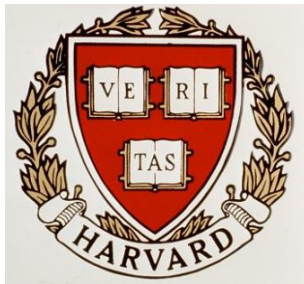


A supersymmetric model for lattice fermions



Liza Huijse – Harvard University
Oct 20, 2011
KITP

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J. Vala (Maynooth), D. Mehta (Syracuse),
B. Bauer (ETH/Station Q), M. Troyer (ETH), E. Berg (Harvard)

Introduction

Strongly interacting electron systems

Key examples:

- High T_c superconductors
- Heavy fermion compounds

Challenge conventional theoretical techniques

Introduction

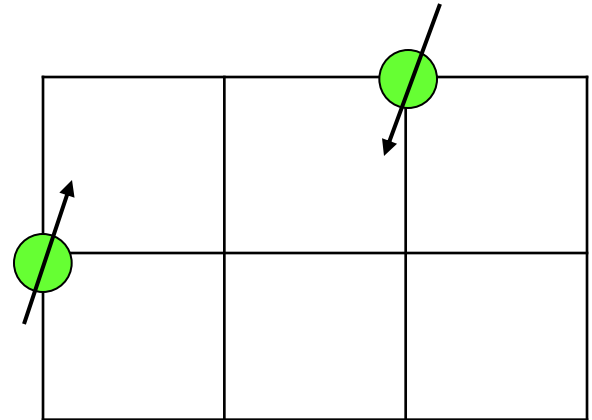
Lattice models

configurations:

electrons located on the sites of an ionic lattice in a solid

Hamiltonian:

typically a sum of kinetic (hopping) terms and short range repulsive interactions



Introduction

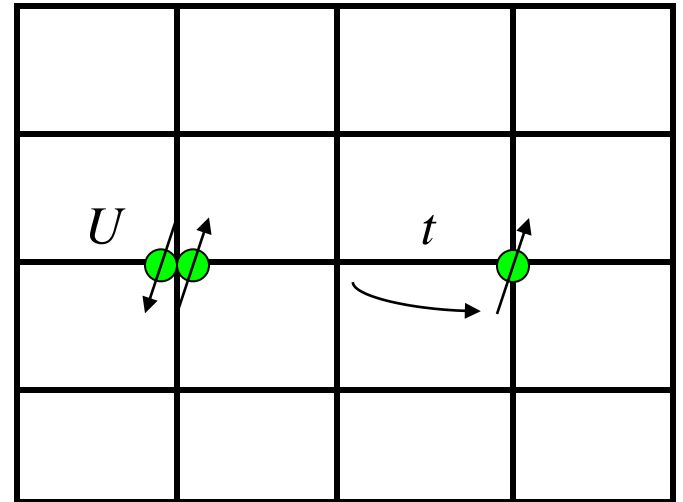
Hubbard model (1963)

Coulomb repulsion \rightarrow onsite repulsion U

$$H = -t \sum_{\langle ij \rangle} c_{i\sigma}^\dagger c_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow}$$

$$n_{i\sigma} = c_{i\sigma}^\dagger c_{i\sigma} \quad \{c_{i\sigma}^\dagger, c_{j\sigma}\} = \delta_{ij}$$

$$\sigma = \uparrow, \downarrow$$



Introduction

Hubbard model

- Kinetics dominated $n \ll 1$, $U \ll t \rightarrow$ Fermi liquid
- Interaction dominated $U \gg t$
 \rightarrow Mott insulator at half filling



Introduction

Strongly interacting electron systems

Challenge: Intermediate densities

Conventional techniques fail

- Mean field results are unreliable
- Bethe Ansatz does not work in $D > 1$
- Quantum Monte Carlo suffers from sign problem
- ...

→ Too difficult

Our work

A model for strongly interacting fermions

1. Simplifications/adjustments
2. Fine tuning

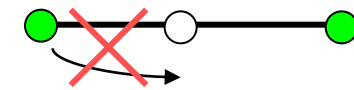
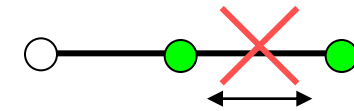
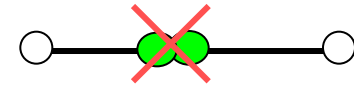
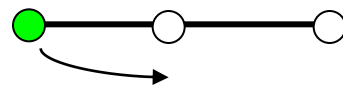
→ Exact result for strongly interacting fermions in $D > 1$
(not accessible via conventional techniques)

The model

Hardcore spinless fermions

- spinless fermions
- hardcore
- hopping t

$$V_1 \rightarrow \infty$$

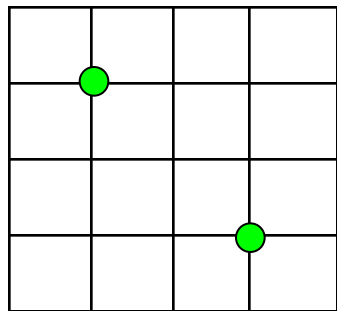
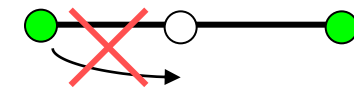
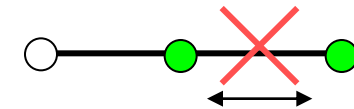
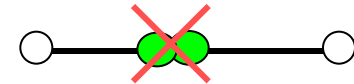
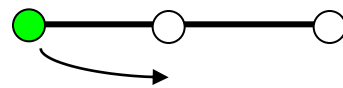


The model

Hardcore spinless fermions

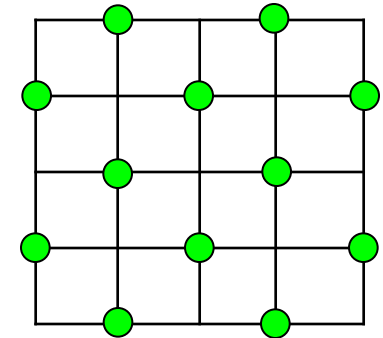
- spinless fermions
- hardcore
- hopping t

$$V_1 \rightarrow \infty$$



Fermi liquid

Stripe phase
[Henley, et. al. '01]



Insulator

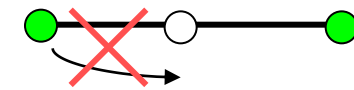
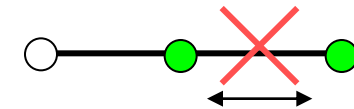
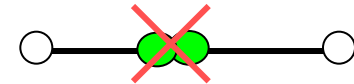
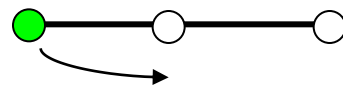
μ

The model

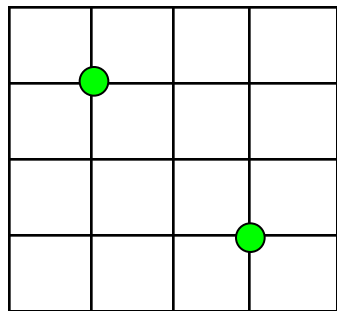
Hardcore spinless fermions

- spinless fermions
- hardcore
- hopping t

$$V_1 \rightarrow \infty$$



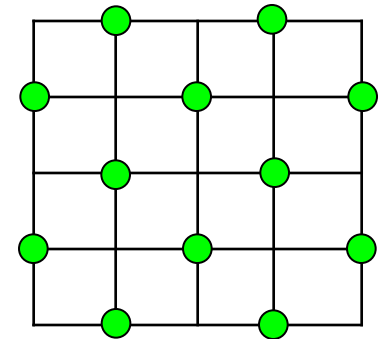
μ



Fermi liquid

Supersymmetry

[Fendley, et. al. '03]



Insulator

Plan of the talk

Benefit of supersymmetry is twofold:

- Powerful tools
- Subtle interplay between kinetic and potential terms leading to quantum criticality and superfrustration
- The model: definition & supersymmetry basics
- Superfrustration: extensive gs entropy
- Quantum criticality (1+1D)
- Quantum ground states as tilings
- Conclusions

Supersymmetry

Algebraic structure

supercharges Q^+ , $Q^{\square}=(Q^+)^{\dagger}$ and fermion number N_f

:

$$(Q^+)^2 = 0, \quad (Q^-)^2 = 0, \quad [N_f, Q^{\pm}] = \pm Q^{\pm}$$

Hamiltonian defined as

$$H = \{Q^+, Q^-\}$$

satisfies

$$[H, Q^+] = [H, Q^-] = 0, \quad [H, N_f] = 0$$

Supersymmetry

Spectrum:

- $E \geq 0$ for all states
- $E > 0$ pair into doublets (superpartners)
 $(|\psi\rangle, Q^+ |\psi\rangle), \quad Q^- |\psi\rangle = 0$
- $E = 0$ states are singlets $Q^+ |\psi\rangle = Q^- |\psi\rangle = 0$

High energy physics:

symmetry between bosonic and fermionic particles

Here:

- particles are spinless fermions (f)
- symmetry between “bosonic” (f even) and “fermionic” ($f \pm 1$ odd) states

The model

The supersymmetric lattice model

Supercharges for hardcore spinless fermions:

$$Q^+ = \sum_i c_i^\dagger \prod_{j \text{ next to } i} (1 - n_j), \quad Q^- = (Q^+)^\dagger, \quad n_j = c_j^\dagger c_j$$

Hamiltonian for 1D chain $H = \{Q^+, Q^-\}$

$$H = \sum_i [(1 - n_{i-1})c_i^\dagger c_{i+1}(1 - n_{i+2}) + \text{h.c.}] + \sum_i n_{i-1}n_{i+1} - 2N_f + L$$

