Topological Defects

Mark Bowick

Symmetry Breaking : $\underbrace{G}_{\substack{\text{symmetry of} \\ \text{Hamiltonian/free energy/} \\ \text{action}}} \longrightarrow \underbrace{H}_{} \subset G$

$H \rightarrow$ symmetry of the ground state / vacuum

<u>Manifold of inequivalent ground states</u>

$$\mathcal{M} = G/H$$ ($\Rightarrow$ typically inf. # of inequivalent ground states)

Spontaneous Symmetry Breaking (SSB) picks one but the topology of $\mathcal{M}$ has important information.

Simplest examples:

Global symmetries

Φ = order field = order parameter for Φ = constant

$\Phi : \mathbb{R}^d \to \mathcal{M}$

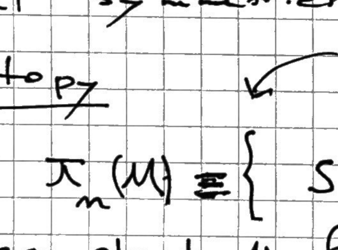
<u>Homotopy</u> $\{ \quad \}$ means set of homotopically equivalent maps

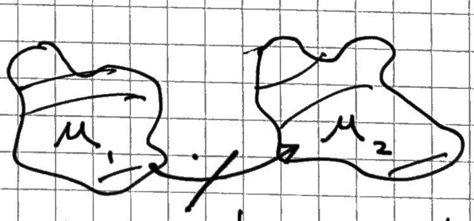
$$\pi_n(\mathcal{M}) \equiv \left\{ S^n \xrightarrow{\Phi} \mathcal{M} \right\}$$ (nth homotopy group)

<u>idea</u> learn about $\mathcal{M}$ from well-understood properties of S^n & the map $S^n \to \mathcal{M}$

* $\underline{\pi_0(\mathcal{M})}$ measures the connectedness of $\mathcal{M}$

$\pi_0(\mathcal{M}) \neq 0 \Rightarrow$ domain wall

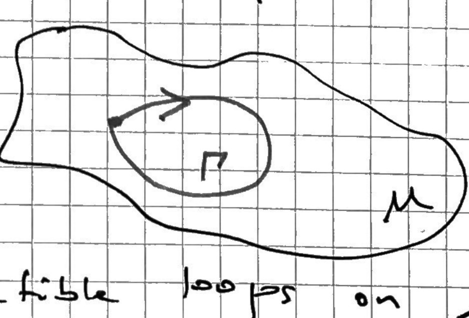




no continuous deformation connects $\mathcal{M}_1$ & $\mathcal{M}_2$ the connected components of $\mathcal{M}$

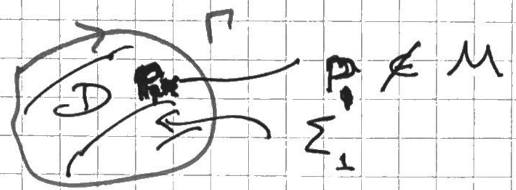
* $\pi_1(\mathcal{M}) = \{ S^1 \to \mathcal{M} \}$

= set of closed loops/paths on $\mathcal{M}$



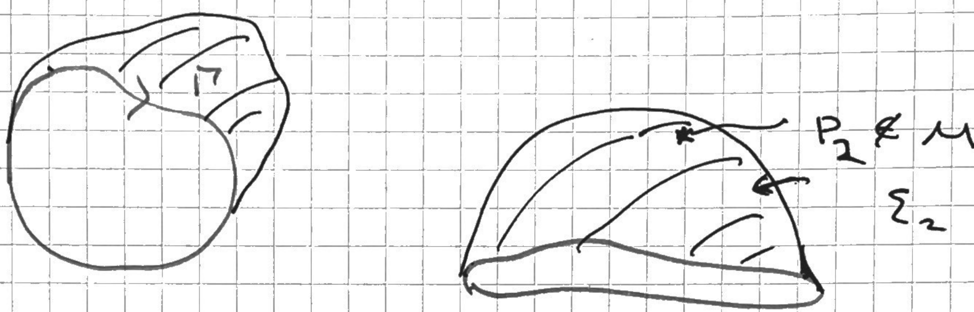
$\pi_1(\mathcal{M}) \neq 0$ if $\exists$ non-contractible loops on $\mathcal{M}$

A non-contractible loop on $\mathcal{M}$ $\Rightarrow$ $\exists$ a point $p_0 \in \mathcal{M}$ inside the domain enclosed by Γ that is not a member of $\mathcal{M}$ $\Rightarrow$ not in the ground state



