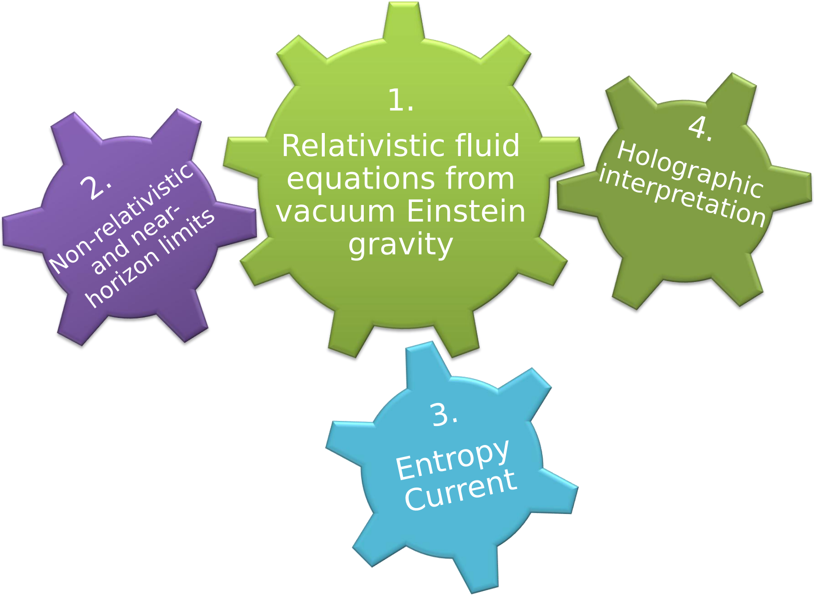
In which sense is gravity “holographic” away from AdS/CFT ?

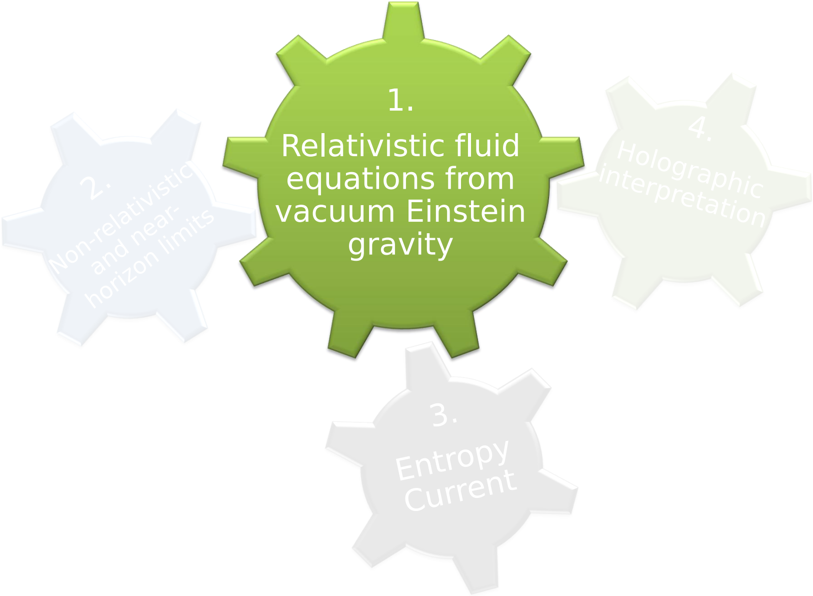
Strategy Study vacuum Einstein gravity in a regime where holography can be expected.

Objective Show that a universal subset of the dynamics of vacuum Einstein gravity describes a fluid in one lower dimension with special properties. Interpret holographically.

[N.B. Inspired from AdS/CFT but logically independent. Results are not obtained from a flat limit of AdS/CFT.]

Plan of the talk




1.
Relativistic fluid equations from vacuum Einstein gravity
2.
Non-relativistic and near-horizon limits
3.
Entropy Current
4.
Holographic interpretation

Domain : Rindler wedge with cutoff

Consider a Rindler wedge in flat spacetime

Isolate the dynamics using a Dirichlet boundary condition

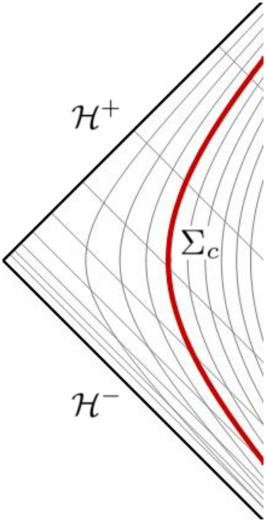
$$g_{ab}|_{\Sigma_c} = \eta_{ab}$$

and impose regularity at the Rindler horizon $\mathcal{H}^+$.