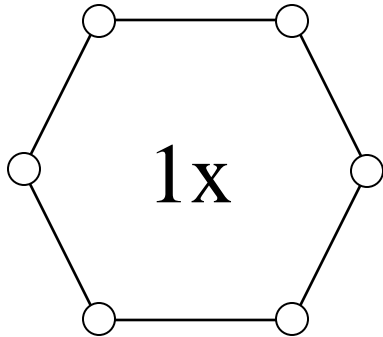
Hamiltonian for general lattice

$$H = \sum_{\langle ij \rangle} P_{\langle i \rangle} c_i^\dagger c_j P_{\langle j \rangle} + \sum_i P_{\langle i \rangle} \quad P_{\langle i \rangle} = \prod_{j \text{ next to } i} (1 - n_j)$$

Supersymmetry: example

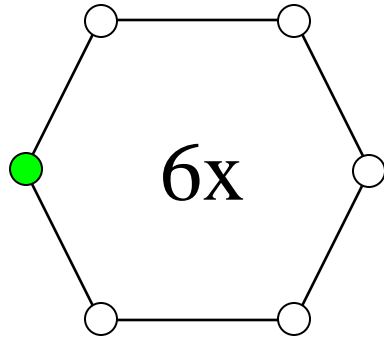
6-site chain

Possible configurations for hardcore spinless fermions



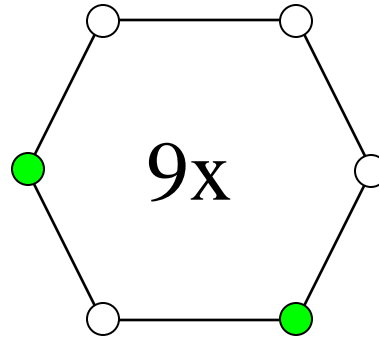
1x

$|0\rangle$



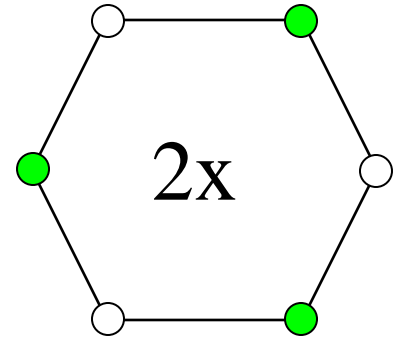
6x

$c_i^+ |0\rangle$



9x

$c_i^+ c_{i+2}^+ |0\rangle$



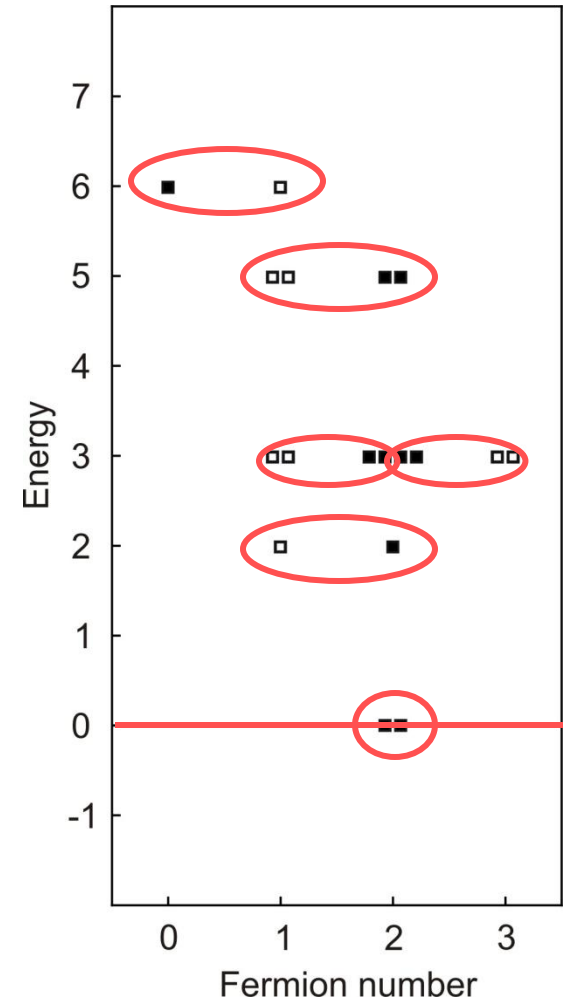
2x

$c_i^+ c_{i+2}^+ c_{i+4}^+ |0\rangle$

Supersymmetry: example

Manifestly supersymmetric spectrum

- Energy is positive definite
- $E > 0$ states form pairs between “fermionic” and “bosonic” states
- $E = 0$ states are singlets



Witten index: superfrustration

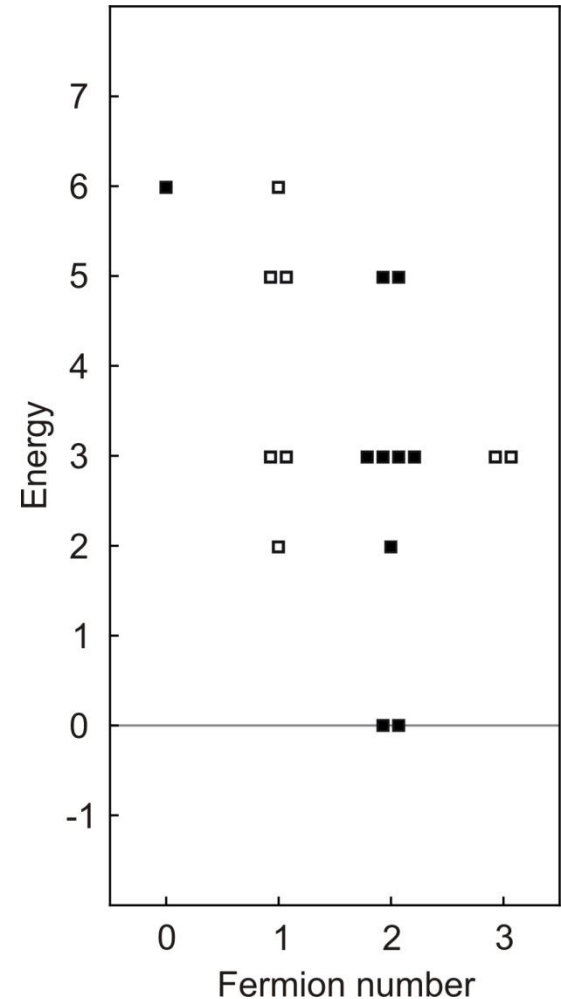
Powerful tool: Witten index

$$W = \text{Tr} (-1)^{N_f}$$

“bosonic” states
contribute +1,
“fermionic” states
contribute -1, so all
superpartners cancel

$$\Rightarrow W = \#GS_B - \#GS_F$$

|W| is lower bound to
number of ground states



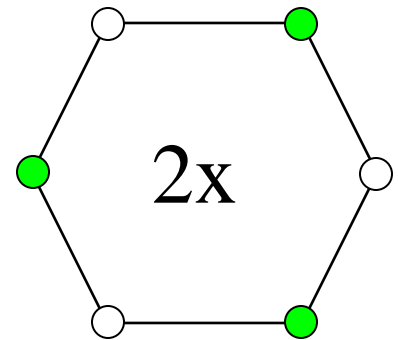
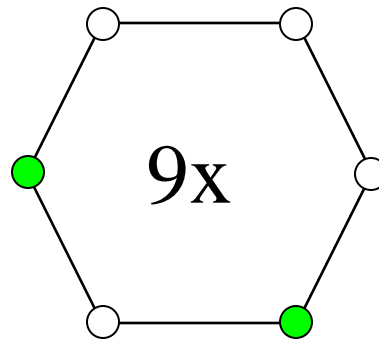
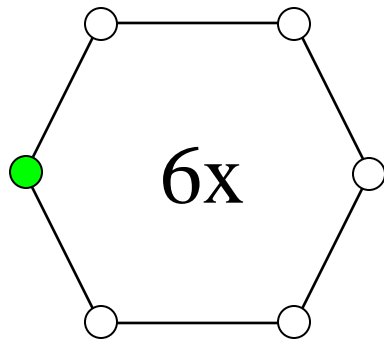
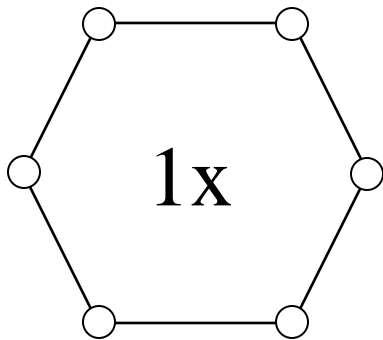
[Witten, '82]