In 3d

Now deform the domain D



Connect the points $\{p_i\}$ $\Rightarrow$ line of points $\notin \mathcal{M}$

$\Rightarrow$ line defect

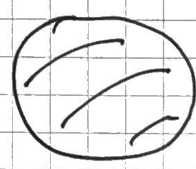
Typically the way you get off the manifold $\mathcal{M}$ is the have ϕ vanish $\phi(p_i) = 0$ $\Rightarrow$ zero of ϕ

ϕ vanishing $\Rightarrow$ $\nabla\phi\big|_{p_i}$ diverges (at p_i) since must "wind" by a fixed amount in a vanishing small circumference (not yet explained but will do so in examples where the topological invariant is calculated)

$\pi_2(\mathcal{M}) = \{ S^2 \to \mathcal{M} \}$

$S^2 \equiv$ 2-sphere = surface of a solid ball

$\pi_2(\mathcal{M}) \neq 0 \Rightarrow \mathcal{M}$ has non-contractible 2-spheres

i.e. $\exists\, \phi \ni$  $\xrightarrow{\phi} \mathcal{M}$ cannot be continuously deformed to a point $\Rightarrow \exists$ a point inside $\mathcal{M}$ which cannot be in the ground state i.e. a zero of $\phi \Rightarrow$ point defect in 3d

$d=3$

$\pi_3(\mathcal{M}) = \{ S^3 \to \mathcal{M} \} \Rightarrow$ non-trivial 3-spheres in $\mathcal{M}$

$\Rightarrow$ 3d texture that is topologically non-trivial

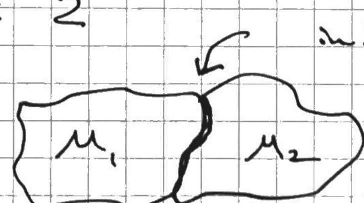
Thus in 3d

$\pi_0(\mathcal{M}) \neq 0$	domain wall	(2-dimensional)
$\pi_1(\mathcal{M}) \neq 0$	line-defect	(1-dimensional)
$\pi_2(\mathcal{M}) \neq 0$	point-defect	(0-dimensional)

$$\dim(\text{defect}) = 2 - n \quad \text{for} \quad \pi_n(\mathcal{M}) \neq 0$$
$$= \underline{\underline{d - 1 - n}}$$

π_3 is different (bulk texture) = 3-dim = d

Two-dimensions $d = 2$

$\pi_0(\mathcal{M}) \neq 0$ 

$\Rightarrow$ 1-dim defect

$\pi_1(\mathcal{M}) \neq 0 \Rightarrow$ non-contractible loop on a surface

$\Rightarrow$ point defect

$d=2$ (spatial)

$\pi_2(\mathcal{M}) \neq 0 \Rightarrow$ point-defect again

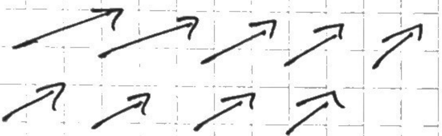
Examples

X-Y Models = Spins in a plane $(\vec{S}_i)$ = vector field on a plane

Order field / parameter = total Magnetization

$$\vec{M} = \sum_i \vec{S}_i$$

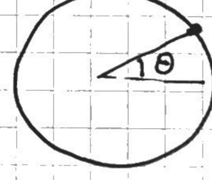
$\vec{M} \neq 0 \Rightarrow$ uniform magnetization in some direction



Look at fixed $\vec{M}$ $|\vec{M}| = 1$

$\Rightarrow \vec{M}$ lives on a circle (S^1)

$\mathcal{M} = S^1 = \{e^{i\theta}\}$



$\pi_0(\mathcal{M}) = 0$ 1-connected component

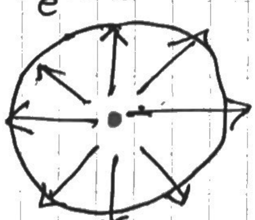
$\Rightarrow$ no domain lines (called walls colloquially here as well)

$\pi_1(S1) = \mathbb{Z} = \{n\}$ $n = -\infty, \ldots, -1, 0, 1, \ldots, \infty$

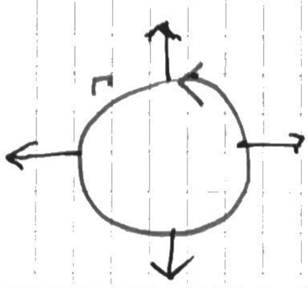
What does the integer n measure

It measures the number of times ϕ winds around the circle

$n=1$ $e^{i\theta}$



"magnetic" monopole



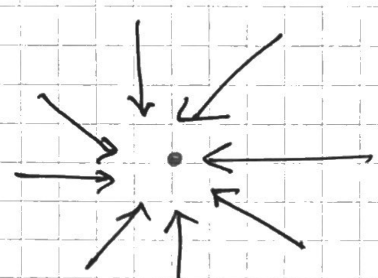
$\vec{M}(\theta)$ winds by 2π as θ goes from $0 \to 2\pi$ in the same sense