What is the remaining dynamics ?

Express it in terms of the stress-tensor on Σ_c !

$$T_{ab} \equiv -K_{ab} + g_{ab}K, \qquad K_{ab} \equiv \frac{1}{2}\mathcal{L}_N g_{ab} \, .$$



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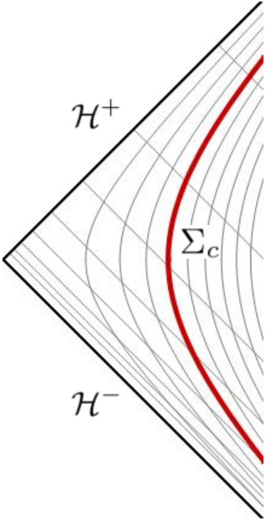

Structure of Einstein's equations

Decomposition :

- $d+1$ coordinates intrinsic to the brane : $x^a \equiv (\tau, x_1, \dots x_d)$
- 1 coordinate extrinsic to the brane : r.

Resulting Gauss-Codazzi equations :

- $R^{rr} = 0$ on Σ_c is a constraint

$$d\, T_{ab} T^{ab} - (T^c_c)^2 = 0\,.$$

- $R^{ra} = 0$ on Σ_c are conservation equations

$$\partial_a T^{ab} = 0\,.$$

- $R^{ab} = 0$ everywhere lead to radial integration,

$$\eta_{ab}|_{\Sigma_c},\ T_{ab}|_{\Sigma_c} \qquad \Rightarrow \qquad g_{\mu\nu}(r, x^a)\,.$$

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Reformulation of Einstein's equations

Define a conserved stress-tensor T_{ab} obeying the equation of state

$$d\,T_{ab}T^{ab} - (T^c_c)^2 \;\; = \;\; 0\,,$$

integrate in the bulk and impose regularity everywhere.

- Existence and unicity of a solution is not clear.
- General solution out of reach. Look close at equilibrium and near-equilibrium.

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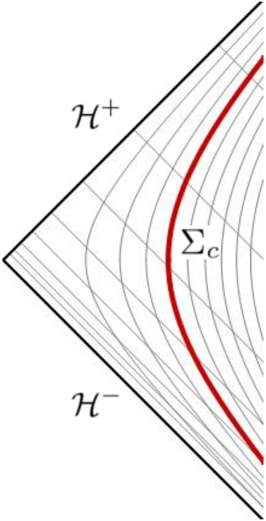
Strategy : solve in the Hydrodynamics regime

We have a dissipative system that relaxes to thermal equilibrium

- We will first look at global equilibrium. This is the thermodynamic regime.
- We will then look then at local equilibrium with long wavelength, low energy perturbations.



Global equilibrium : Rindler spacetime

The metric of Rindler spacetime is

$$ds^2 = 2d\tau dr - r d\tau^2 + dx^i dx^i \,.$$

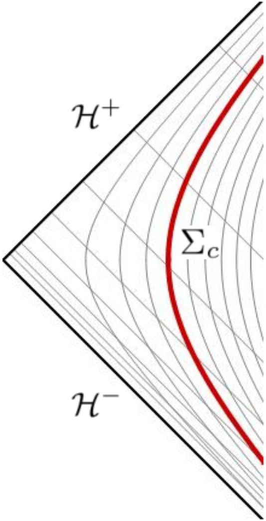
Evaluation reveals the perfect fluid form

$$T_{ab} = 0 u_a u_b + \frac{1}{\sqrt{r_c}} h_{ab},$$

where $h_{ab} = \eta_{ab} + u_a u_b$ and $u_a = \delta_a^\tau$.
Note that the constraint $d\, T_{ab} T^{ab} - (T^c_c)^2 = 0$ implies either

$$\rho_{eq} = 0 \;\; \text{or} \;\; \rho_{eq} = -\frac{2d}{d-1} p_{eq} \,.$$

The second solution is the singular Taub spacetime [Eling, Meyer, Oz, 2012]