Witten index: example

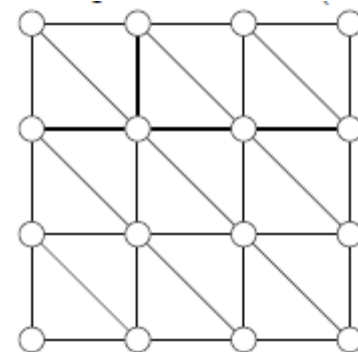
Purely combinatorial problem

$$W = \text{Tr} (-1)^{N_f}$$

$$W = 1 - 6 + 9 - 2 = 2$$



Witten index



Triangular lattice

$N \times M$ sites with periodic BC

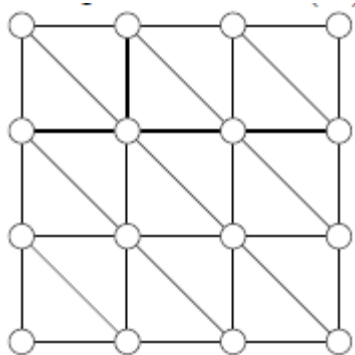
	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	-3	-5	1	11	9	-13	-31	-5	57
3	1	-5	-2	7	1	-14	1	31	-2	-65
4	1	1	7	-23	11	25	-69	193	-29	-279
5	1	11	1	11	36	-49	211	-349	811	-1064
6	1	9	-14	25	19	100	-19	-415	1462	-4911
7	1	-13	1	-69				3403	-7055	5237
8	1	-31	31	193				881	-28517	50849
9	1	-5	-2	-29	881	1462	-7055	-28517	31399	313315
10	1	57	-65	-279	-1064	-4911	5237	50849	313315	950592
11	1	67	1	859	1651	12607	32418	159083	499060	2011307
12	1	-47	130	-1295	-589	-26006	-152697	-535895	-2573258	-3973827
13	1	-181	1	-77	-1949	67523	330331	-595373	-10989458	-49705161
14	1	-87	-257	3641	12611	-139935	-235717	5651377	4765189	-232675057
15	1	275	-2	-8053	-32664	272486	-1184714	-1867189	134858383	-702709340

$$|W| \sim 1.14^{NM}$$

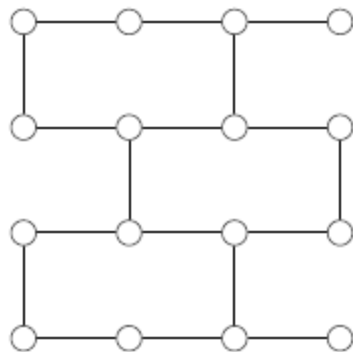
[van Eerten, '05]

Superfrustration - examples

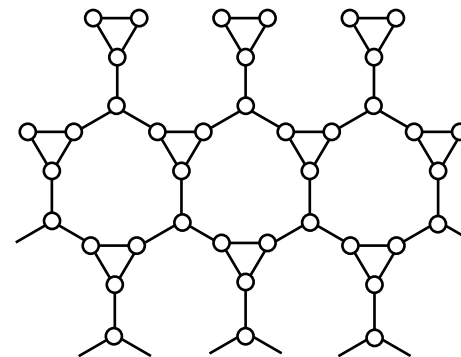
Tri



Hex



Martini



$$|W| \sim 1.14^L$$

$$|W| \sim 1.2^L$$

$$S_{gs} \sim 0.16 L$$

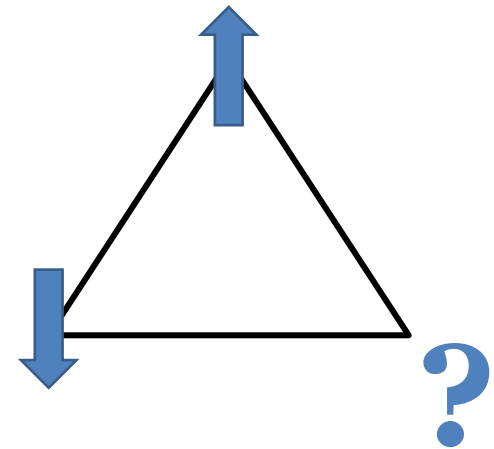
L is number of sites

[van Eerten, '05;
Fendley, Schoutens, '05]

Superfrustration

Frustration

Competing terms in hamiltonian
→ multiple ground states



Supersymmetry

Subtle competition between kinetic and potential terms
→ for 2D lattices exponential ground state degeneracy

Violation of 3rd law of thermodynamics

Exponential number of ground states
→ finite zero temperature entropy

Superfrustration

'3-rule'

- repulsive interactions favor 3-site interparticle distance
- chemical potential favors higher densities

Combined with kinetic terms

→ quantum charge frustration at *intermediate densities*

Quantum criticality

1D chain

Periodic chain of length L :

$$\begin{array}{ll} 2 \text{ gs} & \text{for } L \bmod 3 = 0; \\ 1 \text{ gs} & \text{otherwise} \end{array}$$

- Fermion number in ground state: $f = [L/3]$
- Bethe Ansatz solution
- Continuum limit: $N=(2,2)$ SCFT with central charge $c=1 \rightarrow$ quantum critical, emergent spacetime supersymmetry

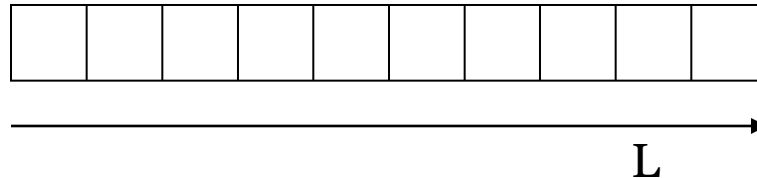
[Fendley-Schoutens-deBoer '03,
Fendley-Nienhuis-Schoutens '03,
Beccaria-DeAngelis '05,
Fendley-Hagendorf '10 & '11, LH '11]

Relations/extensions

- XXZ spin chain – exact mapping
- SUSY Matrix models (via mapping to XXZ spin chain) of Veneziano-Wosiek
- Generalize hard-core constraint
 - Allow k particles to be nearest neighbors, but not $k+1$:
 M_k susy model \leftrightarrow k -th SCFT minimal model

Square ladder

Periodic ladder of length L :



3 gs for $L \bmod 4 = 0$;
1 gs otherwise

- Fermion number in ground state: $f = [L/2]$
- Continuum theory: $\mathbf{N}=(2,2)$ SCFT with $c=3/2$
 - Witten index: $k+1$
 - Spectral flow
 - Entanglement entropy consistent with $c=3/2$
 - Operator content (scaling dimensions)

Entanglement entropy

Von Neumann entropy:

$$S_A = -\text{Tr} \rho_A \ln \rho_A$$

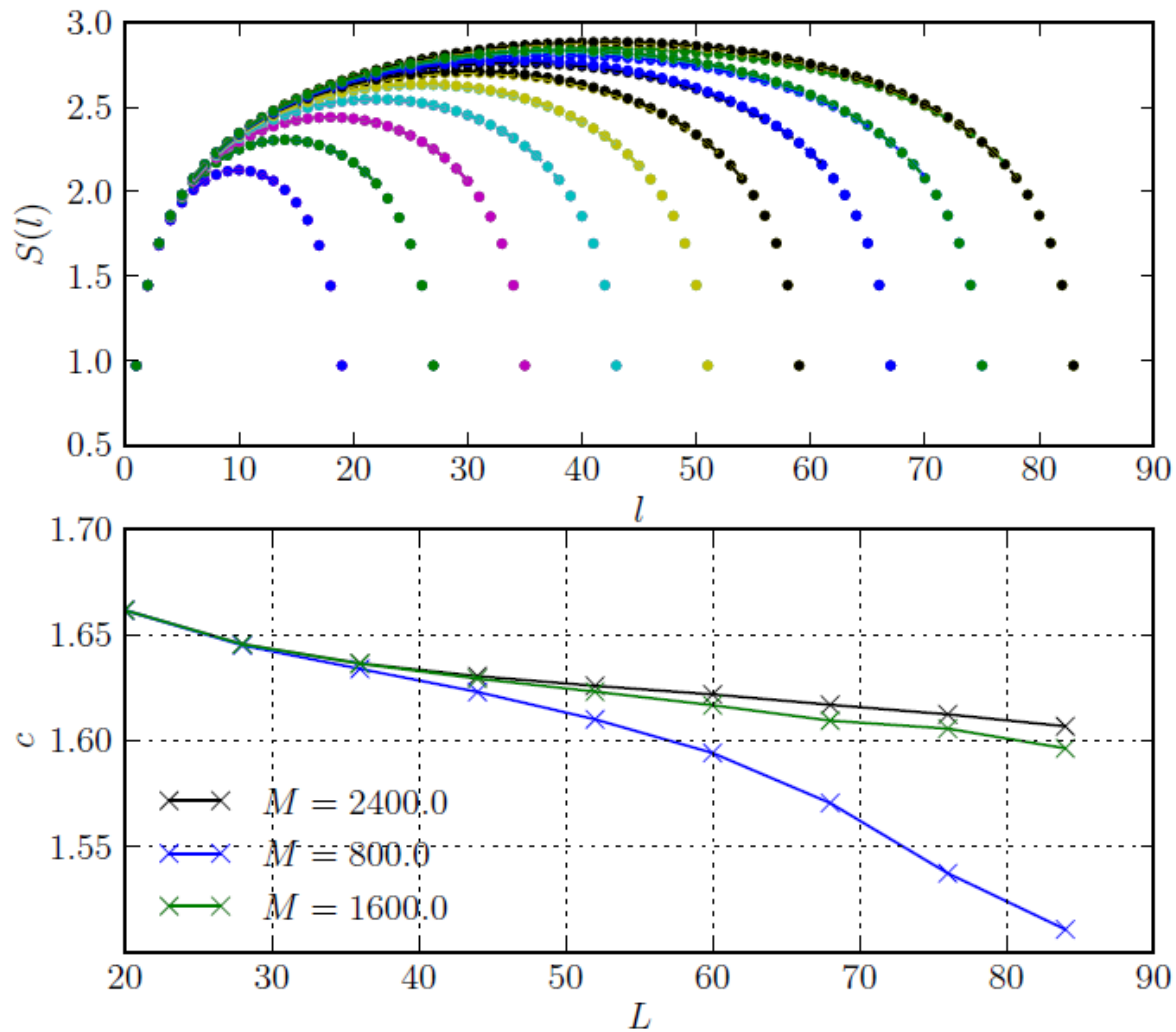
With reduced density matrix:

$$\rho_A = \text{Tr}_B \rho$$

For a critical system with finite length L :

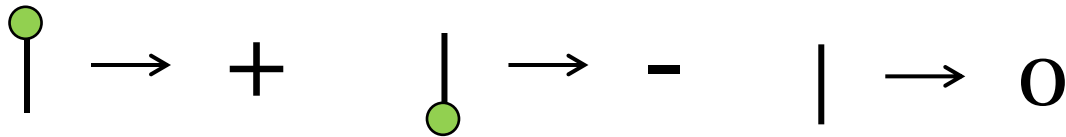
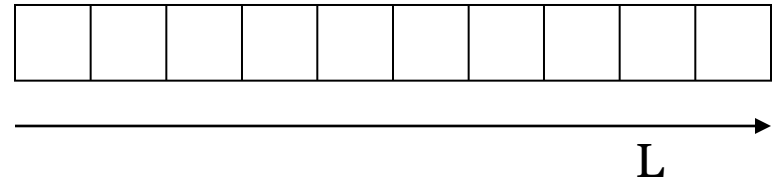
$$S(l_A) = \frac{c}{3} \ln\left(\frac{L}{\pi} \sin\left(\frac{l_A \pi}{L}\right)\right) + b$$

Entanglement entropy



[with B. Bauer, M. Troyer, K. Schoutens (in progress)]

Square ladder



Maps particle configurations to: ...+00-0-+0+...