How to compute n:

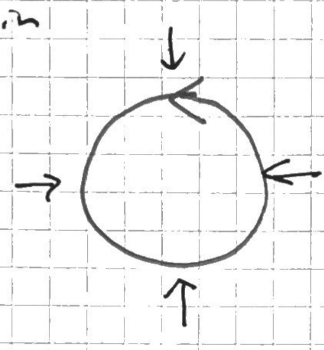
NB This does not depend on the sense of the loop Γ

What about ?

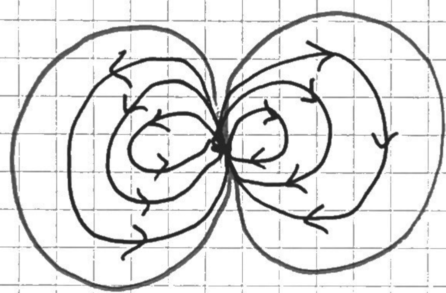
Quiz

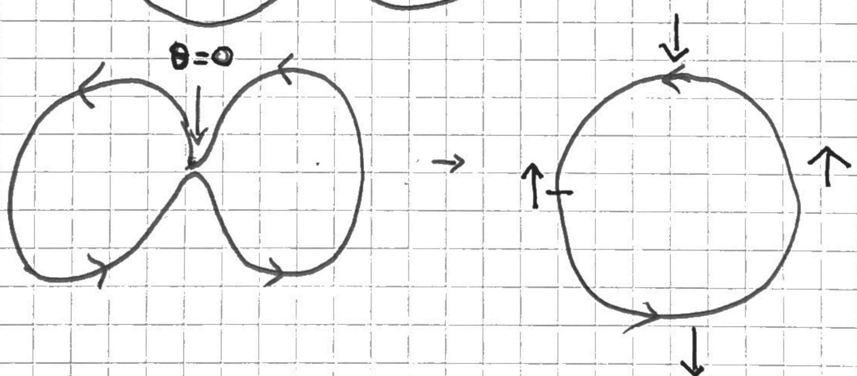


Ans. $n = +1$ again



$n = 2$ "magnetic dipole"

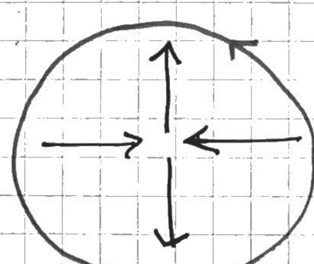




$\vec{M}$ winds 4π as θ winds 2π

$n = 2$

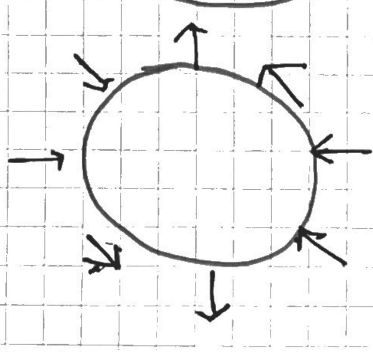
$n = -1$



$\vec{M}$ winds in opposite sense to θ

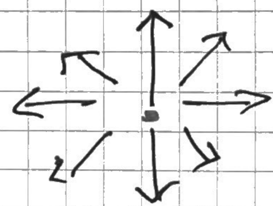
Winding # $= -1$

$n = -1$

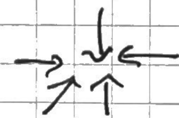


Terminology

$n = +1$

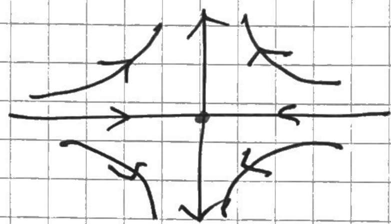


monopole or aster or "source"



"sink"

$n = -1$



hyperbolic point / flow
"cross-hair"

vortex

anti-vortex

N.B. Very different symmetries of monopole and cross-hair

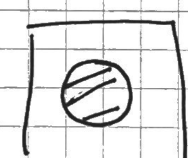
monopole – circularly symmetric

hyperbolic point / flow : 4-fold symmetry (2 reflection symmetries)

vortex and anti-vortex both have circular-symmetry

All vector-fields on the plane

$\pi_2(S^1) = 0$



No π_2 defects

X–Y model point π_1 defects only!