Global equilibrium : Rindler spacetime

- Generate (p, u_a) by acting with diffeos.
- No energy density

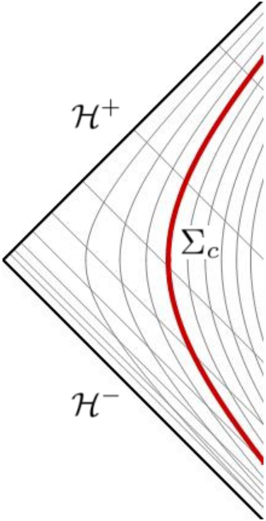
$$\rho_{eq} = 0\,.$$

- Gibbs-Duhem relation

$$s\,\delta T = \delta\left(\frac{p}{16\pi G}\right)$$

where

$$s = \frac{1}{4G}, \qquad T = \frac{p}{4\pi}$$



Local equilibrium configurations

Start with the seed

$$ds^2_{(0)} \quad = \quad \text{equilibrium solution } (p, u_a)$$

where we promote $p = p(x)$ and $u_a = u_a(x)$.

Conservation of the stress-energy tensor

$$0 = \partial^a T_{ab} = \partial^a \left(p(x) h_{ab}(x) \right)$$

is equivalent to the ideal relativistic fluid equations

$$\begin{aligned} \partial_a u^a &= 0, \\ u^b \partial_b u_a &= -h_a^b \partial_b \ln p \quad \Leftrightarrow \quad a_a \equiv D u_a = -D_a^\perp \ln p \,. \end{aligned}$$

Local equilibrium configurations

- We solve for small gradients using the relativistic scaling

$$\partial_a \sim \epsilon, \quad \partial_r \sim \epsilon^0, \quad u_a \sim \epsilon^0, \quad p \sim \epsilon^0.$$

- We solve iteratively Einstein's equations

$$g_{\mu\nu}^{(0)}, \ldots, g_{\mu\nu}^{(n-1)} \quad \rightarrow \quad g_{\mu\nu}^{(n)}$$

by solving

$$0 = R_{\mu\nu}^{(n)} \sim \partial_r^2 g_{\mu\nu}^{(n)} + \hat{R}_{\mu\nu}^{(n)}[g^{(<n)}]$$

Gauge freedom ? Integration constants ? Field redefinitions ?

Existence and Unicity

We fix

- Radial gauge (ingoing null coordinates)

$$g_{rr} = 0, \qquad g_{ra} = p u_a \,.$$

- Regularity at the horizon $\mathcal{H}^+$ (or equivalently analyticity). It fixes a traceless $d \times d$ tensor of integration constants to zero.
- Gauge conditions on the stress-tensor

$$\begin{aligned} (i) \qquad & T_{ab} u^a h^b_c = 0 \\ (ii) \qquad & T_{cd} h^c_a h^d_b = p\, h_{ab} + \text{non-isotropic tensors} \end{aligned}$$

The condition (ii) replaces the gauge $T_{ab} u^a u^b = \rho$.

⇒ Resulting solution is unique and regular at each perturbative order. This extends [Bredberg, Keeler, Lysov, Strominger, 2011].

How to express the solution ?

In terms of the bulk metric

$$ds^2 = p\, u_a dr\, dx^a + g_{ab}(r,x) dx^a dx^b \,.$$

In terms of boundary data on Σ_c :

- Metric

$$g_{ab}|_{\Sigma_c} = \eta_{ab} \,.$$

- Stress-energy tensor

$$T_{ab} = \rho u_a u_b + p h_{ab} + \Pi^{\perp}_{ab}, \qquad \Pi^{\perp}_{ab} u^a = 0.$$

We need to find a basis of fluid scalars/tensors

Note that ρ is completely determined by $\Pi^{\perp}_{ab}$ through

$$d\, T_{ab} T^{ab} = (T^c_c)^2 \,.$$

Results at first order

- Stress-energy tensor

$$T_{ab} = \rho u_a u_b + p h_{ab} + \Pi^{\perp}_{ab}, \qquad \Pi^{\perp}_{ab} u^a = 0.$$

The first order corrections are

$$\begin{aligned} \rho &= \zeta' D \ln p + O(\partial^2) \\ \Pi^{\perp}_{ab} &= -2\eta \mathcal{K}_{ab} + O(\partial^2) \end{aligned}$$

The fluid dual to vacuum Einstein gravity admits

$$\frac{\zeta'}{s} = 0, \qquad \frac{\eta}{s} = \frac{1}{4\pi}, \qquad s \equiv \frac{1}{4G}\,.$$

[Chirco,Eling,Liberati] No higher order corrections.