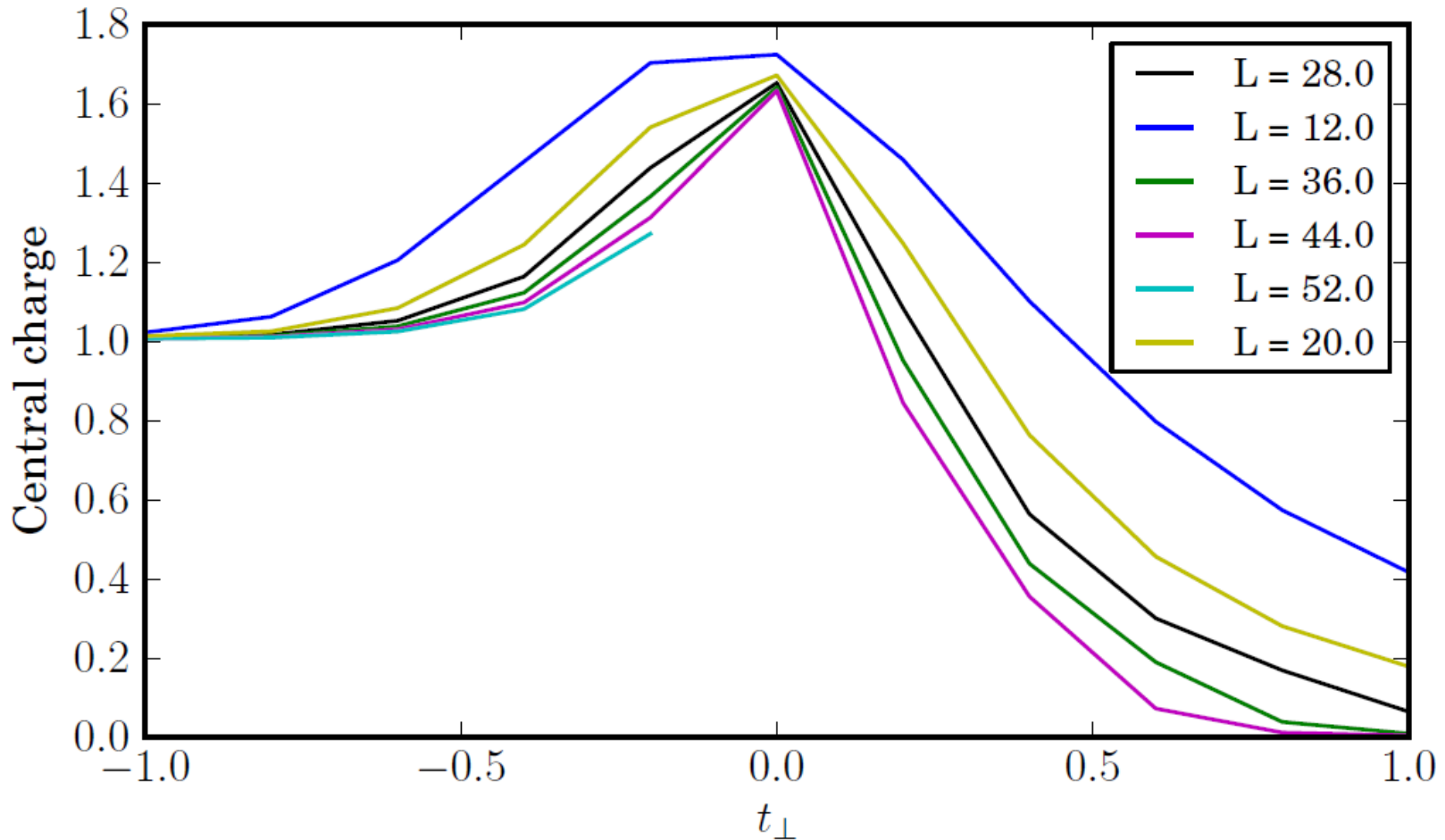
$c=3/2$:

$c=1$ bosonic mode \sim charge, $c=1/2$ Ising mode \sim +/-

Perturbing away from $c=3/2$

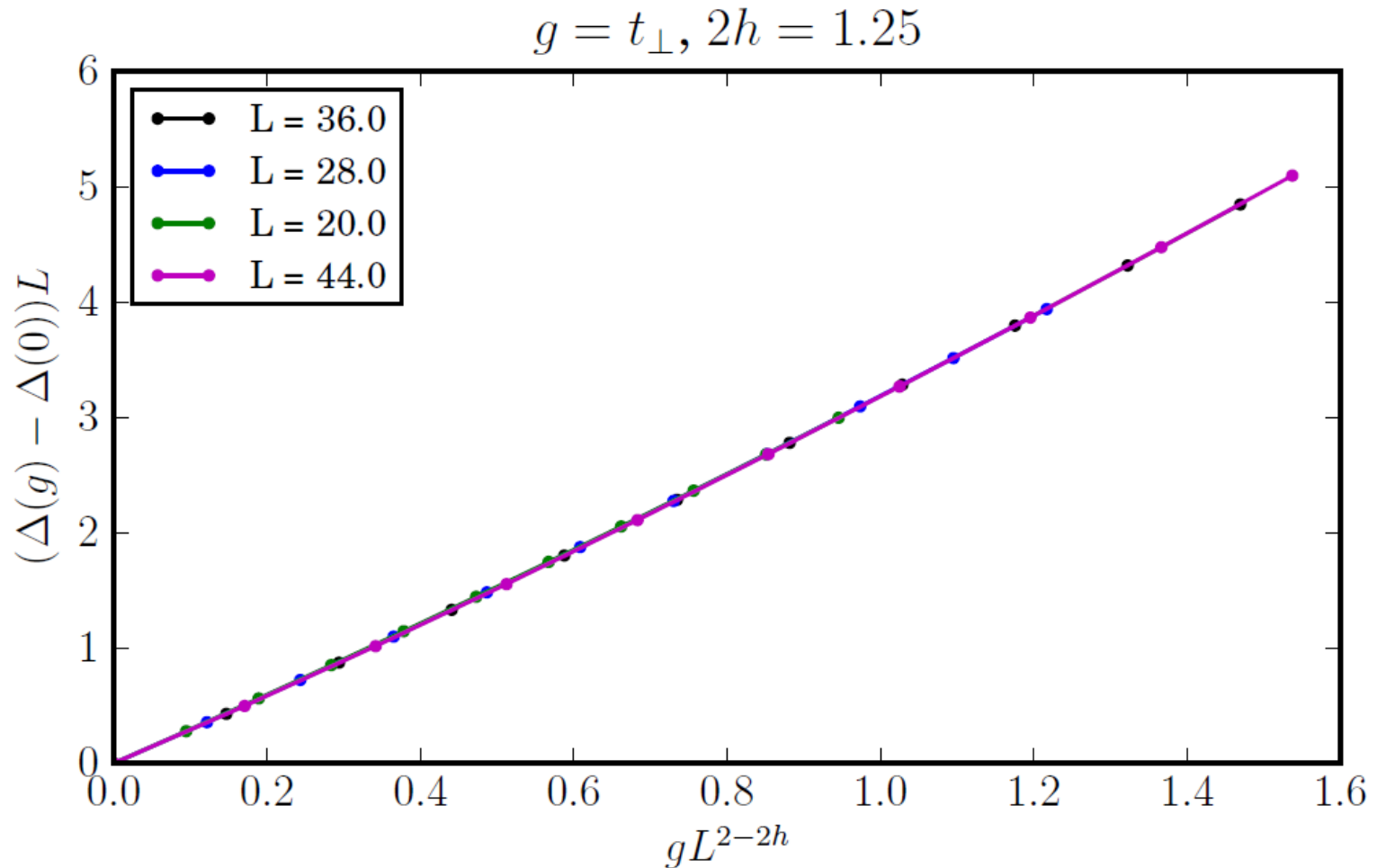
- Change rung hopping: favors/disfavors singlets on rungs \rightarrow gaps Ising mode
- Stagger chemical potential on even/odd rungs: breaks translational invariance \rightarrow gaps bosonic charge mode