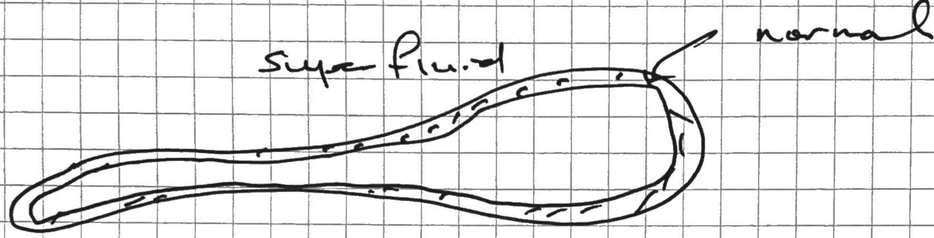
<u>Superfluid</u> $\quad \psi = \psi_0 e^{i\theta}$

$|\psi_0|$ = superfluid order/density (magnitude)

$\nabla\theta$ = superfluid velocity

$\mathcal{M} = S^1$ (symmetry of $U(1)$)

so again only $\pi_1(\mathcal{M}) \neq 0 \Rightarrow$ superfluid flux lines / loops



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Ferromagnet in 3d

Fix $|\vec{M}|$ (3-vector) = const M_0 $= O(3)/O(2)$

$\Rightarrow M_x^2 + M_y^2 + M_z^2 = M_0^2$ $\boxed{\mathcal{M} = SO(3)/SO(2)}$

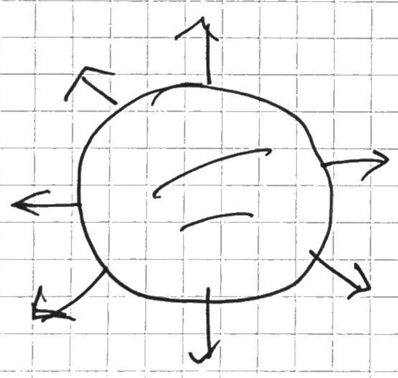
$\Rightarrow$ 2-sphere (radius M_0) $= SU(2)/U(1)$

$\pi_0(S^2) = 0$ no domain walls

$\pi_1(S^2) = 0$ The 2-sphere is contractible (you cannot lasso a [(2)-sphere] basketball)

no line-defects = magnetic flux lines

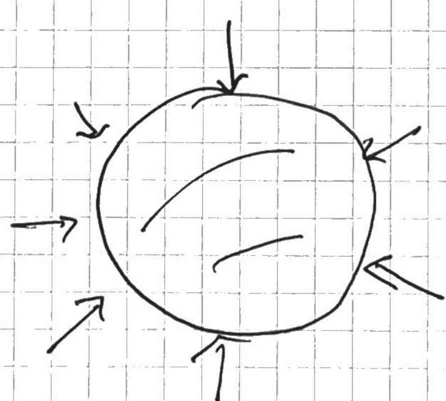
$\pi_2(S^2) = \mathbb{Z}$ (like $\pi_1(S_1)$) - you can wrap a 2-sphere an integer # of times on (itself) a 2-sphere



"hedgehog" $n = +1$
monopole
point charge (positron ~~electron~~)
Source in 3d

$\vec{M} \to -\vec{M}$

$\underline{n = -1}$



anti-monopole
negative point charge (electron)
sink in 3-d

both spherically-symmetric

<u>π_3</u> $\qquad \pi_3(S^2) = \mathbb{Z}$!!!

Hopf <u>fibration</u> $\qquad$ Hopf 1931
Actually known to Gauss

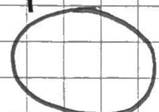
<u>Amazing</u> - you can wrap a 3-sphere on a 2-sphere

What does the Hopf fibration look like

$$S^3 \xrightarrow{f} S^2 \qquad \begin{aligned} &f(a,b,c,d) \\ &= (a^2+b^2-c^2-d^2,\ 2(ad+bc),\ 2(bd-ac)) \end{aligned}$$

$f^{-1}(\vec{x})$ is a loop Γ on S^3 (fibre of the Hopf map over $\vec{x}$)

Γ



$f^{-1}(\vec{x}')$ is another loop Γ'

They can link an integer # of times

-1 $\qquad$ $+1$

link # = 1
relative orientation determines sign

The linking # for Hopf fibration is independent of the 2 points $(\vec{x}, \vec{x}')$ one chooses