Results at second order

- Stress-energy tensor

$$T_{ab} = \rho u_a u_b + p h_{ab} + \Pi^{\perp}_{ab}, \qquad \Pi^{\perp}_{ab} u^a = 0.$$

The first and second order corrections are

$$\begin{aligned} \rho &= \zeta' D \ln p + \frac{1}{p}\Big(d_1 \mathcal{K}_{ab}\mathcal{K}^{ab} + d_2 \Omega_{ab}\Omega^{ab} + d_3 (D \ln p)^2 \\ &\quad + d_4 DD \ln p + d_5 (D_\perp \ln p)^2\Big) \\ \Pi^{\perp}_{ab} &= -2\eta \mathcal{K}_{ab} + \frac{1}{p}\Big(c_1 \mathcal{K}_a^c \mathcal{K}_{cb} + c_2 \mathcal{K}^c_{(a}\Omega_{|c|b)} + c_3 \Omega_a{}^c \Omega_{cb} \\ &\quad + c_4 h_a^c h_b^d \partial_c \partial_d \ln p + c_5 \mathcal{K}_{ab} D \ln p + c_6 D^{\perp}_a \ln p \, D^{\perp}_b \ln p\Big). \end{aligned}$$

The fluid dual to vacuum Einstein gravity admits

$$\zeta' = 0, \qquad d_1 = -2, \qquad d_2 = d_3 = d_4 = d_5 = 0,$$
$$\eta = 1, \qquad c_1 = -2, \quad c_2 = c_3 = c_4 = c_5 = -c_6 = -4.$$

Results at second order

- Stress-energy tensor

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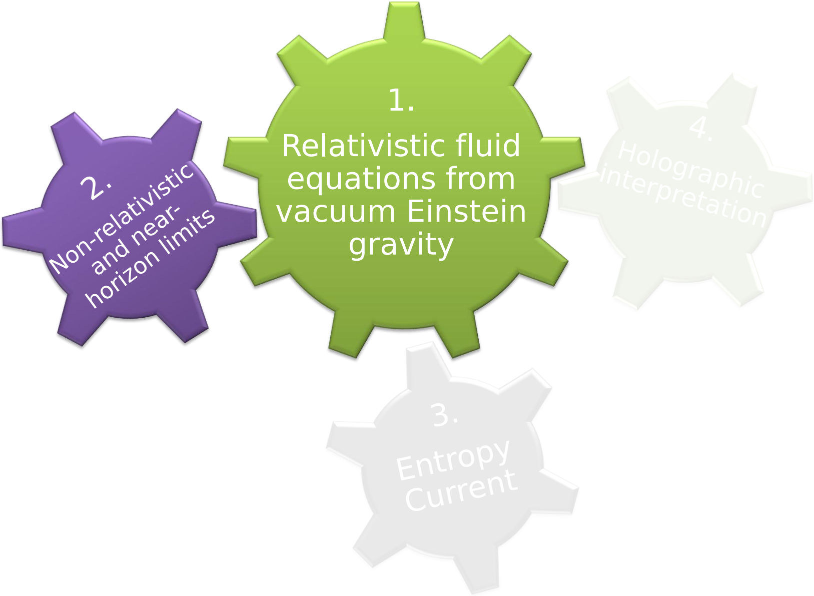
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Near-horizon limit

What happens in the near-horizon limit

$$r_c \to r_h \to 0 \qquad ?$$

We observe that the following near-horizon limit :

$$NH_{①} : \qquad \frac{r_h}{r_c} = \text{fixed}, \qquad \frac{v^2}{r_c} = \text{fixed}$$

can be written as

$$NH_{①} = \text{Weyl rescaling} + \text{Relativistic scaling}$$

⇒ The ideal relativistic fluid appears in the near-horizon limit.

Incompressible Navier-Stokes equations

The non-relativistic scaling defined by

$$\partial_i \to \epsilon\partial_i, \quad \partial_\tau \to \epsilon^2\partial_\tau, \quad v_i \to \epsilon v_i, \quad p \to \bar{p} + \epsilon^2 P$$

preserves the incompressible Navier-Stokes equations

$$\partial_\tau v_i + v^j \partial_j v_i - \eta\partial^2 v_i + \partial_i P \sim \epsilon^3, \qquad \partial_i v^i \sim \epsilon^2.$$

Corrections to incompressibility scale as $O(\epsilon^4)$.

Corrections to Navier-Stokes scale as $O(\epsilon^5)$.

The non-relativistic scaling of any relativistic fluid gives the incompressible Navier-Stokes equation in the limit $\epsilon \to 0$.