Central charge vs rung hopping perturbation



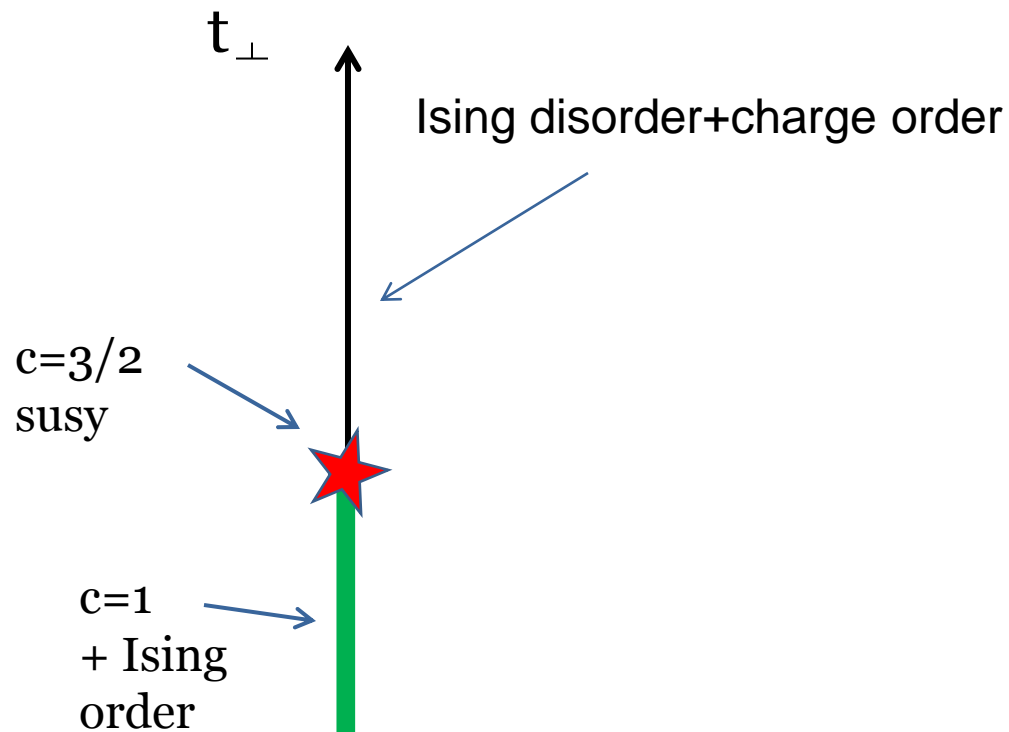
[with B. Bauer, M. Troyer, K. Schoutens (in progress)]

Gap scaling for rung hopping



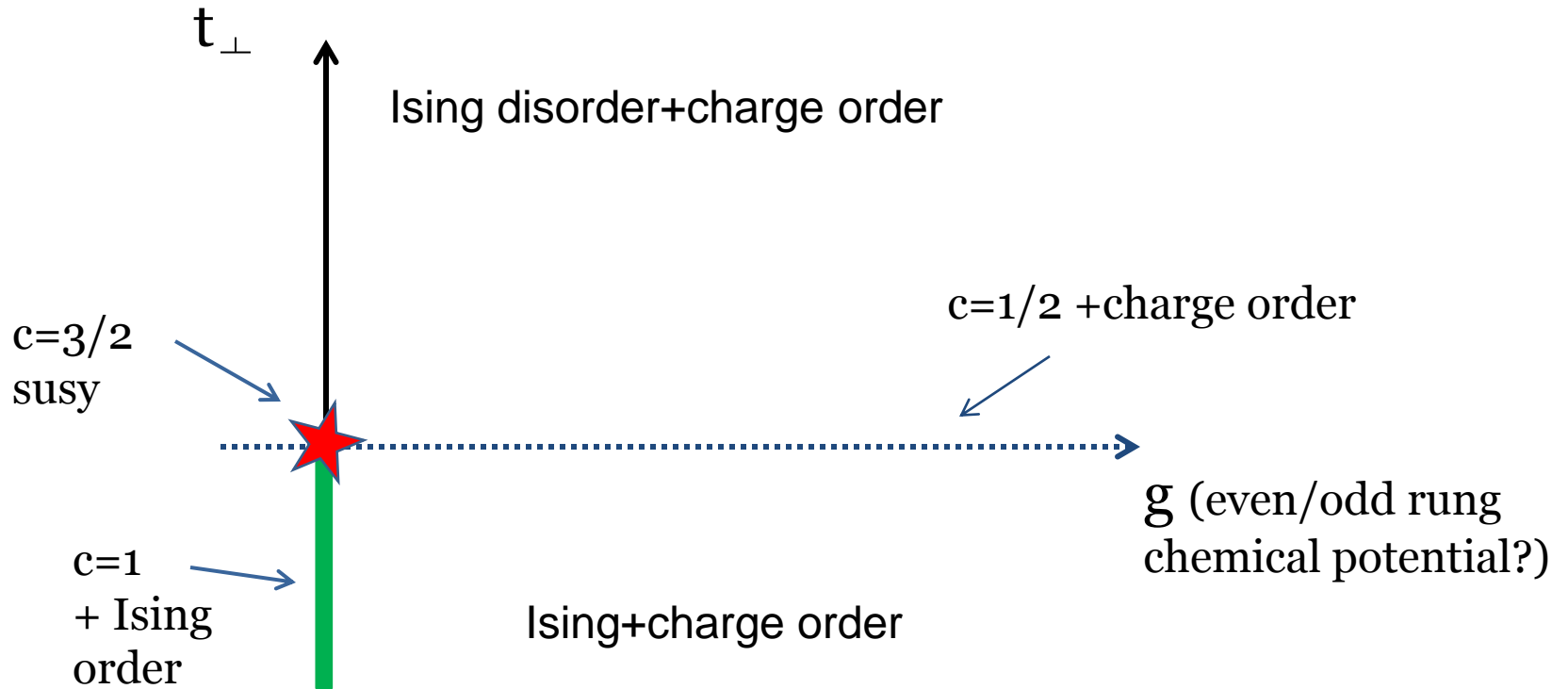
[with B. Bauer, M. Troyer, K. Schoutens (in progress)]

Phase diagram (tentative)



[with B. Bauer, M. Troyer, K. Schoutens (in progress)]

Phase diagram (very tentative)

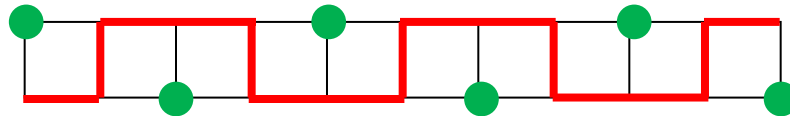


[with B. Bauer, M. Troyer, K. Schoutens (in progress)]

Susy staggering

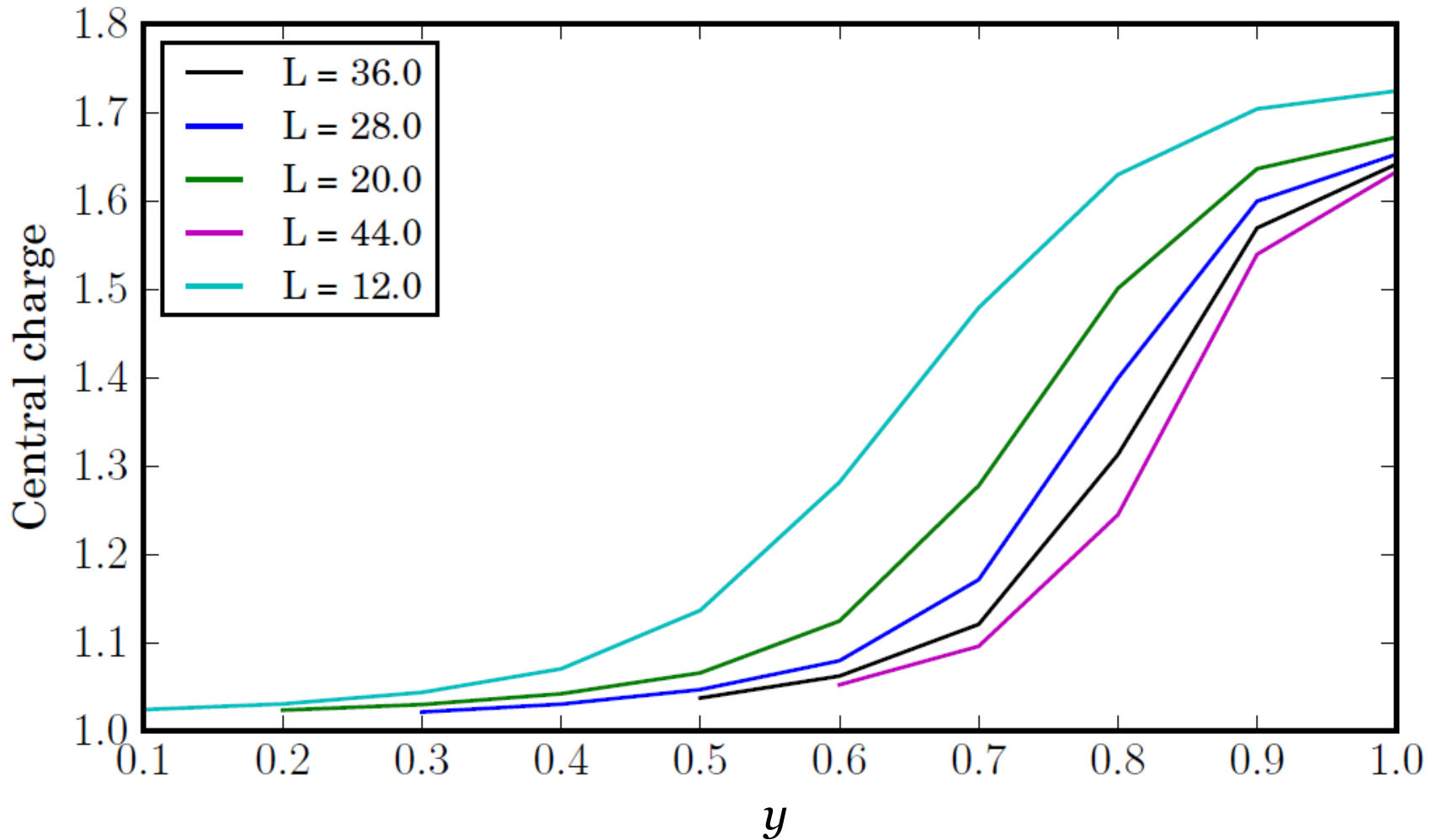
Susy preserving perturbation:

$$Q^+ = \sum \lambda_i^* c_i^\dagger P_{\langle i \rangle} \quad \lambda_i = \begin{cases} 1 & \text{for } i \in S_1 \text{ —} \\ y & \text{for } i \in S_2 \bullet \end{cases}$$
$$Q^- = \sum \lambda_i c_i P_{\langle i \rangle}$$