$\Rightarrow$ <u>invariant</u>

Mathematically the Hopf fibration is a fibre <u>bundle</u>

This S^1 fibre bundle over S^2 is <u>twisted</u>

$$S^1 \to S^3 \xrightarrow{\pi} S^2 \qquad (F, E, B)$$

<u>fibre</u> $\quad$ total space $\quad$ base space

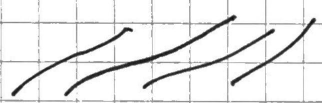
$\pi : E \to B$
projection

$$\begin{array}{ccc} F & \to & E \\ & & \downarrow \pi \\ & & B \end{array}$$

Liquid Crystals

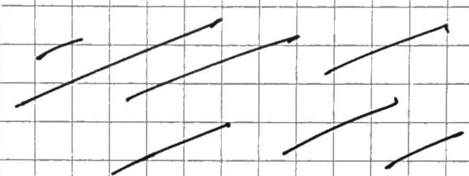
Now the relevant order field is an orientation in space rather than a direction.

By this I mean a line field rather than a vector field



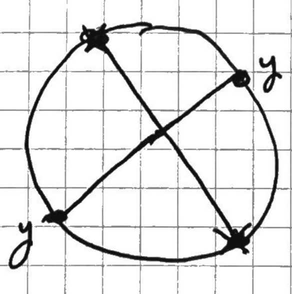
Nematic order means rotational invariance is broken but not translation invariance i.e. liquid wrt translations & crystalline wrt rotations

Uniform nematic



As a space if we take the orientational order constant we have the set of lines passing through a fixed point (the origin) as the set of all possible ordered states $\equiv$ projective space

Thus $\quad \mathcal{M} = S^2/\mathbb{Z}_2 \quad$ i.e. antipodal points are identified



N.B. $S^2/\mathbb{Z}_2 \equiv \mathbb{R}P^2$

$\mathcal{M} = SO(3)/D_\infty$

d=3

Defects/homotopy (d=3)

$\pi_0(S^2/\mathbb{Z}_2) = 0$ still a connected space since a 2-sphere but w/ identification

$\Rightarrow$ no domain walls

$\pi_1(S^2/\mathbb{Z}_2)$ Theorem $\pi_1(G/H) = \pi_0(H)$ if $\pi_1(G) = 0$

Thus $\pi_1(S^2/\mathbb{Z}_2) = \pi_0(\mathbb{Z}_2) = \mathbb{Z}_2$ since $\pi_1(S^2) = 0$

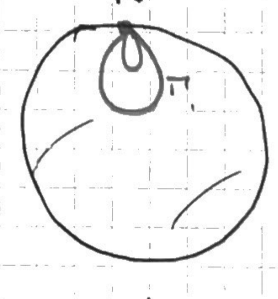
$$\mathbb{Z}_2 = (+1, -1) = (+1, e^{i\pi}) = (e^0, e^{i\pi})$$
$$= \{ e^{i\theta} : \theta = 0, \pi \}$$

One trivial defect (identity)

One non-trivial defect = -1 element of $\mathbb{Z}_2$

<u>What is it?</u>

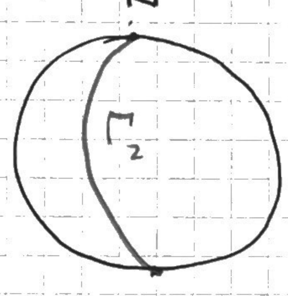
On $S^2/\mathbb{Z}_2$



take North pole as the origin

Γ_1 is a contractible loop (trivial)

<u>but</u>

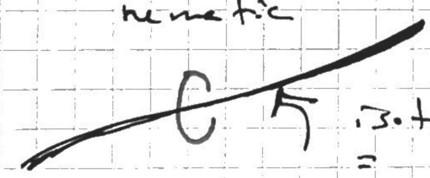


is also a loop!

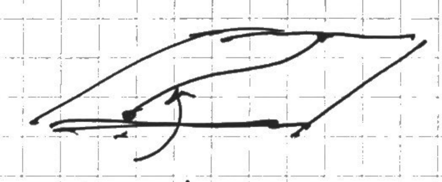
Γ_2 cannot be continuously contracted to the identity b/c it would break the identification $N \equiv S$

This is the -1 element of $\mathbb{Z}_2$

Since π_1 is a line-defect we have a nematic line defect called a disclination defect



For the loop above to be non-contractible the defect line must terminate at boundaries or be a closed loop