Near-horizon limit

Now, there are other near-horizon limits

$$r_c \to r_h \to 0 \qquad !$$

One can define the alternative near-horizon limit

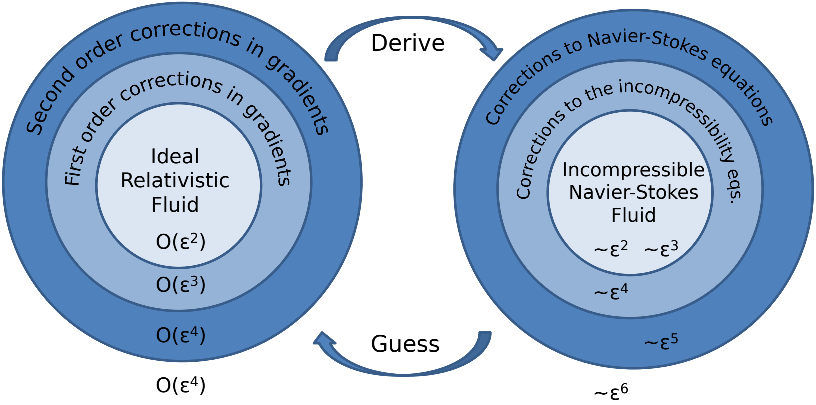
$$NH_{②}: \qquad \frac{r_h}{r_c^2} = \text{fixed}, \qquad \frac{v_i}{r_c} = \text{fixed}$$

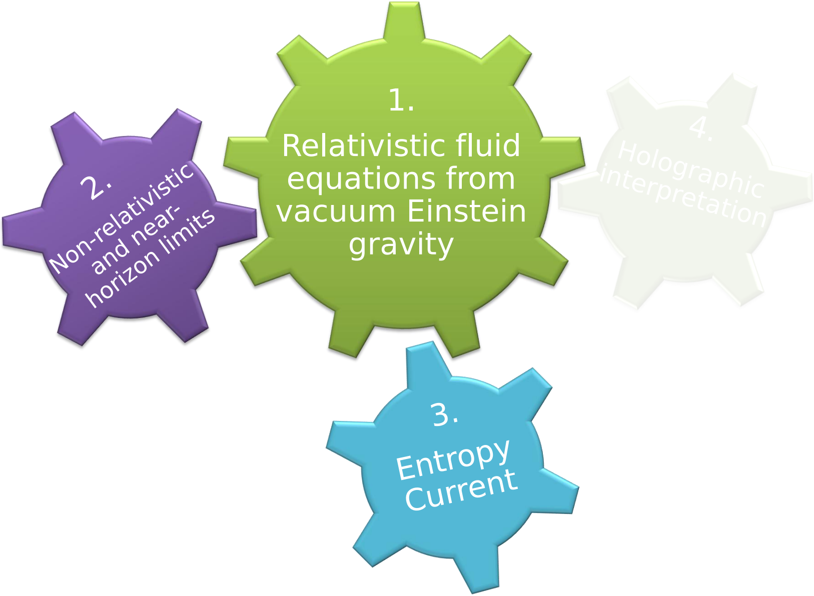
It has been observed in [Bredberg et al., 2011] that the near-horizon limit can be written as

$$NH_{②} = \text{Weyl rescaling} + \text{Non-relativistic scaling}$$

⇒ Navier-Stokes fluid and relativistic fluid both appear in near-horizon limits.

From relativistic expansion to non-relativistic expansion

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Entropy current : Fluid analysis

Classify the non-negative currents

$$\mathcal{J}^a = s_{eq} u^a + O(\partial), \qquad s_{eq} = \frac{1}{4G}\,.$$

The condition $\partial_a \mathcal{J}^a \geq 0$ implies

$$\mathcal{J}^a = s_{eq} u^a + (3 \text{ terms} \sim \partial^2) + \partial_b \mathcal{K}^{[ab]} + O(\partial^3)\,.$$

There is therefore a 3-parameter family of non-trivial entropy currents.

Can we define from the bulk an entropy current in that class ?