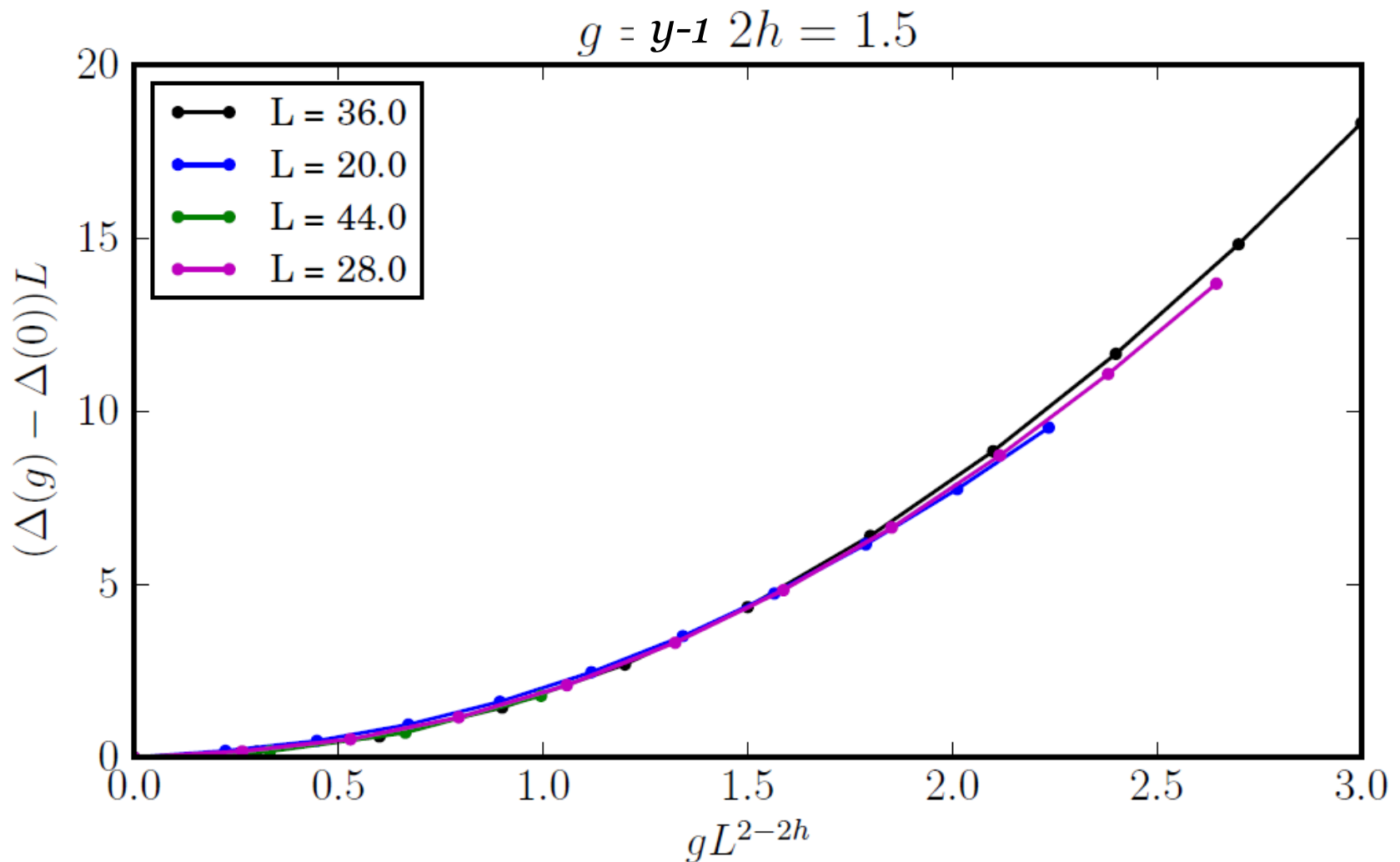
[with B. Bauer, M. Troyer, K. Schoutens (in progress)]

Central charge vs susy staggering



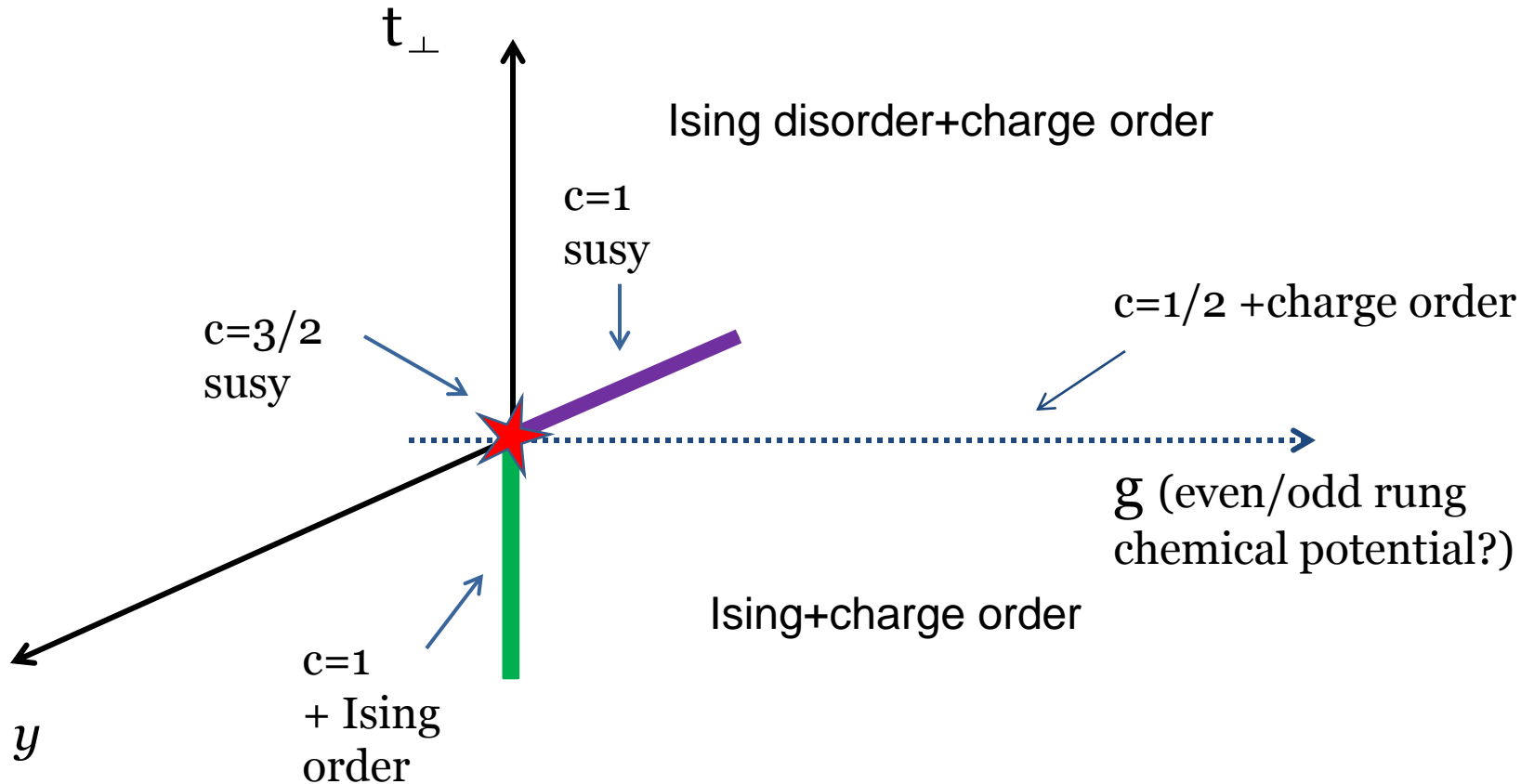
[with B. Bauer, M. Troyer, K. Schoutens (in progress)]