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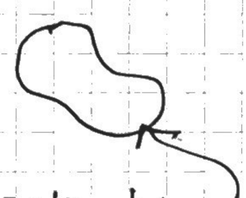
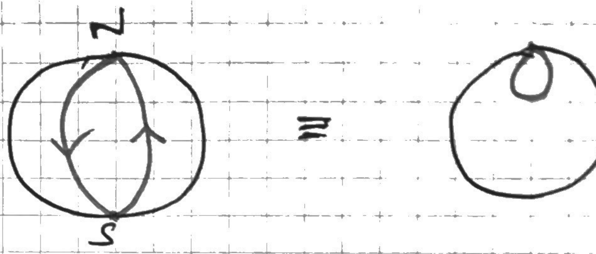


Algebra $\quad (-1)^2 = +1$

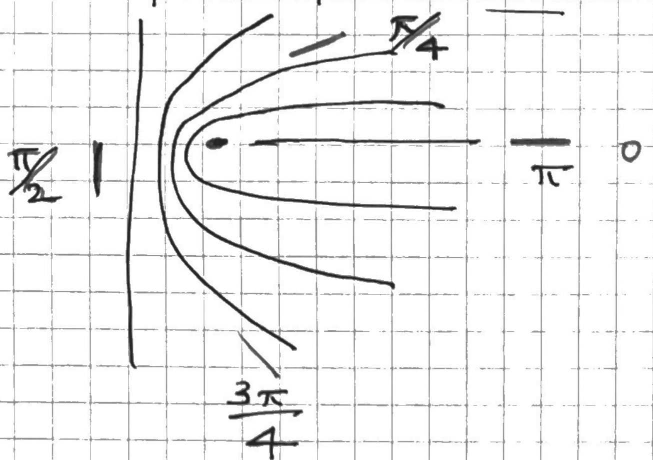


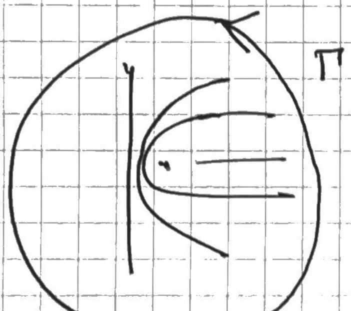
Line field is also called a director field $\vec{n}(\vec{x})$ w/

$\vec{n} \equiv -\vec{n}$ inversion symmetry $\Longleftrightarrow$

Non-contractible loops – basic disclination loop

In 2d-plane / cross-section





Director (line) field winds π in same sense as π winds 2π

strength of defect = topological charge

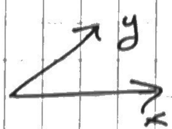
$= \pi/2\pi = +\frac{1}{2}$

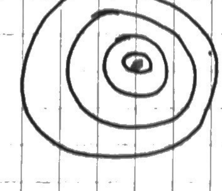
$\boxed{s = +\frac{1}{2}}$ $\quad$ index

$n = |n| e^{2i\theta}$

Continue point in 3d w/ cross-section as above

+1 defect





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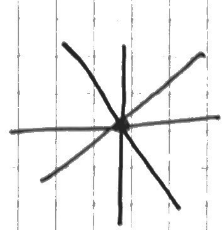
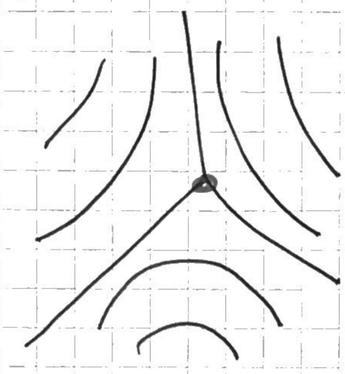


Can "escape" into the 3rd dimension (z) to point vertically and deform to the identity $\Rightarrow$ topologically trivial

$-\frac{1}{2}$ cross-section



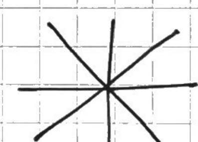
<u>Show</u> $+\frac{1}{2}$ can be continuously deformed into $-\frac{1}{2}$

$$+\tfrac{1}{2} \equiv -\tfrac{1}{2}$$

Thus only <u>one</u> topologically non-trivial class of defect $|s| = \frac{1}{2}$

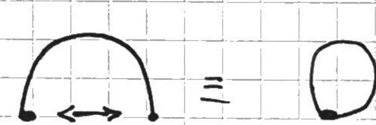
<u>2d</u> In <u>2d</u> the situation is different