Entropy current : Bulk analysis

We consider the horizon $\mathcal{H}_+$ defined by $r = r_{\mathcal{H}}(x)$. Let l^μ be its affine generator

$$l^\mu \nabla_\mu l^\nu = 0\,.$$

The expansion of geodesics at the horizon is non-negative if the area law is obeyed :

$$\theta \equiv (\nabla_\mu l^\mu)_{\mathcal{H}} \geq 0$$

After some algebraic manipulations, one can show that

$$(\nabla_\mu l^\mu)_{\mathcal{H}} \geq 0 \text{ if and only if } \partial_a \mathcal{J}^a \geq 0$$

where

$$\mathcal{J}^a = \frac{1}{4G}\sqrt{-g_{\mathcal{H}}}\,\xi^a, \qquad \xi_\mu \equiv \partial_\mu(r - r_{\mathcal{H}}(x))$$

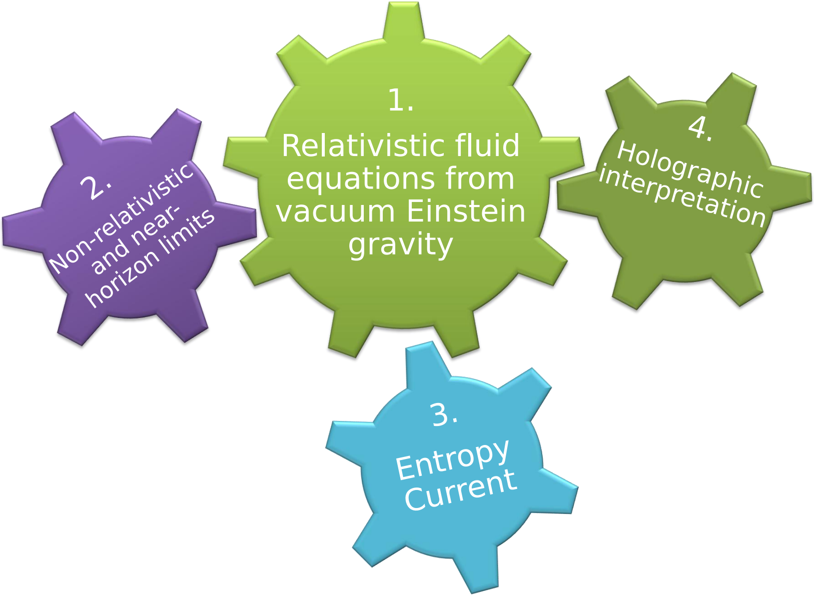
The entropy current can be pulled-back from the horizon to Σ_c using the preferred null geodesics $x^\mu = constant$.

Entropy current : Bulk analysis

Result : we obtain one particular current out of the 3-parameter family.

The divergence of the entropy current is indeed non-negative.

The fluid description is consistent with the second law of thermodynamics.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In AdS/CFT

Thermal state	$\Leftrightarrow$	AdS black brane
Relativistic hydrodynamics	$\Leftrightarrow$	Relativistic gradient expansion solution

From general expectations of holography :

Rindler / dual QFT

Thermal state	$\Leftrightarrow$	Rindler equilibrium solution
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Clues on the QFT ? Idea ! Reproduce the equation of state of the equilibrium stress-tensor.

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In AdS/CFT

What is the equation of state ?

- Dirichlet boundary condition at infinity :

$$T_a^a = 0.$$

Dual implementation : super Yang-Mills theory.

- Dirichlet boundary condition at cutoff radius

$$(T_a^a)^2 - dT_{ab}T^{ab} = -\frac{d^2(d+1)}{l^2}.$$

Dual implementation : non-local irrelevant multi-trace deformation of the CFT. Non-local QFT. [Brattan, Camps, Loganayagam, Rangamani, 2011]

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  Dual implementation ?

Preliminary holographic model

The action

$$S = \int d^d x \sqrt{-\gamma}\sqrt{-\partial_a \phi \partial^a \phi}\,.$$

has a stress-tensor obeying

$$(T^a_a)^2 - d T_{ab} T^{ab} = 0.$$

Assuming $S(\phi, (\partial\phi)^2)$, there are only two solutions for such a Lagrangian. One of which is the square-root action(*).

(*) The other action is a model for a fluid in Taub spacetime [Eling, Meyer, Oz, 2012]

Conclusion

- Einstein's equations around Rindler horizon is described by a relativistic fluid with

$$\rho_{eq} = 0, \qquad p_{eq} > 0\,,$$

  and specific dissipative coefficients.

  The fluid stress-tensor and the bulk solution are both regular and uniquely defined.
- Near-horizon limits give either the ideal relativistic fluid or the incompressible Navier-Stokes fluid.
- The fluid is consistent with the second law of thermodynamics up to second order in gradients.
- A Sqrt action reproduces key characteristics of the fluid equation of state.