Gap scaling for susy staggering



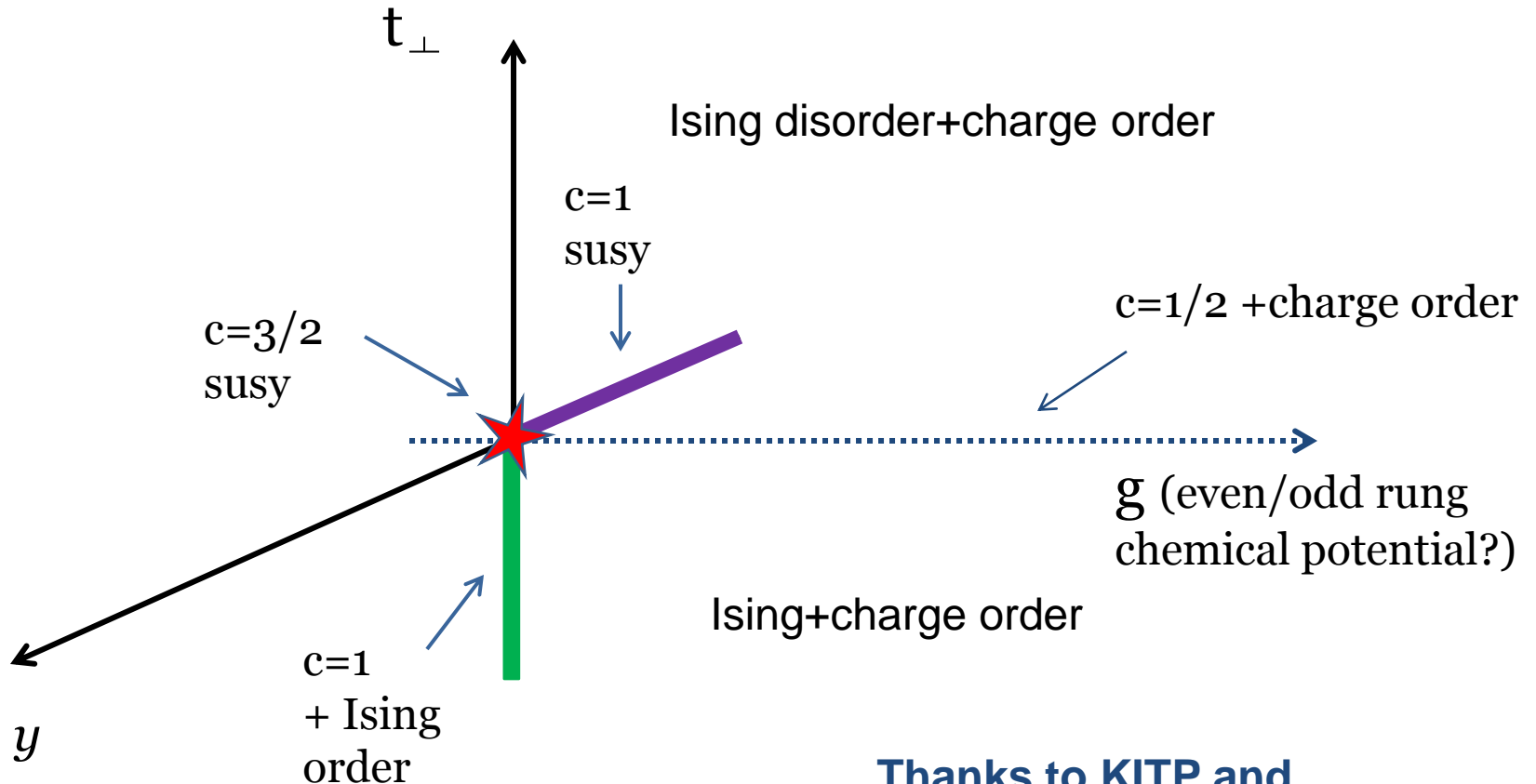
[with B. Bauer, M. Troyer, K. Schoutens (in progress)]

Phase diagram (very tentative)



[with B. Bauer, M. Troyer, K. Schoutens (in progress)]

Phase diagram (very tentative)



Thanks to KITP and
E. Berg, S. Trebst, J. McGreevy

[with B. Bauer, M. Troyer, K. Schoutens (in progress)]

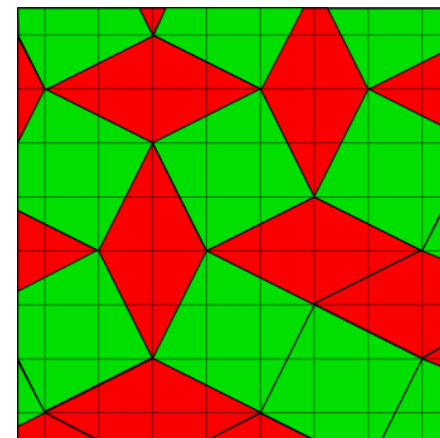
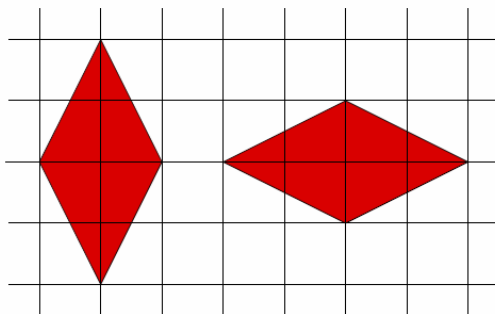
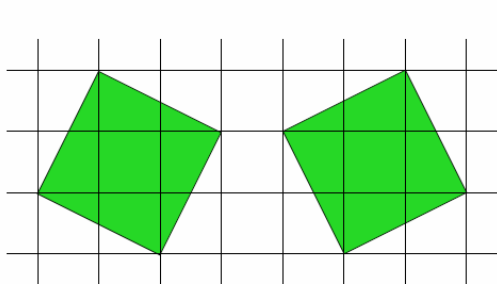
**Cohomology:
quantum ground states as tilings**

Powerful tool: Cohomology

- Cohomology of Q
 - GS are in 1-1 correspondence with cohomology elements
- More difficult to compute than Witten index
- But gives more information:
 - gives total number of gs
 - gives fermion number of gs
 - often gives relation between gs and geometric object

Square lattice (periodic BC)

- Cohomology of Q gives direct relation between ground states and rhombus tilings
- # of tiles = # of fermions



[LH - Schoutens '10]

Square lattice (periodic BC)

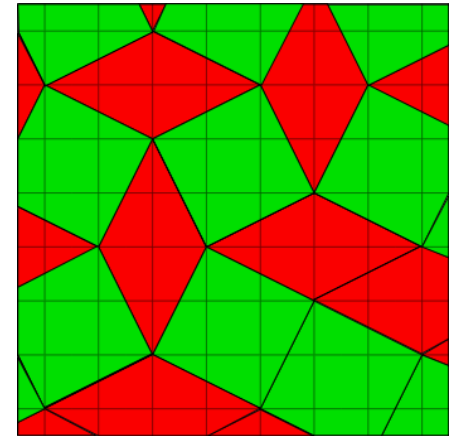
- Cohomology of Q gives direct relation between ground states and rhombus tilings
- # of tiles = # of fermions

- GS at intermediate filling

$$\frac{N_f}{L} \in \left[\frac{1}{5}, \frac{1}{4} \right] \cap \mathbb{Q}$$

- Sub-extensive GS entropy

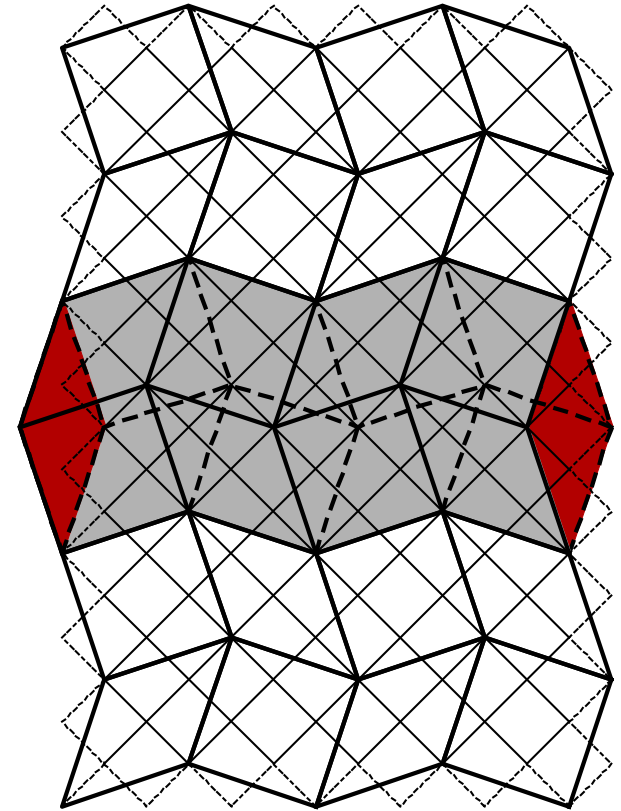
$$S_{gs} \sim 0.7\sqrt{L}$$



[LH - Schoutens '10]

Square lattice: edge states

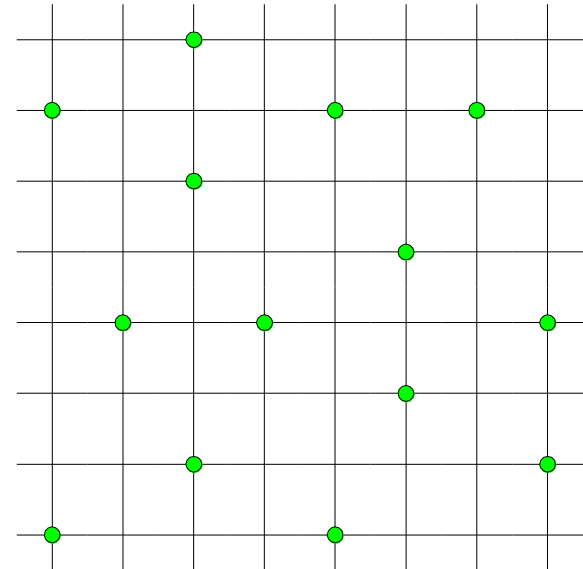
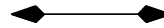
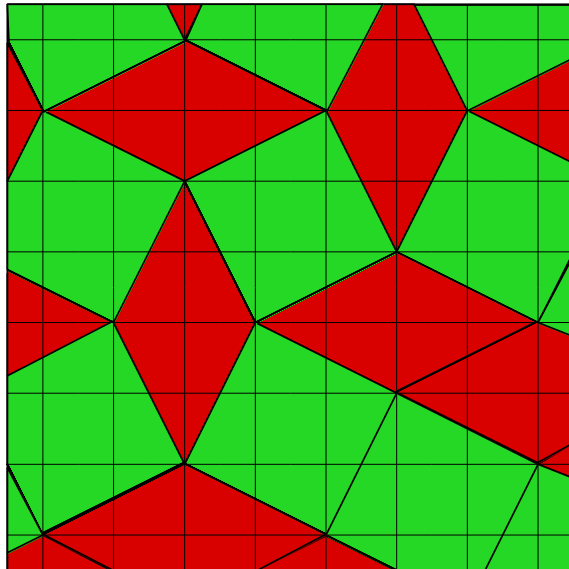
- for ‘diagonal’ open boundary conditions there is a unique gs; expect that ‘vanished’ torus gs’s form band of edge modes
- explicit evidence for critical modes from ED studies of various ladder geometries