Lines thru origin in 2d of fixed length



$\Rightarrow \quad \mathcal{M} = S^1/\mathbb{Z}_2 = \mathbb{RP}^1$

= hemisphere w/ ends identified

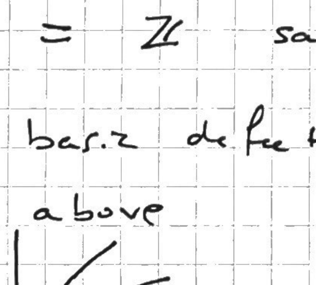


$= S^1$

$\pi_0(\mathcal{M}) = 0$ no domain walls

$\pi_1(S^1/\mathbb{Z}_2) = \mathbb{Z}$ same as for $\pi_1(S^1)$

but the basic defect (elementary defect) is
≡ minimal defect
the one above



$+\frac{1}{2}$

Thus we have all $\mathbb{Z}$ multiples of $\frac{1}{2}$

$$-\frac{n}{2}, \dots, -1, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 1, \dots \frac{n}{2}$$

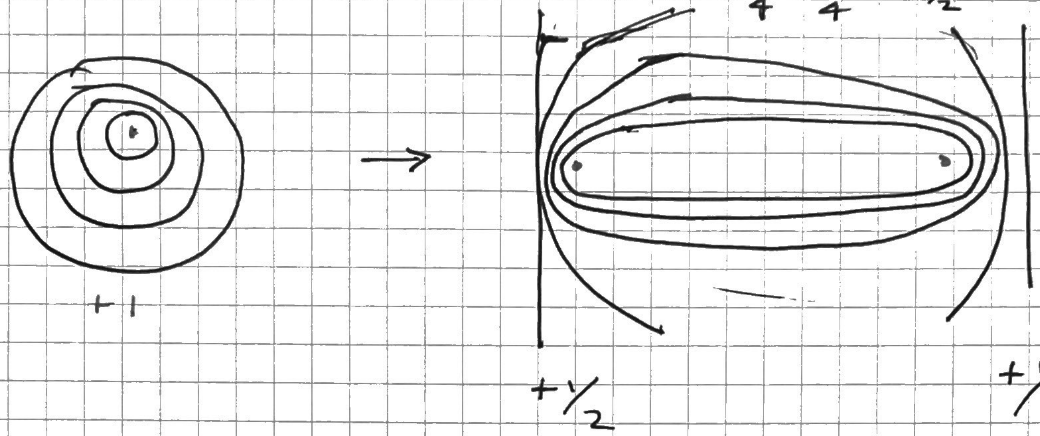
Since there is no escape into the 3rd dimension the integer strength defects are stable topologically and $+\frac{1}{2}$ cannot be deformed into $-\frac{1}{2}$ etc

In this respect 2d is richer than 3d

of course the energy of a defect $E_s \sim s^2$

so it pays for $+1 \to \underline{\underline{+\frac{1}{2}}} + \underline{\underline{\frac{1}{2}}}$

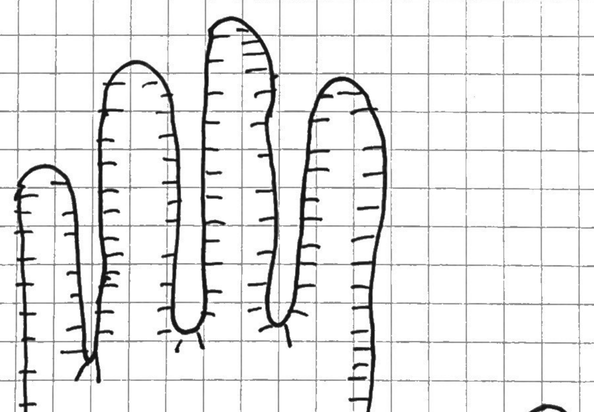
$E \sim 1$ $\qquad E \sim \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$



Far field is the same in both cases!

N.B. Since topological charge is an invariant we can compute the total charge inside a region with any contour including one @ the boundary.

Example Finger prints = ridge system



Around boundary director rotates $-\pi$ 4 times or $D-1$ for D fingers.

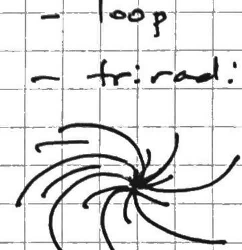
$$\sum_{\text{defects}} (n_{+1/2} - n_{-1/2}) = -(D-1)$$

Finger prints

$+\frac{1}{2}$ – loop

$-\frac{1}{2}$ – triradius

whorl



whorl $\equiv$ 2 $+\frac{1}{2}$s

Biaxial Nematics

symmetry of molecule is that of a rectangular box
— proper point group D_2

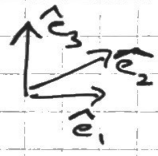
Order parameter is a real symmetric matrix with three (fixed) distinct eigenvalues

Requires almost the entire 3-d rotation group to specify the orientation of such a molecule.