[LH - Halverson - Fendley - Schoutens '08]

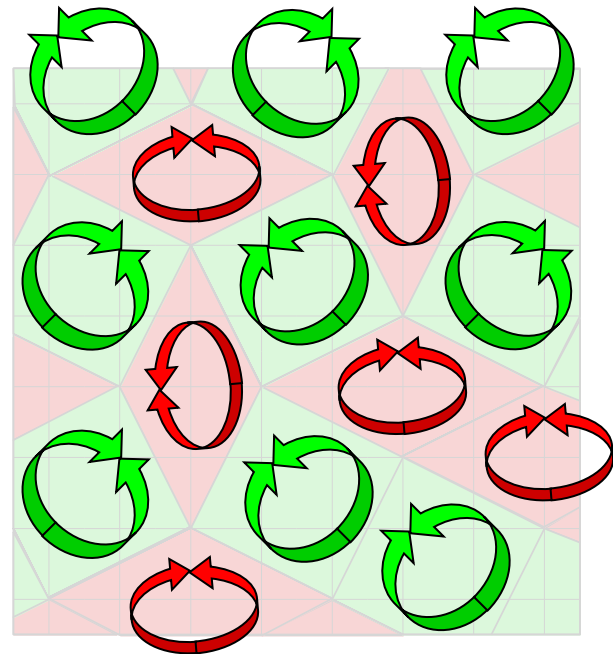
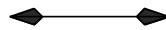
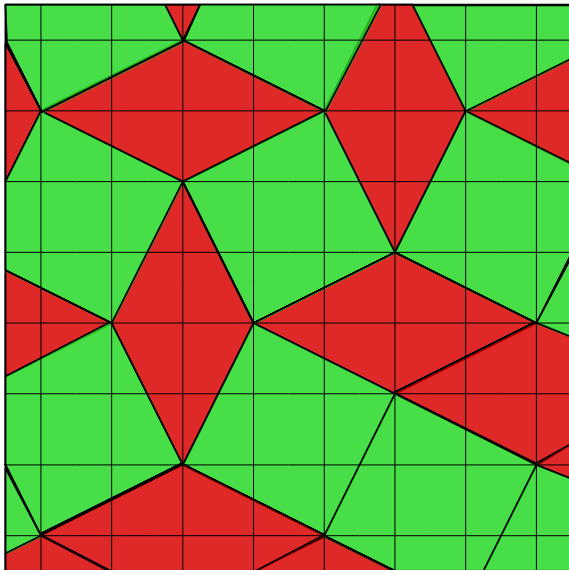
Square lattice: ground states

- # gs grows exponentially with the **linear** size of the system
- zero energy ground states found at **intermediate** filling
- compelling evidence for **critical edge modes**
- what is the nature of these states?



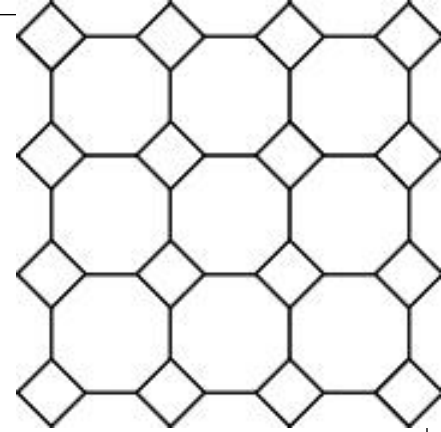
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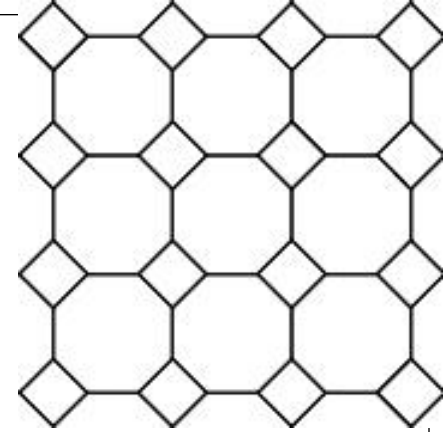
Octagon-square lattice

- $N \times M$ plaquettes with open bc : unique gs with one fermion per plaquette
- $N \times M$ plaquettes with closed bc: $2^M + 2^N - 1$ gs



[Fendley, Schoutens, '05]

Octagon-square lattice

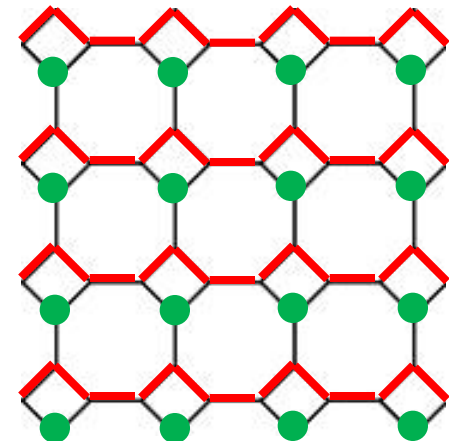


- $N \times M$ plaquettes with open bc : unique gs with one fermion per plaquette
- $N \times M$ plaquettes with closed bc: $2^M + 2^N - 1$ gs
- Distinct topological sectors in extreme susy staggering limit ($y \ll 1$) (preliminary)

$$Q^+ = \sum \lambda_i^* c_i^\dagger P_{\langle i \rangle}$$

$$Q^- = \sum \lambda_i c_i P_{\langle i \rangle}$$

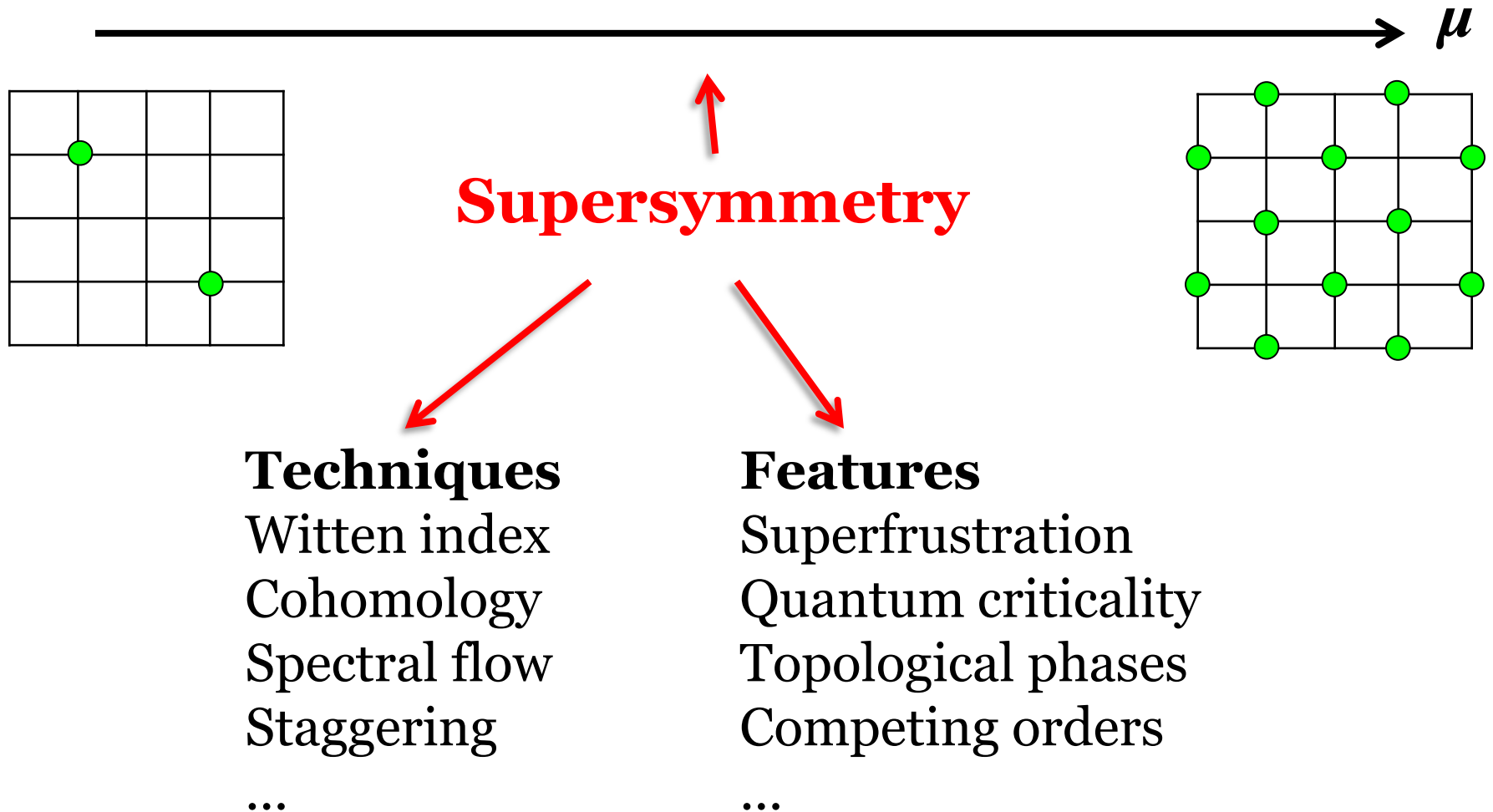
$$\lambda_i = \begin{cases} 1 & \text{for } i \in S_1 \text{ ———} \\ y & \text{for } i \in S_2 \text{ ●} \end{cases}$$



[Fendley, Schoutens, '05,
With E. Berg (in progress)]

Conclusions

Exact results for strongly interacting fermions



Connection to AdS/CFT ??

- Extensive gs entropy is typical feature
- In 1+1D emergent spacetime supersymmetry
- Low energy effective theories in 1+1D
N=2 superconformal minimal models
- ...

Thank you