<u>But</u> π rotations about any of the 3 perpendicular symmetry axes are indistinguishable

$$\mathcal{M} = SO(3)/D_2 \qquad D_2 = \{1, \pi \text{ rotations about } \hat{e}_1, \hat{e}_2, \hat{e}_3\}$$

$$= SU(2)/Q$$



Q = group of quaternions

$$Q = \{\pm 1, \pm i, \pm \hat{j}, \pm k : i^2 = \hat{j}^2 = k^2 = ijk = -1\}$$

$$i = i\sigma_x \qquad \hat{j} = i\sigma_y \qquad k = i\sigma_z$$

$$Q = \{\pm 1, \pm i\sigma_x, \pm i\sigma_y, \pm i\sigma_z\}$$

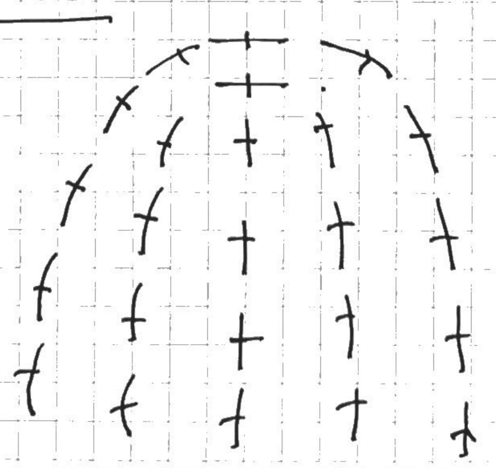
$\pi_1(\mathcal{M}) = Q$ which is non-abelian !

$\Rightarrow$ network junctions for line defects !!

Non-abelian fundamental groups

<u>3 classes of defect</u>

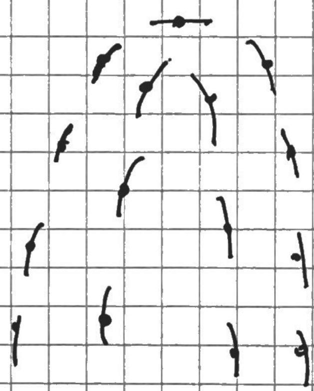
(a)



C_2

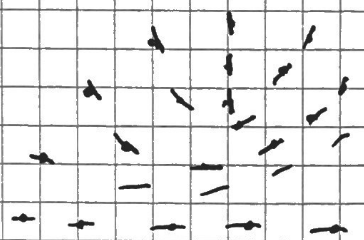
Biaxial Nematic

(b)



C_x

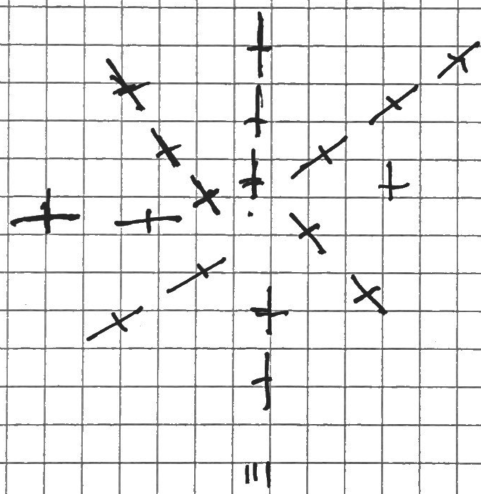
(c)

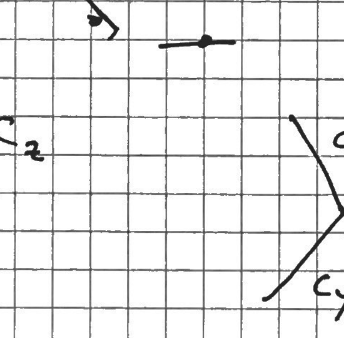


C_y

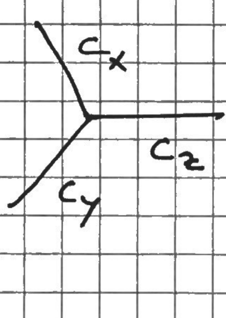
$\bar{C}_0 = -1$

(d)





$C_x C_y = C_z$



He-3 Very rich

Dipole-locked A phase $G = SO(3)$ $H = Id$

$$\boxed{M = SO(3)} \approx SU(2)/\mathbb{Z}_2 = S^3/\mathbb{Z}_2$$

$\pi_0(S^3) = 0$ $\pi_1(S^3/\mathbb{Z}_2) = \mathbb{Z}_2$ $\pi_2(S^3) = \mathbb{Z}$ $\pi_3(S^3) = \mathbb{Z}$